International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I4P103
Volume 70 Issue 4, 27-32, April 2024

Original Article

On Local Strong Solutions to the 2D MHD Equations with Navier-Type Boundary Condition in a Strip Domain

Songshang Yang

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China.

Corresponding Author : yf10251208@163.com

Received: 15 February 2024 Revised: 17 March 2024 Accepted: 08 April 2024 Published: 28 April 2024

Abstract *- This paper presents the local well-posedness of the strong solutions to the 2D incompressible magnetohydrodynamics (MHD) equations without magnetic diffusion in a strip domain. Via a semi-discrete Galerkin scheme, we construct approximate solutions with Navier-type boundary conditions, and can have solutions by passing the limit. Moreover, our results are valid for the Cauchy problem.*

Keywords *- Magnetohydrodynamics (MHD) equations, A strip domain, Semi-discrete Galerkin scheme, Navier-type boundary condition.*

1. Introduction

Magnetohydrodynamics(MHD) describes the motion of conductive fluids such as plasmas, liquid metals and electrolytes in magnetic fields and has a wide range of applications in astrophysics, geophysics, high-speed aerodynamics and engineering. Under the continuity hypothesis, the motion of the fluid satisfies the conservation of mass, momentum and energy.

The incompressible magnetohydrodynamics (MHD) can be described as follows:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b - \nu \Delta b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \nabla \cdot b = 0 \end{cases} \tag{1.1}$$

where $u = u(x_1, x_2, t)$ denotes the fluid velocity, $b = b(x_1, x_2, t)$ is the magnetic field and $p = p(x_1, x_2, t)$ is the pressure of the field, μ is the kinematic viscosity, $\frac{1}{\nu}$ is the magnetic Reynolds number. μ and ν are positive constants.

2. Literature Review

Magnetohydrodynamic(MHD) has been extensively studied by physicists and mathematicians because it plays an important role in simulating many phenomena in astrophysics, geophysics, and plasma physics, as detailed in [3-5,11,13,15] and the literature cited therein. There is a strong coupling and interaction between fluid motion and magnetic field in MHD systems, which makes the study of well-posedness and dynamical behavior of the systems very complicated. Nonetheless, many important advances have been made in the mathematical analysis of these topics for MHD systems in recent years.

Among them, we briefly review the results of the high-dimensional incompressible MHD equations that are more relevant to this problem. For the homogeneous incompressible MHD equations, Durant and Lions[7] established the existence of a global weak solution with finite energy. In the case that the given initial data is smooth, the smoothness and uniqueness in the 2D case are also proved. Sermange and Teman[13] respectively constructed local strong solutions and global strong solutions (with small initial data) in the three-dimensional case. For the nonhomogeneous incompressible MHD equations, Gerbeau, Le Bris [8] and Desjardins, Le Bris [6] established respectively the global existence of weak solutions with finish energy on three-dimensional bounded domains and on the torus. For the initial density ρ_0 bounded away from zero, Abidi, Hmidi[1] and Abidi, Paicu[2] have established the existence of local and global (with small initial data) strong solutions in some Besov Spaces.

If the magnetic field $b = 0$, the MHD equations reduce to the classical Navier-Stokes equations. Ren, Xiang and Zhang investigated the local existence of the two-dimensional MHD systems with no-slip boundary condition in a strip domain in [12]. In the present paper it studys the local existence for 2D MHD equation with Navier-slip boundary condition in a strip domain.

3. Main Results

This paper will study small perturbations of equation (1.1) around the equilibrium $e_1 = (1,0)$. Let $B = b - e_1$, and substitute it into (1.1), the two-dimensional incompressible magnetohydrodynamics (MHD) system without magnetic diffusion can be written as follows:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = \partial_1 B + B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B = \partial_1 u + B \cdot \nabla u, \\ \nabla \cdot u = 0, \nabla \cdot B = 0 \end{cases} \tag{1.2}$$

The system (1.2) will be considered in a strip domain:

$$\Omega = \{x = (x_1, x_2) | x_1 \in \mathbb{R}, x_2 \in (0,1)\} \subset \mathbb{R}^2. \tag{1.3}$$

We need to pay attention that u satisfies the Navier-slip boundary condition:

$$u \cdot n = 0, \quad curlu = 0 \quad on\ \partial\Omega, \tag{1.4}$$

and the container is perfectly conductive to the magnetic field B, i.e.,

$$B \cdot n = 0 \quad on\ \partial\Omega, \tag{1.5}$$

where $n = (n_1, n_2)$ denotes the unit outward normal vector.

