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Research Article

A Classification of 4-Degree Tri-Cayley Graphs Over a Group of Order pq

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Abstract *- Symmetry properties are of vital importance for graphs. Meanwhile, graphs with the vertex transitivity are a class of highly symmetrical graphs. A graph Φ is said to be a tri-Cayley graph over a group H if it has a semi-regular automorphism group which acts on the vertex set with three orbits of equal length and is isomorphic to H. In this paper, the vertex transitivity, edge transitivity and arc transitivity of the 4-degree 0-type and 2-type tri-Cayley graphs over a group $\mathbb{Z}_{pq}$ are discussed and give the automorphism group of the corresponding vertex transitive graph.*

Keywords *- Group $\mathbb{Z}_{pq}$, Tri-Cayley graph, Vertex transitive, Automorphism group, Edge transitive.*

1. Introduction

A graph is said to be a *tri-Cayley graph* if it admits a semiregular subgroup of automorphisms having three orbits of equal length. Let L_0, L_1, L_2, S_0, S_1 and S_2 be subsets of a group H with identity element e such that $L_0 = L_0^{-1}, L_1 = L_1^{-1}, L_2 = L_2^{-1}$ and $e \notin L_0 \cup L_1 \cup L_2$. Then we let $\Phi = \mathrm{TCay}(H; L_0, L_1, L_2; S_0, S_1, S_2)$ be the graph with vertex set $\mathbb{Z}_3 \times H$, and edge set the union of $\{\{(0,h),(0,l_0h)\} \mid l_0 \in L_0\}, \{\{(0,h),(1,s_0h)\} \mid s_0 \in S_0\}, \{\{(1,h),(1,l_1h)\} \mid l_1 \in L_1\}, \{\{(1,h),(2,s_1h)\} \mid s_1 \in S_1\}, \{\{(2,h),(2,l_2h)\} \mid l_2 \in L_2\}$ and $\{\{(2,h),(0,s_2h)\} \mid s_2 \in S_2\}$. For the case when $|S_0| = |S_1| = |S_2| = 1$, $\mathrm{TCay}(H; L_0, L_1, L_2; S_0, S_1, S_2)$ is also called one-matching tri-Cayley graph. Also, if $|L_0| = |L_1| = |L_2| = t$, then $\mathrm{TCay}(H; L_0, L_1, L_2; S_0, S_1, S_2)$ is said to be t -type tri-Cayley graph.

The concept of the Cayley graph was proposed by Cayley [1] in 1878, and we can use it to construct graphs with special symmetry due to its simple construction and high symmetry. In [2], it investigated the normality of Cayley graphs of order pq, where p, q are distinct primes and $p > q \geq 3$, and determined all non-normal Cayley graphs of order pq. In [3], it showed that every abelian Cayley graph is edge-hamiltonian and every Cayley graph of order pq is also edge-hamiltonian, where p, q are primes.

The n -Cayley graph is a natural generalization of the Cayley graph. There are many research results on the edge connectivity, characteristic polynomial, normality and other properties of n -Cayley graph. For example, in [4], bounds for the edge connectivity of n -Cayley graphs were found, and also several structural conditions were given for a connected k -regular bi-abelian graph to have edge connectivity strictly less than k. In 2013, Arezoomand et al. [5] represented the adjacency matrix of n -Cayley graph as a diagonal block matrix in terms of irreducible representations of G and determined its characteristic polynomial. In [6], it determined the characteristic polynomial of quasi-abelian n -Cayley graphs and exactly determined the eigenvalues. In [7], it proved that every finite group admits a vertex-transitive normal n -Cayley graph for every $n \geq 2$. In [8], it investigated properties of symmetric n -Cayley graphs in the special case of valency 3, and used these properties to develop a computational method for classifying connected cubic core-free symmetric n -Cayley graphs. Especially, the tri-Cayley graph has also been a hot topic. For example, in 2009, Kutnar et al. [9] studied the structure of strongly regular tri-Cayley graphs and a structural description of strongly regular tri-Cayley graphs of cyclic groups was given; it gave that the complete bipartite graph $K_{3,3}$, the Pappus graph, Tutte's 8-cage and the unique cubic symmetric graph of order 54 are the only connected cubic symmetric tricirculants in [10]; all finite connected cubic vertex-transitive tricirculants were classified in [11].

