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Original Article

A Classical Blow up Criterion to Cauchy Problem for the Micropolar Fluid Flows

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Abstract - *In this paper, we discuss the cauchy problem of the viscous micropolar fluid flow model in 2D. This note obtains a classical regularity blow up criterion for the two-dimensional micropolar fluid flows. When the inital data is allowed to the suitable Sobolev space, for the life span* T^{max}*, it is worth noting that the result holds* $\int_0^{T^{max}} \|\nabla u(t)\|_{L^\infty} dt = 0$.

Index Terms - *Micropolar fluid flow, blow up criterion, suitable Sobolev space, the life span.*

1. Introduction

In this paper, we deals with the system of equations for motion of micropolar fluid. To describe the motion of the incompressible conductive micropolar fluids, Eringen first introduced the micropolar equations in [5]. The 3D incompressible micropolar fluid equations can be written as:

$$\begin{cases}\partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + 2\chi \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega - \alpha \nabla \nabla \cdot \omega + 4\chi \omega = \kappa \Delta \omega + 2\chi \nabla \times u, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

Where $u = u(x_{1,} x_2, x_3, t)$ denotes the fluid velocity, $\omega = \omega(x_{1,} x_2, x_3, t)$ is the field of microrotation representing the angular velocity of the rotation of the particles of the fluid and $\pi = \pi(x_{1,} x_2, x_3, t)$ is the scalar pressure, μ is the kinematic viscosity, χ is the vortex viscosity, κ and α is the micro-rotation viscosity. μ, χ, κ and α are positive constants. Specially, when

$$u = (u_1(x_{1,} x_2, t), u_2(x_{1,} x_2, t), 0), \quad \omega = \big(0,0, \omega_3(x_{1,} x_2, t)\big), \quad b = (b_1(x_{1,} x_2, t), b_2(x_{1,} x_2, t), 0).$$

Here the following 2D micropolar equations which we will consider in this paper can be deduced by 3D micropolar equations

$$\begin{cases}\partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + 2\chi \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega + 4\chi \omega = \kappa \Delta \omega + 2\chi \nabla \times u, \\ \nabla \cdot u = 0, \end{cases}$$

Where $\nabla \times \omega = (\partial_2 \omega, -\partial_1 \omega)$ with ω is the concise form of ω_3, and $\nabla \times u = \partial_1 u^2 - \partial_2 u^1$.

This model was first proposed by [5] in 1966. The existences of weak and strong solutions were proved by Galdi and Rionero [6] and Yamaguchi [16], respectively. To go directly to the main points of the present paper, in what follows we only review some known results which are closely related to our main result. Galdi and Rionero [6], Lukaszewicz [11] (and references therein) proved the global existence of weak solutions of micropolar flows (1.1) with the methods of Ladyzhenskaya [12] and Temam [15]. Chen and Price [4], Rojas-Medar et al. [2,3,13,14] investigated the local existence and uniqueness of strong solutions to the micropolar flows (or magneto-micropolar flows) by some different methods. If further, letting $\omega = 0$ and $\chi = 0$, the magneto-micropolar fluids equations reduce to the classical Navier--Stokes equations [7,10].

In this paper, we will investigate the classical blow up criterion above system in $\mathbb{R}^2$. Let $\mu = \chi = \frac{1}{2}$ for simplicity, substitute it into the equation, we will have the following 2D incompressible micropolar fluid equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u = \Delta u - \nabla \pi + \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega + 2\omega = \kappa \Delta \omega + \nabla \times u, \\ \nabla \cdot u = 0, \end{cases} \tag{1.2}$$

Then, we state our main result as follows.

Theorem1.1 Let $u^0 \in \mathrm{H}^2(\mathbb{R}^2)$ and $\omega^0 \in \mathrm{H}^2(\mathbb{R}^2)$, (1.2) has a unique solution (u, ω) for some $T > 0$ so that $u \in C([0,T]; \mathrm{H}^2(\mathbb{R}^2))$, $\omega \in C([0,T]; \mathrm{H}^2(\mathbb{R}^2))$, with $\nabla u \in L^2([0,T]; \mathrm{H}^2(\mathbb{R}^2))$ and $\nabla \pi \in C([0,T]; \mathrm{H}^1(\mathbb{R}^2))$. Moreover, if T^{max} is the life span to this solution, and $T^{max} < \infty$, one has

$$\int_0^{T^{max}} \|\nabla u(t)\|_{L^\infty} dt = \infty \tag{1.3}$$

The rest of the paper is organized as follows. In Section 2, some known facts and elementary inequalities will be given which will be needed in later analysis. In Section 3, it is devoted to deriving the priori estimates of solutions, then the result can guarantee the extension of the local strong solution to be a global one, it is easy to see that this contradicts the results of Theorems 1.1 in this article i.e., the definition of T^{max}, and Theorems 1.2 can be obtain.