Especially since Ω is a strip domain, the boundary condition (1.4) and (1.5) are equivalent to the boundary condition

$$(v_2, \partial_2 v_1, B_2)|_{\partial\Omega} = 0. \tag{1.6}$$

Here the following initial boundary value problem can be deduced:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = \partial_1 B + B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B = \partial_1 u + B \cdot \nabla u, \\ \nabla \cdot u = 0, \nabla \cdot B = 0 \\ (v_2, \partial_2 v_1, B_2)|_{\partial\Omega} = 0 \\ u(x_1, x_2, 0) = u_0(x_1, x_2), B(x_1, x_2, 0) = B_0(x_1, x_2) \end{cases} \tag{1.7}$$

Then, The main results are stated as follows.

Theorem 1.1 Assume that the initial data $(u_0, B_0) \in H^2(\Omega) \times H^2(\Omega)$ such that $div\ u_0 = 0$ in Ω Then there exists a $T > 0$ such that the MHD system (1.7) admits a unique solution (u, ω) on $[0, T]$ satisfying

$$(u, B) \in C\big([0, T]; H^2(\Omega)\big).$$

In the second section, we will give some known facts and basic inequalities, which will be used later in the analysis. In the third section, a priori estimate for understanding (u, B) is given, thus ensuring the extension of the

local strong solution, i.e., theorem 1.1, via the semi-discrete Galerkin scheme.
Section 3 is devoted to deriving the priori estimates of solutions (u, B), and then the result can guarantee the extension of the local strong solution, i.e., Theorems 1.1, via a semi-discrete Galerkin scheme.
Notations In this paper, the norms of L^p and H^s are $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$, respectively. For simplicity, $I_t := (0,t)$, $\overline{I_t} := [0,t]$ and $\int := \int_\Omega$.

4. Preliminaries

This section will introduce some known facts and essential inequalities that will be used frequently in the future. Firstly, the product estimate as follows [10].

Lemma 2.1 Product estimate:

$$\|gh\|_{H^i} \lesssim \begin{cases} \|g\|_{H^1}\|h\|_{H^1} & for \quad i=0; \\ \|g\|_{H^i}\|h\|_{H^2} & for\ 0 \le i \le 2\,; \\ \|g\|_{H^2}\|h\|_{H^i} + \|g\|_{H^i}\|h\|_{H^2} & for \quad i=3. \end{cases} \tag{2.1}$$

Lemma 2.2 ([14]) let $\varphi(\cdot)$be a nonnegative, absolutely continuous function on $\overline{I_T}$, for all $t \in \overline{I_T}$, which satisfies for a.e. t the differential inequality

$$f'(t) \le\ f(t)g(t) + h(t),$$

where $g(t)$ and $h(t)$ are nonnegative, summable functions on $\overline{I_T}$. Then

$$f(t) \le e^{\int_0^t g(s)ds}\left[f(0) + \int_0^t h(s)ds\right]. \tag{2.2}$$

Lemma 2.3 ([12]) Let Ω be the strip domain defined by (1.3) and $f \in\ H^k(\Omega)$ for $k \ge\ 0$. If $u \in\ H^1(\Omega)$is a weak solution to the following Stokes system with $u \cdot n = 0$ and $curlu\ = 0$ on $\partial\Omega$,

$$\begin{cases} -\Delta u + \nabla p = f, x \in \Omega, \\ divu = 0, \qquad x \in \Omega, \end{cases}$$

then $u \in\ H^{k+2}(\Omega)$ and there exists a positive constant C such that

$$\|\nabla u\|_{H^{k+1}} + \|\nabla p\|_{H^k} \le C\|f\|_{H^k}. \tag{2.3}$$

5. The Main Proof

The content of this section is divided into two parts. The one part provides a prior estimate, and the other part uses the regularity of a prior estimate to proof Theorem 1.1.