Moreover, it is well known that the symmetric graph is an important graph not only in algebraic graph theory, but also has a wide range of applications in real life. For example, more efficient algorithms can be realized by using the symmetry of the graph in the field of the Internet models. Therefore, it is necessary for us to study the vertex transitive graphs. In this paper, the vertex transitivity, edge transitivity and arc transitivity of the 4-degree 0-type and 2-type tri-Cayley graphs over a group $\mathbb{Z}_{pq}$ are discussed and give the automorphism group of the corresponding vertex transitive graph.

2. Definition and Preliminaries

In this paper, we define that all graphs are finite, connected, simple, regular, undirected and all groups are finite. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [12,13].

For any $x, y \in V(\Phi)$, if there exists a walk connects x to y, then Φ is said to be *connected*. A graph Φ is connected, finite, undirected and simple, then we use $V(\Phi), E(\Phi), A(\Phi), \mathrm{Aut}(\Phi)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. A graph Φ is *vertex transitive*, *edge transitive* and *arc transitive* (or *symmetric*) if $\mathrm{Aut}(\Phi)$ acts transitively on $V(\Phi), E(\Phi)$ and $A(\Phi)$, respectively. The set of neighbours of a vertex x in a graph Φ is denoted by $N(x)$. Meanwhile, we denote that $N_y(x)$ is the set of vertices at a distance of y from the vertex x, called the y*-step neighborhood* of the vertex x, where y is a positive integer. The *degree* of a vertex x in a graph Φ, denoted by $d_\Phi(x)$, is the number of edges of Φ incident with x. A graph Φ is said to be t*-regular* if $d(x) = t$ for any $x \in V(\Phi)$. Let G be a permutation group on a set Ω and $\beta \in \Omega$, the *vertex-stabilizer* of β in G is denoted by $G_\beta = \{g \in G \mid \beta^g = \beta\}$, that is to say, the subgroup of G fixing the vertex β.

In this section, we always assume that $\Phi = \mathrm{TCay}(H; L_0, L_1, L_2; S_0, S_1, S_2)$ is a connected tri-Cayley graph over a group H.

Definition 2.1 *Take any $j \in \mathbb{Z}_3 = \{0,1,2\}$ and $x, y \in H$, we define $R(y): \mathbb{Z}_3 \times H \mapsto \mathbb{Z}_3 \times H$ by*

$$R(y): (j, x) \mapsto (j, xy).$$

Clearly, $R(y)R(y') = R(yy')$. Set $R(H) = \{R(y) \mid y \in H\}$. Then $R(H)$ is a semiregular subgroup of $\mathrm{Aut}(\Phi)$ isomorphic to H.

Lemma 2.2 *([14]) For a finite group H, if it acts on the finite set Ω and $\beta \in \Omega$, then we have*

$$|\beta^H| = |H : H_\beta|.$$

Lemma 2.3 *Assume that $\Phi = \mathrm{TCay}(H; L_0, L_1, L_2; S_0, S_1, S_2)$ is a connected tri-Cayley graph over a group H, meanwhile, Φ is also regular, then $|S_0| = |S_1| = |S_2|$. Furthermore, $S_j \neq \emptyset$, where $j = 0,1,2$.*

Lemma 2.4 *The following hold.*
(1) Up to graph isomorphism, S_j can be chosen to contain the identity element of H, where $j = 0,1,2$.
(2) H is generated by $L_0 \cup L_1 \cup L_2 \cup S_0 \cup S_1 \cup S_2$.

Assume that $\Phi = \mathrm{TCay}(H; L_0, L_1, L_2; S_0, S_1, S_2)$ is a connected tri-Cayley graph over a group $H = \mathbb{Z}_{pq}$, where p, q are distinct primes and $p > q$. According to the definition of tri-Cayley graph and Lemma 2.4, we can construct 4-degree tri-Cayley graphs over group $H = \mathbb{Z}_{15} = \langle c \rangle$. As shown below:
(1) $\Phi_1 = \mathrm{TCay}(H; \emptyset, \emptyset, \emptyset; \{1, c\}, \{1, c\}, \{1, c\})$;
(2) $\Phi_2 = \mathrm{TCay}(H; \emptyset, \emptyset, \emptyset; \{1, c\}, \{1, c^{-1}\}, \{1, c^{-1}\})$;
(3) $\Phi_3 = \mathrm{TCay}(H; \{c, c^{-1}\}, \{c, c^{-1}\}, \{c, c^{-1}\}; \{1\}, \{1\}, \{1\})$.