Notation. In this paper, $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$ mean the norm of L^p, H^s, respectively. For simplicity, $I_t := (0,t)$, $\overline{I_t} := [0,t]$ and $\int := \int_{\mathbb{R}^2}$.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later.

First, we recall the local existence of the strong solutions to (1.1), the proof is similar to [5].

Lemma 2.1 Assume that the initial data $(u_0, \omega_0) \in \mathrm{H}^m(\mathbb{R}^2) \times \mathrm{H}^m(\mathbb{R}^2) (m \geq 2)$ such that $div\, u_0 = 0$ in $\mathbb{R}^2$. Then there exists a $T > 0$ such that system (1.1) admits a unique solution (u, ω) on $[0, T]$ satisfying

$$(u, \omega) \in C([0,T]; \mathrm{H}^m(\mathbb{R}^2)).$$

Lemma 2.2([14]) Gronwall's inequality (differential form): let $\varphi(\cdot)$ be a nonnegative, absolutely continuous function on $\overline{I_T}$, for all $t \in \overline{I_T}$, which satisfies for a.e. t the differential inequality

$$f'(t) \leq f(t)g(t) + \xi(t),$$

where $g(t)$ and $\xi(t)$ are nonnegative, summable functions on $\overline{I_T}$. Then

$$f(t) \leq e^{\int_0^t g(s)ds}\left[f(0) + \int_0^t \xi(s)\, ds\right]. \tag{2.1}$$

Lemma 2.3 ([1])Assume Ω is a domain in $\mathbb{R}^2$, the integer $k \geq 0$ and $1 < q < \infty$, for $v \in W^{k+1,q}$, then there exists a positive constant C depending only on q, k such that

$$\|\nabla v\|_{L^q} \leq C(\|\mathrm{div} v\|_{L^q} + \|\mathrm{curl} v\|_{L^q}).$$

3. The Main Proof

This section is mainly divided into two parts. The first part provides a prior estimate, and the second part uses the method of proof by contradiction and the priori estimate to obtain Theorem 1.1.

3.1. A Priori Estimates

Proposition 3.1 Let the (u, ω) be the solution of system of (1.1) satisfies

$$(u^0, \omega^0) \in \mathrm{H}^2(\mathbb{R}^2) \times \mathrm{H}^2(\mathbb{R}^2),$$

then there exists the result such that

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^2}^2 + \|\omega\|_{H^2}^2\right) + \frac{1}{4}\|\nabla u\|_{H^2}^2 + \frac{2}{3}\|\omega\|_{H^2}^2 + \kappa\|\nabla \omega\|_{H^2}^2 \lesssim \|\nabla u\|_{L^\infty}\left(\|u\|_{H^2}^2 + \|\omega\|_{H^2}^2\right) \tag{3.1}$$

for any $t \in [0, T]$.

Proof. The proof of proposition 3.1 is divided into several energy estimates. First, we will have the L^2 energy estimates.

Step 1. L^2 estimate of (u,ω).

Taking the L^2 inner product of $(1.2)_1$ with u, due to

$$\int (u\cdot\nabla u+\nabla\pi)\cdot u dx=\int \frac{1}{2}u\cdot\nabla|u|^2dx-\int divu\pi dx$$

$$=-\int\left(\frac{1}{2}|u|^2+\pi\right)divudx$$

$$=0,$$

we can obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2+\|\nabla u(t)\|_{L^2}^2=\int(\nabla\times\omega)\cdot udx. \tag{3.2}$$

Making the L^2 inner product of $(1.2)_2$ with ω, it gives

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2+2\|\omega(t)\|_{L^2}^2+\kappa\|\nabla\omega(t)\|_{L^2}^2=\int(\nabla\times u)\omega dx \tag{3.3}$$

where we have used

$$\int u\cdot\nabla\omega\cdot\omega dx=-\frac{1}{2}\int divu\omega^2\,dx=0$$

The integration by parts together with $Hol\ddot{d}er$ inequality and Young's inequality gives that

$$\int(\nabla\times\omega)\cdot udx+\int(\nabla\times u)\omega dx$$

$$=\int(\partial_2\omega u^1-\partial_1\omega u^2)dx+\int(\nabla\times u)\omega dx$$

$$=2\int(\nabla\times u)\omega dx$$

$$\le\frac{3}{4}\|\nabla u\|_{L^2}^2+\frac{4}{3}\|\omega\|_{L^2}^2 \tag{3.4}$$

Then, for any $t\in[0,T]$, by $(3.2)-(3.4)$, we can obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2\right)+\frac{1}{4}\|\nabla u\|_{L^2}^2+\frac{2}{3}\|\omega\|_{L^2}^2+\kappa\|\nabla\omega\|_{L^2}^2\le 0. \tag{3.5}$$

Step 2. H^1 estimate of (u,ω).

