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Original Article

Quasi-Ultrafilter on the Connectivity System: Its Relationship to Branch-Decomposition

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Abstract *- The exploration of graph width parameters, spanning both graph theory and algebraic frameworks, has captured substantial attention. Among these, branch width has distinctly emerged as a key metric. The Quasi-Ultrafilter serves as an axiomatic tool for scrutinizing incomplete social judgments. In this concise study, we outline a coherent definition of Quasi-Ultrafilters within the connectivity system and clarify its dual association with branch width.*

Keywords *- Filter, Ultrafilter, Quasi-Ultrafilter, Branch-width, Branch-decomposition.*

1. Definitions and Notations in this Paper

This section provides mathematical definitions of each concept.

1.1. Filters on Boolean Algebras

In the Boolean algebra $(X, \cup, \cap)$, a filter is defined as outlined below. Filters and Ultrafilters stand as cornerstone concepts in mathematics, with a wealth of research and related studies on them available in references [30-40]. Within this algebraic structure, the complement of a filter is termed an ideal.

Definition 1: In a Boolean algebra $(X, \cup, \cap)$, a set family $F \subseteq 2^X$ satisfying the following conditions is called a filter on the carrier set X.

(FB1) $A, B \in F \Rightarrow A \cap B \in F$,

(FB2) $A \in F, A \subseteq B \subseteq X \Rightarrow B \in F$,

(FB3) $\emptyset$ is not belong to F.

In a Boolean algebras $(X, \cup, \cap)$, A maximal filter is called an ultrafilter and satisfies the following axiom (FB4):

(FB4) $\forall A \subseteq X$, *either* $A \in F$ *or* $X / A \in F$.

1.2. Quasi-Ultrafilter on Boolean Algebras

In reference [1], the notion of a Quasi-Ultrafilter is introduced. This literature also provides an axiomatic examination of incomplete social judgments. The quasi-ultrafilter plays a pivotal role in the proofs of reference [1].

This concept is illustrated using a Boolean algebra $(X, \cup, \cap)$. While the properties of a Quasi-Ultrafilter closely resemble those of an ultrafilter, they diverge in property (QB1). The significance of the Quasi-Ultrafilter is evident, given its mention in various related studies (e.g., [1-8,25]).

Definition 2: In a Boolean algebra $(X, \cup, \cap)$, a set family $Q \subseteq 2^X$ satisfying the following conditions is called a Quasi-filter on the carrier set X.

(QB1) $A \subseteq X, B \subseteq X, A \notin Q, B \notin Q \Rightarrow A \cup B \notin Q$,

(QB2) $A \in Q, A \subseteq B \subseteq X \Rightarrow B \in Q$,

(QB3) $\emptyset$ is not belong to Q.

(QB4) $\forall A \subseteq X$, *either* $A \in Q$ *or* $X / A \in Q$.

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1.3. Symmetric Submodular Function and Connectivity System

The definition of a symmetric submodular function is given below. The symmetric submodular function is widely utilized and discussed in various scholarly publications (e.g., [9-12]).

Definition 3: Let X be a finite set. A function $f\colon X \to \mathbb{N}$ is called symmetric submodular if it satisfies the following conditions:

· $\forall A \subseteq X,\ f(A) = f(X \backslash A)$.

· $\forall A, B \subseteq X,\ f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

In this short paper, a pair (X, f) of a finite set X and a symmetric submodular function f is called a connectivity system.
It is known that a symmetric submodular function f satisfies the following properties:

Lemma 1[12] : A symmetric submodular function f satisfies:

1. $\forall A \subseteq X,\ f(A) \geq f(\emptyset) = f(X)$.
2. $\forall A, B \subseteq X,\ f(A) + f(B) \geq f(A \backslash B) + f(B \backslash A)$.

In this short paper, we use the notation f for a symmetric submodular function, a finite set X, and a natural number k. A set A is k-efficient if $f(A) \leq k$. Unless otherwise specified, in this paper, the underlying set X is assumed to be a non-empty finite set.