3. 4-Degree 0-Type Tri-Cayley Graph

Theorem 3.1 *Let $\Phi_1 = \mathrm{TCay}(H; \emptyset, \emptyset, \emptyset; \{1, c\}, \{1, c\}, \{1, c\})$ be a connected 4-degree 0-type tri-Cayley graph over a group H. Then Φ_1 is vertex-transitive.*

Proof: Take any $c^t \in H$, we define a mapping λ from $V(\Phi_1)$ to $V(\Phi_1)$ as follows:

$$\lambda: (0, c^t) \mapsto (1, c^t),\ (1, c^t) \mapsto (2, c^t),\ (2, c^t) \mapsto (0, c^t),$$

where $t = 0, 1, \cdots, 14$. Firstly, we will prove that λ is a bijection. Take any $(1, c^t)$, $(2, c^t)$, $(0, c^t) \in V(\Phi_1)$, there exist $(0, c^t)$, $(1, c^t)$, $(2, c^t) \in V(\Phi_1)$ such that

$$(0, c^t)^\lambda = (1, c^t), (1, c^t)^\lambda = (2, c^t) \text{ and } (2, c^t)^\lambda = (0, c^t).$$

Therefore, λ is a surjection. Take any $(0, c^t)$, $(0, c^{t'})$, $(1, c^t)$, $(1, c^{t'})$, $(2, c^t)$ and $(2, c^{t'}) \in V(\Phi_1)$, then

$$(0, c^t) = (0, c^{t'}) \Leftrightarrow \lambda((0, c^t)) = \lambda((0, c^{t'})) \Leftrightarrow (1, c^t) = (1, c^{t'}),$$
$$(1, c^t) = (1, c^{t'}) \Leftrightarrow \lambda((1, c^t)) = \lambda((1, c^{t'})) \Leftrightarrow (2, c^t) = (2, c^{t'}),$$
$$(2, c^t) = (2, c^{t'}) \Leftrightarrow \lambda((2, c^t)) = \lambda((2, c^{t'})) \Leftrightarrow (0, c^t) = (0, c^{t'}).$$

Thus, λ is a bijection. Next, we claim that $\lambda \in \mathrm{Aut}(\Phi_1)$. Since

$$N((0,c^t))^{\lambda} = \{(1,c^t),(1,c^{t+1}),(2,c^t),(2,c^{t-1})\}^{\lambda} = \{(2,c^t),(2,c^{t+1}),(0,c^t),(0,c^{t-1})\} = N((1,c^t)),$$
$$N((1,c^t))^{\lambda} = \{(2,c^t),(2,c^{t+1}),(0,c^t),(0,c^{t-1})\}^{\lambda} = \{(0,c^t),(0,c^{t+1}),(1,c^t),(1,c^{t-1})\} = N((2,c^t)),$$
$$N((2,c^t))^{\lambda} = \{(0,c^t),(0,c^{t+1}),(1,c^t),(1,c^{t-1})\}^{\lambda} = \{(1,c^t),(1,c^{t+1}),(2,c^t),(2,c^{t-1})\} = N((0,c^t)).$$

Then $\lambda \in \mathrm{Aut}(\Phi_1)$. Furthermore, $\langle R(H),\lambda\rangle$ acts transitively on $V(\Phi_1)$. Hence, Φ_1 is vertex-transitive.

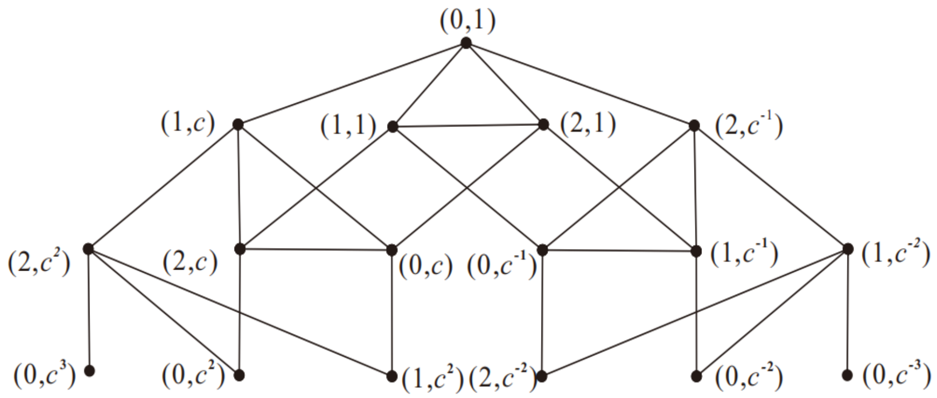

Fig. 1 Induced subgraph of $\Phi_1 = \mathrm{TCay}(H;\emptyset,\emptyset,\emptyset;\{1,c\},\{1,c\},\{1,c\})$