Taking the L^2 product of equation $(1.2)_1$ with $-\Delta u$, then we can obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2=-\int\nabla\times\omega\cdot\Delta udx+\int u\cdot\nabla u\cdot\Delta udx. \tag{3.6}$$

Here we use the fact $-\int u_t\cdot\Delta udx=\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2$, which follows from the integration by parts.

For ω, taking the L^2 inner product of the $(1.2)_2$ with $-\Delta\omega$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2+2\|\nabla\omega\|_{L^2}^2+\kappa\|\Delta\omega\|_{L^2}^2=\int(u\cdot\nabla\omega)\Delta\omega dx-\int(\nabla\times u)\,\Delta\omega dx. \tag{3.7}$$

Since u is divergence-free, we have

$$\int u\cdot\nabla u\cdot\Delta udx+\int(u\cdot\nabla\omega)\Delta\omega dx=-\int(\partial_i u\cdot\nabla u\cdot\partial_i u+u\cdot\nabla\partial_i u\cdot\partial_i u)dx$$

$$-\int(\partial_i u\cdot\nabla\omega\partial_i\omega+u\cdot\nabla\partial_i\omega\partial_i\omega)dx$$

$$\le\|\nabla u\|_{L^\infty}\left(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2\right), \tag{3.8}$$

where we have used the fact that

$$-\int(u\cdot\nabla\partial_i u\cdot\partial_i u+u\cdot\nabla\partial_i\omega\partial_i\omega)dx=-\frac{1}{2}\int u\cdot\nabla\left(|\nabla u|_{L^2}^2+|\nabla\omega|_{L^2}^2\right)dx$$

$$=\frac{1}{2}\int divu\left(|\nabla u|_{L^2}^2+|\nabla\omega|_{L^2}^2\right)dx$$

$$= 0.$$

Along the same line as (3.4), thus together with (3.6), (3.7) and (3.8), we conclude that

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2\right) + \frac{1}{4}\|\Delta u\|_{L^2}^2 + \frac{2}{3}\|\nabla\omega\|_{L^2}^2 + \kappa\|\Delta\omega\|_{L^2}^2$$
$$\lesssim \|\nabla u\|_{L^\infty}\left(\|\nabla u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2\right) \quad (3.9)$$

Step 3. H^1 estimate of (u,ω).

To estimate the third-order derivative of u, we can have after applying operator ∇ to $(1.2)_1$ and $(1.2)_2$,

$$\begin{cases}\partial_t\nabla u + \nabla(u\cdot\nabla u) = \nabla\Delta u - \nabla\nabla\pi + \nabla\nabla\times\omega,\\ \partial_t\nabla\omega + \nabla(u\cdot\nabla\omega) + 2\nabla\omega = \kappa\nabla\Delta\omega + \nabla\nabla\times u,\\ \nabla\cdot u = 0,\end{cases} \quad (3.10)$$

Due to

$$\int \partial_t\nabla u\cdot\nabla\Delta u dx = \frac{1}{2}\frac{d}{dt}\|\nabla^2 u\|_{L^2}^2$$

And

$$\int(\nabla\Delta u - \nabla\nabla\pi)\cdot\nabla\Delta u dx = \|\nabla\Delta u\|_{L^2}^2$$

then multiplying by $-\nabla\Delta u$ in $(3.10)_1$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla^2 u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 = \int(\nabla(u\cdot\nabla u) - \nabla\nabla\times\omega)\cdot\nabla\Delta u dx \quad (3.11)$$

In order to estimate the third-order derivative of ω, we take the L^2 inner product of equations $(3.10)_2$ with $-\nabla\Delta\omega$, and integrate by parts to have

$$\frac{1}{2}\frac{d}{dt}\|\nabla^2\omega\|_{L^2}^2 + 2\|\nabla^2\omega\|_{L^2}^2 + \kappa\|\nabla\Delta\omega\|_{L^2}^2 = \int(\nabla(u\cdot\nabla\omega) - \nabla\nabla\times u)\cdot\nabla\Delta\omega dx \quad (3.12)$$