1.4. Branch-Decomposition of a Connectivity System

In graph theory, branch width stands as a pivotal graph width parameter. It entails a branch decomposition wherein the decomposition's leaves align with the graph's edges. Every edge is paired with a value derived from a symmetric submodular function, gauging the connectivity between edges. Branch width notably extends the breadth of symmetric submodular functions applied to graphs.

The definition of branch-decomposition is shown below. Due to its significance, branch-decomposition has been the subject of various research studies [13-29].

Definition 5: Let (X, f) be a connectivity system. The pair (T, μ) is a branch decomposition tree of (X, f) if T is a ternary tree such that $|L(T)| = |X|$ and μ is a bijection from $L(T)$ to X, where $L(T)$ denotes the leaves in T. For each $e \in E(T)$, we define $bw(T, \mu, e)$ as $f(\cup_{v \in L(T1)} \mu(v))$, where T_1 is a tree obtained by removing e from T (taking into account the symmetry property of f). The width of (T, μ) is defined as the maximum value among $bw(T, \mu, e)$ for all $e \in E(T)$. The branch-width of X, denoted by $bw(X)$, is defined as the minimum width among all possible branch decomposition trees of X.

2. Quasi-Ultrafilter on Connectivity System

We introduce the Quasi-Ultrafilter on the Connectivity System (X,f) as an extension of the Quasi-Ultrafilter on Boolean Algebras. Subsequently, we elucidate its dual relationship with branch-width. The primary distinction in this definition, compared to the one on Boolean Algebras, is the inclusion of the Symmetric Submodular Function condition.

Definition 4: Let X be a finite set and f be a symmetric submodular function. In a connectivity system, the set family $Q \subseteq 2^X$ is called a Quasi ultrafilter of order $k+1$ if the following axioms hold true:

(Q0) $\forall A \in Q,\ f(A) \leq k$

(Q1) $A \subseteq X,\ B \subseteq X,\ A \notin Q,\ B \notin Q \Rightarrow A \cup B \notin Q$

(Q2) $A \in Q,\ A \subseteq B \subseteq X$, $f(B) \leq k \Rightarrow B \in Q$

(Q3) $\emptyset$ is not belong to Q.

(Q4) $\forall A \subseteq X,\ f(A) \leq k \Rightarrow$ *either* $A \in Q$ *or* $X / A \in Q$.

And Quasi-Ultrafilter is non-principal if the Quasi-Ultrafilter satisfies following axiom:

(Q5) $A \notin Q$ for all $A \subseteq X$ with $|A| = 1$.

The main theorem of this paper is presented as follows. This proof utilizes techniques from the paper [19]. At first glance, the concepts that seem unrelated possess an extremely intriguing duality when specific conditions are applied. Moving forward, I plan to continue exploring such interconnected concepts.

Theorem 2: Let X be a finite set and f be a symmetric submodular function. Branch-width of the connectivity system (X, f) is at most k if and only if no (non-principal) Quasi Ultrafilter of order $k+1$ exists.

Proof. This proof utilizes techniques from the paper [19]. So the proof will be presented concisely, focusing primarily on the key points or highlights.

Let X be a finite set and f be a symmetric submodular function. Assume that the branch-width of the connectivity system (X, f) is at most k. Note that A set $A \subseteq X$ is called k-branched if the connectivity system obtained from f by identifying $X \setminus A$ has branch-width at most k.

Consider the set I defined by $I = \{A \mid X \setminus A \in Q\}$. If the branch-width of the connectivity system (X, f) is bounded above by k, then the set X is classified as k-branched. It's evident that any k-branched set, provided it consists of at least two elements, can be expressed as the union of two distinct, proper subsets that are both k-branched. Given axiom (Q3) and axiom (Q4) in definition of non-principal Quasi Ultrafilter, we have $X \in Q$, implying $X \notin I$. Although I is expected to encompass all k-branched sets, the absence of X from I creates a contradiction. Thus, there cannot exist a non-principal Quasi Ultrafilter. And if the branch-width of the connectivity system (X, f) is greater than k, then there exists a non-principal Quasi Ultrafilter. This proof is completed.

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