Theorem 3.2 *Let $\Phi_1 = \mathrm{TCay}(H;\emptyset,\emptyset,\emptyset;\{1,c\},\{1,c\},\{1,c\})$ be a connected 4-degree 0-type tri-Cayley graph over a group$H$. Then$\mathrm{Aut}(\Phi_1) = \langle R(H),\lambda\rangle\mathbb{Z}_2 \times \mathbb{Z}_2$, where λ is defined in Theorem 3.1.*

Proof: We can get that $|A| = |A_{(0,1)}||(0,1)^A|$by Lemma 2.2, where$A = \mathrm{Aut}(\Phi_1)$. By Theorem 3.1, one has$|(0,1)^A| = 45$.

(1) The action of $A_{(0,1)}$ on set $N((0,1))$ is faithful.

Let K be the kernel of the action of $A_{(0,1)}$ on set $N((0,1))$, then K fixes $(0,1),(1,1),(1,c),(2,1)$ and $(2,c^{-1})$. From Fig 1, we can find that $[(0,1),(1,1),(0,c^{-1}),(2,c^{-1})]$ is the unique 4-cycle passing through $(0,1),(1,1)$ and$(2,c^{-1})$, then K fixes $(0,c^{-1})$; we can find that $[(0,1),(2,1),(1,c^{-1}),(2,c^{-1})]$ is the unique 4-cycle passing through $(0,1),(2,1)$ and$(2,c^{-1})$, then Kfixes $(1,c^{-1})$; we can find that $[(0,1),(1,c),(0,c),(2,1)]$ is the unique 4-cycle passing through $(0,1),(1,c)$ and$(2,1)$, then Kfixes $(0,c)$; we can find that $[(0,1),(1,1),(2,c),(1,c)]$ is the unique 4-cycle passing through $(0,1),(1,1)$ and$(1,c)$, then Kfixes $(2,c)$. Meanwhile, it is easy to find that there are two 4-cycles passing through $(1,c)$ and $(0,c)$, namely $[(1,c),(0,c),(1,c^2),(2,c^2)]$ and$[(1,c),(0,c),(2,1),(0,1)]$, and that there are two 4-cycles passing through $(2,c^{-1})$ and$(1,c^{-1})$, namely $[(2,c^{-1}),(1,c^{-1}),(0,c^{-2}),(1,c^{-2})]$ and $[(2,c^{-1}),(1,c^{-1}),(0,1),(2,1)]$. Since $(0,1)$ and $(2,1)$ are fixed, it follows that $(2,c^2)$ and $(1,c^{-2})$ are also fixed. Thus, Kfixes$N_2(1_0)$. Note that the graph Φ_1 is connected and vertex-transitive, then K fixes all the vertices in it. Hence, $K = 1$.

(2) The action of $A_{(0,1)}$ on set $N((0,1))$ is not transitive and$A_{(0,1)} \cong \mathbb{Z}_2$.

Assume that the action of $A_{(0,1)}$ on set $N((0,1))$ is transitive. Then there exists $\pi_1 \in A_{(0,1)}$ such that$(1,c)^{\pi_1} = (1,1),(1,c)^{\pi_1} = (2,1),(2,1)^{\pi_1} = (2,c^{-1})$, where$o(\pi_1) = 4$. From Fig 1, we can get that

$$|N((1,c)) \cap N_2((0,1))| = 3 \neq |N((1,1)) \cap N_2((0,1))| = 2,$$

a contradiction. Thus, the action of $A_{(0,1)}$ on set $N((0,1))$ is not transitive.

Next, we will prove that$A_{(0,1)} \cong \mathbb{Z}_2$. Take any$c^t \in H$, we define a mapping π_2 from $V(\Phi_1)$ to $V(\Phi_1)$ as follows:

$$\pi_2\colon (0,c^t) \mapsto (0,c^{-t}),(1,c^t) \mapsto (2,c^{-t}),(2,c^t) \mapsto (1,c^{-t}),$$

where$t = 0,1,\cdots,14$. It is easy to see that π_2 is a bijection. Next, we claim that$\pi_2 \in \mathrm{Aut}(\Phi_1)$. Since