A direct computation implies

$$\begin{aligned}\int\nabla(u\cdot\nabla u)\cdot\nabla\Delta u dx &= -\int\partial_i\nabla(u\cdot\nabla u)\cdot\nabla\partial_i u dx\\ &= -\int\left(\partial_i\nabla_j u\cdot\nabla u\cdot\nabla_j\partial_i u + \partial_i u\cdot\nabla\nabla_j u\cdot\nabla\partial_i u\right)dx\\ &\quad -\int\left(\nabla_j u\cdot\nabla\partial_i u\cdot\nabla_j\partial_i u + u\cdot\nabla\nabla_j\partial_i u\cdot\nabla\partial_i u\right)dx\\ &\le \|\nabla u\|_{L^\infty}\|\nabla^2 u\|_{L^2}^2,\end{aligned}$$

where in the last inequality, we have used

$$-\int u\cdot\nabla\nabla_j\cdot\nabla\partial_i u dx = \frac{1}{2}\int divu|\nabla^2 u|^2\ dx = 0$$

By the same way, it also gives

$$\begin{aligned}\int\nabla(u\cdot\nabla\omega)\cdot\nabla\Delta\omega dx &= \int\partial_i\nabla(u\cdot\nabla\omega)\cdot\nabla\partial_i\omega dx\\ &= \int(\partial_i\nabla u\cdot\nabla\omega\cdot\nabla\partial_i\omega + \nabla u\cdot\nabla\partial_i\omega\cdot\nabla\partial_i\omega)dx\\ &\quad -\int(\partial_i u\cdot\nabla\nabla\omega\cdot\nabla\partial_i\omega + u\cdot\nabla\nabla\partial_i\omega\cdot\nabla\partial_i\omega)dx\\ &\le \|\nabla u\|_{L^\infty}\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\omega\|_{L^2}^2\right).\end{aligned}$$

Thus together with (3.11) and (3.12), we conclude that

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\omega\|_{L^2}^2\right) + \frac{1}{4}\|\nabla\Delta u\|_{L^2}^2 + \frac{2}{3}\|\nabla^2\omega\|_{L^2}^2 + \kappa\|\nabla\Delta\omega\|_{L^2}^2$$
$$\lesssim \|\nabla u\|_{L^\infty}\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\omega\|_{L^2}^2\right). \quad (3.13)$$

3.2. Proof of Theorems 1.1 and 1.2

The following proof mainly relies on by contradiction. If the conclusion in Theorem 1.1 is false, then there exist some constant $C_1 > 0$ such that

$$\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} dt \leq C_1$$

Due to Lemma 2.2 and (3.1), it implies

$$\|u\|_{H^2}^2 + \|\omega\|_{H^2}^2 \leq \left(\|u^0\|_{H^2}^2 + \|\omega^0\|_{H^2}^2\right) e^{\int_0^T \|\nabla u(t)\|_{L^\infty} dt}$$

$$\leq \left(\|u^0\|_{H^2}^2 + \|\omega^0\|_{H^2}^2\right) e^{C_1} \tag{3.14}$$

By Lemma Lemma 2.1, there exists a $T^* > 0$ such that the problem (1.2) has a unique local strong solution (u, ω) on $\mathbb{R}^2 \times (0, T^*]$. One will use the a priori estimates (3.14) to extend the classical solution (u, ω)globally in time.

From (3.14), we can set

$$T^* = sup\left\{T \in I_{T^{max}} \middle| \|u\|_{H^2}^2 + \|\omega\|_{H^2}^2 \leq C \ \ for\ any\ \ t \leq T\right\} \tag{3.15}$$

it is easy to see that the definition of T^* makes sense and $T^* > 0$.

Next, we claim that

$$T^* = \infty \tag{3.16}$$

Otherwise, $T^* < \infty$. It follows from (3.14) that $\left(u(x, T^*), \omega(x, T^*)\right)$ satisfy the initial data condition. Hence, Lemma 2.1 shows that there exists some $T^{**} > T^*$, such that (3.15) holds for $T = T^{**}$, which contradicts the definition of T^*[8].

Finally, it is easy to see that the above results contradict the definition of T^{max} in Theorem 1.1, thus (1.3) holds. The proof of Theorem 1.1 is finished.

In view of

$$\|\nabla u\|_{L^\infty} \lesssim \|\nabla u\|_{H^2},$$

then the global existence of strong solutions can be established by local solutions and continuity method, due to the argument is standard, we ignore the proof. For details, please refer to [9].

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