$$N((0,c^t))^{\pi_2} = \{(1,c^t),(1,c^{t+1}),(2,c^t),(2,c^{t-1})\}^{\pi_2} = \{(2,c^{-t}),(2,c^{-t-1}),(1,c^{-t}),(1,c^{-t+1})\} = N((0,c^{-t})),$$
$$N((1,c^t))^{\pi_2} = \{(2,c^t),(2,c^{t+1}),(0,c^t),(0,c^{t-1})\}^{\pi_2} = \{(1,c^{-t}),(1,c^{-t-1}),(0,c^{-t}),(0,c^{-t+1})\} = N((2,c^{-t})),$$
$$N((2,c^t))^{\pi_2} = \{(0,c^t),(0,c^{t+1}),(1,c^t),(1,c^{t-1})\}^{\pi_2} = \{(0,c^{-t}),(0,c^{-t-1}),(2,c^{-t}),(2,c^{-t+1})\} = N((1,c^{-t})),$$

then $\pi_2 \in \mathrm{Aut}(\Phi_1)$. Since $(0,1)^{\pi_2} = (0,1)$, then$\pi_2 \in A_{(0,1)}$. And $o(\pi_2) = 2$, so $A_{(0,1)} \cong \langle\pi_2\rangle \cong \mathbb{Z}_2$.

Consequently, $\mathrm{Aut}(\Phi_1) = \langle R(H),\lambda\rangle\mathbb{Z}_2 \times \mathbb{Z}_2$, where λ is defined in Theorem 3.1.

Theorem 3.3 *Let $\Phi_1 = \mathrm{TCay}(H;\emptyset,\emptyset,\emptyset;\{1,c\},\{1,c\},\{1,c\})$ be a connected 4-degree 0-type tri-Cayley graph over a group$H$. Then Φ_1 is not edge-transitive. Furthermore, Φ_1 is not arc-transitive.*

Proof: We can find from Fig 1 that there exists a 3-cycle $[(0,1),(2,1),(1,1)]$ passing through the edge$\{(2,1),(1,1)\}$. But for the edge$\{(1,c),(0,1)\}$, there is no a 3-cycle passing through it. Thus, Φ_1 is not edge-transitive. Furthermore, $\Phi_1$is not arc-transitive.

Theorem 3.4 *Let* $\Phi_2 = \mathrm{TCay}(H;\emptyset,\emptyset,\emptyset;\{1,c\},\{1,c^{-1}\},\{1,c^{-1}\})$ *be a connected 4-degree 0-type tri-Cayley graph over a group*H. *Then* Φ_2 *is vertex-transitive.*

Proof: Take any$c^t \in H$, we define a mapping $\omega$from $V(\Phi_2)$ to $V(\Phi_2)$ as follows:

$$\omega: (0,c^t) \mapsto (2,c^t), (2,c^t) \mapsto (1,c^t), (1,c^t) \mapsto (0,c^{t-1}),$$

where$t = 0,1,\cdots,14$. Firstly, we will prove that ω is a bijection. Take any$(2,c^t),(1,c^t),(0,c^t) \in V(\Phi_2)$, there exist $(0,c^t),(2,c^t),(1,c^{t+1}) \in V(\Phi_2)$ such that

$$(0,c^t)^\omega = (2,c^t), (2,c^t)^\omega = (1,c^t) \text{ and } (1,c^{t+1})^\omega = (0,c^t).$$

Therefore, ω is a surjection. Take any $(0,c^t),(0,c^{t'}),(1,c^t),(1,c^{t'}),(2,c^t)$ and $(2,c^{t'}) \in V(\Phi_2)$, then

$$(0,c^t) = (0,c^{t'}) \Leftrightarrow \omega((0,c^t)) = \omega((0,c^{t'})) \Leftrightarrow (2,c^t) = (2,c^{t'}),$$
$$(1,c^t) = (1,c^{t'}) \Leftrightarrow \omega((1,c^t)) = \omega((1,c^{t'})) \Leftrightarrow (0,c^{t-1}) = (0,c^{t'-1}),$$
$$(2,c^t) = (2,c^{t'}) \Leftrightarrow \omega((2,c^t)) = \omega((2,c^{t'})) \Leftrightarrow (1,c^t) = (1,c^{t'}).$$

Thus, $\omega$is a bijection. Next, we claim that$\omega \in \mathrm{Aut}(\Phi_2)$. Since

$$N((0,c^t))^\omega = \{(1,c^t),(1,c^{t+1}),(2,c^t),(2,c^{t+1})\}^\omega = \{(0,c^{t-1}),(0,c^t),(1,c^t),(1,c^{t+1})\} = N((2,c^t)),$$
$$N((1,c^t))^\omega = \{(2,c^t),(2,c^{t-1}),(0,c^t),(0,c^{t-1})\}^\omega = \{(1,c^t),(1,c^{t-1}),(2,c^t),(2,c^{t-1})\} = N((0,c^{t-1})),$$
$$N((2,c^t))^\omega = \{(0,c^t),(0,c^{t-1}),(1,c^t),(1,c^{t+1})\}^\omega = \{(2,c^t),(2,c^{t-1}),(0,c^{t-1}),(0,c^t)\} = N((1,c^t)),$$

then $\omega \in \mathrm{Aut}(\Phi_2)$. Furthermore, $\langle R(H),\omega\rangle$ acts transitively on $V(\Phi_2)$. Hence, Φ_2 is vertex-transitive.

Theorem 3.5 *Let* $\Phi_2 = \mathrm{TCay}(H;\emptyset,\emptyset,\emptyset;\{1,c\},\{1,c^{-1}\},\{1,c^{-1}\})$ *be a connected 4-degree 0-type tri-Cayley graph over a group*H. *Then* Φ_2 *is not edge-transitive. Furthermore,* Φ_2 *is not arc-transitive.*

Proof: By calculation, we can find from that there exist two 3-cycles passing through the edge$\{(0,1),(1,c)\}$, namely $[(0,1),(1,c),(2,1)]$ and $[(0,1),(1,c),(2,c)]$. But for the edge$\{(0,1),(1,1)\}$, there exists the unique 3-cycle $[(0,1),(1,1),(2,1)]$ passing through it. Thus, $\Phi_2$is not edge-transitive. Furthermore, Φ_2 is not arc-transitive.

4. 4-Degree 2-Type Tri-Cayley Graph

Theorem 4.1 *Let* $\Phi_3 = \mathrm{TCay}(H;\{c,c^{-1}\},\{c,c^{-1}\},\{c,c^{-1}\};\{1\},\{1\},\{1\})$ *be a connected 4-degree 2-type tri-Cayley graph over a group*H. *Then* Φ_3 *is vertex-transitive.*

Proof: Take any$c^t \in H$, we define a mapping φ from $V(\Phi_3)$ to $V(\Phi_3)$ as follows:

$$\varphi: (0,c^t) \mapsto (1,c^{-t}), (1,c^t) \mapsto (2,c^{-t}), (2,c^t) \mapsto (0,c^{-t}),$$

where$t = 0,1,\cdots,14$. Firstly, we will prove that φ is a bijection. Take any $(1,c^t),(2,c^t),(0,c^t) \in V(\Phi_3)$, there exist $(0,c^{-t}),(1,c^{-t}),(2,c^{-t}) \in V(\Phi_3)$ such that

$$(0,c^{-t})^\varphi = (1,c^t), (1,c^{-t})^\varphi = (2,c^t) \text{ and } (2,c^{-t})^\varphi = (0,c^t).$$

Therefore, φ is a surjection. Take any $(0,c^t),(0,c^{t'}),(1,c^t),(1,c^{t'}),(2,c^t)$ and$(2,c^{t'}) \in V(\Phi_3)$, then

$$(0,c^t) = (0,c^{t'}) \Leftrightarrow \varphi((0,c^t)) = \varphi((0,c^{t'})) \Leftrightarrow (1,c^{-t}) = (1,c^{-t'}),$$
$$(1,c^t) = (1,c^{t'}) \Leftrightarrow \varphi((1,c^t)) = \varphi((1,c^{t'})) \Leftrightarrow (2,c^{-t}) = (2,c^{-t'}),$$
$$(2,c^t) = (2,c^{t'}) \Leftrightarrow \varphi((2,c^t)) = \varphi((2,c^{t'})) \Leftrightarrow (0,c^{-t}) = (0,c^{-t'}).$$

Thus, φ is a bijection. Next, we claim that$\varphi \in \mathrm{Aut}(\Phi_3)$. Since

$$N((0,c^t))^\varphi = \{(0,c^{t+1}),(0,c^{t-1}),(1,c^t),(2,c^t)\}^\varphi = \{(1,c^{-t-1}),(1,c^{-t+1}),(2,c^{-t}),(0,c^{-t})\} = N((1,c^{-t})),$$
$$N((1,c^t))^\varphi = \{(1,c^{t+1}),(1,c^{t-1}),(2,c^t),(0,c^t)\}^\varphi = \{(2,c^{-t-1}),(2,c^{-t+1}),(0,c^{-t}),(1,c^{-t})\} = N((2,c^{-t})),$$
$$N((2,c^t))^\varphi = \{(2,c^{t+1}),(2,c^{t-1}),(0,c^t),(1,c^t)\}^\varphi = \{(0,c^{-t-1}),(0,c^{-t+1}),(1,c^{-t}),(2,c^{-t})\} = N((0,c^{-t})),$$

then$\varphi \in \mathrm{Aut}(\Phi_3)$. Furthermore, $\langle R(H),\varphi\rangle$ acts transitively on $V(\Phi_3)$. Hence, Φ_3 is vertex-transitive.

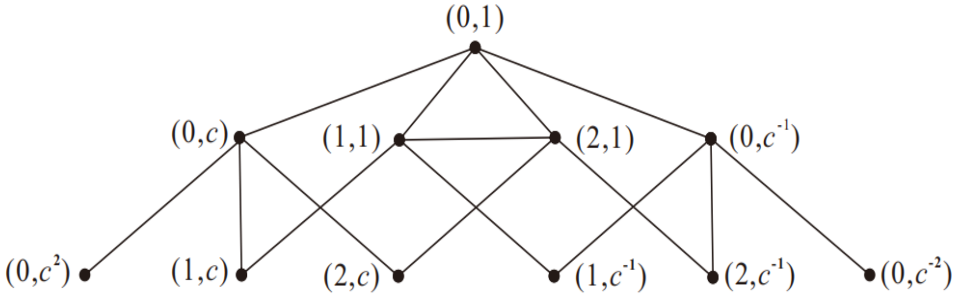

Fig. 2 The induced subgraph of $\boldsymbol{\Phi_3} = \mathbf{TCay}(\boldsymbol{H}; \{\boldsymbol{c}, \boldsymbol{c}^{-1}\}, \{\boldsymbol{c}, \boldsymbol{c}^{-1}\}, \{\boldsymbol{c}, \boldsymbol{c}^{-1}\}; \{\mathbf{1}\}, \{\mathbf{1}\}, \{\mathbf{1}\})$.

Theorem 4.2 *Let* $\Phi_3 = \mathrm{TCay}(H; \{c, c^{-1}\}, \{c, c^{-1}\}, \{c, c^{-1}\}; \{1\}, \{1\}, \{1\})$*be a connected 4-degree 2-type tri-Cayley graph over a group*H. *Then* $\mathrm{Aut}(\Phi_3) = \langle R(H), \varphi\rangle\mathbb{Z}_2$, *where* φ *is defined in Theorem 4.1.*

Proof: We can get from Lemma 2.2 that

$$|A| = |A_{(0,1)}||(0,1)^A| = |A_{(0,1)(1,1)(2,1)}||(2,1)^{A_{(0,1)(1,1)}}||(1,1)^{A_{(0,1)}}||(0,1)^A|,$$

where$A = \mathrm{Aut}(\Phi_3)$. It is clear to see that Φ_3 is vertex-transitive by Theorem 4.1, so$|(0,1)^A| = 45$. Next, we will get $|A|$ in three steps.

(1)$|A_{(0,1)(1,1)(2,1)}| = 2$.

Take any$c^t \in H$, we define a mapping δ_1 from $V(\Phi_3)$ to $V(\Phi_3)$ as follows:

$$\delta_1: (0, c^t) \mapsto (0, c^{-t}), (1, c^t) \mapsto (1, c^{-t}), (2, c^t) \mapsto (2, c^{-t}),$$

where$t = 0,1, \cdots, 14$. Clearly, δ_1 is a bijection. Next, we claim that $\delta_1 \in \mathrm{Aut}(\Phi_3)$. We have

$$N((0, c^t))^{\delta_1} = \{(0, c^{t+1}), (0, c^{t-1}), (1, c^t), (2, c^t)\}^{\delta_1} = \{(0, c^{-t-1}), (0, c^{-t+1}), (1, c^{-t}), (2, c^{-t})\} = N((0, c^{-t})),$$
$$N((1, c^t))^{\delta_1} = \{(1, c^{t+1}), (1, c^{t-1}), (2, c^t), (0, c^t)\}^{\delta_1} = \{((1, c^{-t-1}), (1, c^{-t+1}), 2, c^{-t}), (0, c^{-t})\} = N((1, c^{-t})),$$
$$N((2, c^t))^{\delta_1} = \{(2, c^{t+1}), (2, c^{t-1}), (0, c^t), (1, c^t)\}^{\delta_1} = \{(2, c^{-t-1}), (2, c^{-t+1}), (0, c^{-t}), (1, c^{-t})\} = N((2, c^{-t})).$$

Thus, $\delta_1 \in \mathrm{Aut}(\Phi_3)$. Furthermore, $(0,1)^{\delta_1} = (0,1), (1,1)^{\delta_1} = (1,1)$ and $(2,1)^{\delta_1} = (2,1)$. Then, $\delta_1 \in A_{(0,1)(1,1)(2,1)}and(0, c)^{\delta_1} = (0, c^{-1})$. Meanwhile, $o(\delta_1) = 2$, so $A_{(0,1)(1,1)(2,1)} \cong \langle\delta_1\rangle \cong \mathbb{Z}_2$. Hence, $|A_{(0,1)(1,1)(2,1)}| = 2$.

(2)$|(2,1)^{A_{(0,1)(1,1)}}| = 1$.

Clearly, the vertices $(0,1), (1,1)$ are fixed and $N((0,1))\backslash\{(1,1)\} = \{(0, c), (2,1), (0, c^{-1})\}$. We can find from Fig 2 that $[(0,1), (2,1), (1,1)]$ is the unique 3-cycle passing through $(0,1)$ and $(1,1)$. That is to say, there is no a graph automorphism which causes $(2,1)$ to become $(0, c)$ or $(0, c^{-1})$ and fixes $(0,1)$ and$(1,1)$. Thus, $|(2,1)^{A_{(0,1)(1,1)}}| = 1$.

(3)$|(1,1)^{A_{(0,1)}}| = 2$.

The vertex $(0,1)$ is fixed, meanwhile, $N((0,1)) = \{(1,1), (0, c), (2,1), (0, c^{-1})\}$. We can find from Fig 2 that $[(0,1), (2,1), (1,1)]$ is the unique 3-cycle passing through $(0,1)$. That is to say, there is no a graph automorphism which causes $(1,1)$ to become $(0, c)$ or $(0, c^{-1})$ and fixes$(0,1)$. Take any $c^t \in H$, we define a mapping δ_2 from $V(\Phi_3)$ to $V(\Phi_3)$ as follows:

$$\delta_2: (0, c^t) \mapsto (0, c^{-t}), (1, c^t) \mapsto (2, c^{-t}), (2, c^t) \mapsto (1, c^{-t}),$$

where$t = 0,1, \cdots, 14$. It is easy to see that δ_2 is a bijection. Next, we claim that$\delta_2 \in \mathrm{Aut}(\Phi_3)$. We have

$$N((0, c^t))^{\delta_2} = \{(0, c^{t+1}), (0, c^{t-1}), (1, c^t), (2, c^t)\}^{\delta_2} = \{(0, c^{-t-1}), (0, c^{-t+1}), (2, c^{-t}), (1, c^{-t})\} = N((0, c^{-t})),$$
$$N((1, c^t))^{\delta_2} = \{(1, c^{t+1}), (1, c^{t-1}), (2, c^t), (0, c^t)\}^{\delta_2} = \{(2, c^{-t-1}), (2, c^{-t+1}), (1, c^{-t}), (0, c^{-t})\} = N((2, c^{-t})),$$
$$N((2, c^t))^{\delta_2} = \{(2, c^{t+1}), (2, c^{t-1}), (0, c^t), (1, c^t)\}^{\delta_2} = \{(1, c^{-t-1}), (1, c^{-t+1}), (0, c^{-t}), (2, c^{-t})\} = N((1, c^{-t})).$$

Thus,$\delta_2 \in \mathrm{Aut}(\Phi_3)$. Furthermore,$(0,1)^{\delta_2} = (0,1)$. Then, $\delta_2 \in (1,1)^{A_{(0,1)}}ando(\delta_2) = 2$. Hence, $(1,1)^{A_{(0,1)}} \cong \langle\delta_2\rangle \cong \mathbb{Z}_2$ and so$|(1,1)^{A_{(0,1)}}| = 2$.

Consequently, $\mathrm{Aut}(\Phi_3) = \langle R(H), \varphi\rangle Z_2$, where φ is defined in Theorem 4.1.

Theorem 4.3 *Let* $\Phi_3 = \mathrm{TCay}(H; \{c, c^{-1}\}, \{c, c^{-1}\}, \{c, c^{-1}\}; \{1\}, \{1\}, \{1\})$ *be a connected 4-degree 2-type tri-Cayley graph over a group*H. *Then* Φ_3 *is not edge-transitive. Furthermore,* Φ_3 *is not arc-transitive.*

Proof: We can find from Fig 2 that there exists a 3-cycle $[(0,1), (2,1), (1,1)]$ passing through the edge$\{(2,1), (0,1)\}$. But for the edge$\{(0, c^{-1}), (0,1)\}$, there is no a 3-cycle passing through it. Thus, $\Phi_3$is not edge-transitive. Furthermore, Φ_3 is not arc-transitive.

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