5.1. Proposition 3.1

Assume $(u_0, B_0) \in H^2(\Omega) \times H^2(\Omega)$,then there exists the positive constants C_T may depending on $\mu, T, u_0$and B_0 such that

$$\|u\|_{H^2}^2 + \|B\|_{H^2}^2 + \int_0^T\left(\|\nabla u\|_{H^2}^2 + \|u_t\|_{H^1}^2 + \|B_t\|_{H^1}^2\right)dt \lesssim C_T. \tag{3.1}$$

Proof: There are mainly some energy estimates to prove proposition 3.1. First, there will have the energy estimates for L^2.

Step 1. L^2 estimate of (u, B).

Making the L^2 inner product of $(1.7)_1$ with u, ones have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \mu\|\nabla u(t)\|_{L^2}^2 = \int(\partial_1 B + B\cdot\nabla B)\cdot u dx, \tag{3.2}$$

Where take advantage of the fact that

$$\begin{aligned} \int(u\cdot\nabla u + \nabla p)\cdot u dx &= \int \frac{1}{2} u\cdot\nabla|u|^2 dx - \int divupdx \\ &= -\int\left(\frac{1}{2}|u|^2 + p\right)divudx + \int_{\partial\Omega}\left(\frac{1}{2}|u|^2 + p\right)u\cdot nds \\ &= 0. \end{aligned}$$

Making the L^2 inner product of $(1.7)_2$with B can get

$$\frac{1}{2}\frac{d}{dt}\|B(t)\|_{L^2}^2 = \int(\partial_1 u + B\cdot\nabla u)\cdot B dx, \tag{3.3}$$

where it have used

$$\int u\cdot\nabla B\cdot B dx = -\frac{1}{2}\int divu|B|^2 dx + \frac{1}{2}\int_{\partial\Omega}|B|^2 u\cdot n ds = 0.$$

The integration by parts together with the boundary condition $(1.7)_4$ gives that

$$\int\partial_1 B\cdot u dx + \int\partial_1 u\cdot B dx = \int\partial_1 B\cdot u dx - \int u\cdot\partial_1 B\, dx = 0, \tag{3.4}$$

and

$$\int B\cdot\nabla B\cdot u\, dx + \int B\cdot\nabla u\cdot B\, dx = 0. \tag{3.5}$$

Then, by$(3.2)-(3.5)$, it can obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2+\|B\|_{L^2}^2\right) + \mu\|\nabla u\|_{L^2}^2 = 0. \tag{3.6}$$

Step 2. L^2 estimate of (u_t, B_t).

Making the L^2 product of equation $(1.7)_1$ with u_t, it gives

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 = \int(\partial_1 B + B\cdot\nabla B - u\cdot\nabla u)\cdot u_t dx, \tag{3.7}$$

where it have used $-\mu\int\Delta u\cdot u_t dx = \frac{\mu}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2$. Then by $H\ddot{o}lder$ inequality, Young's inequality and (2.1), it have

$$\begin{aligned}\int(\partial_1 B + B\cdot\nabla B - u\cdot\nabla u)\cdot u_t dx &\le \|\partial_1 B + B\cdot\nabla B - u\cdot\nabla u\|_{L^2}\|u_t\|_{L^2}^2\\ &\lesssim \|B\|_{H^2}^2\left(1+\|B\|_{H^1}^2\right)\|u\|_{H^2}^2\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|u_t\|_{L^2}^2.\end{aligned} \tag{3.8}$$

By the same way, it can obtain

$$\begin{aligned}\|B_t\|_{L^2}^2 &\le \|\partial_1 u + B\cdot\nabla u - u\cdot\nabla B\|_{L^2}\\ &\lesssim \|u\|_{H^2}(1+\|B\|_{H^2}).\end{aligned} \tag{3.9}$$

From (3.7), (3.8) and (3.9), it can get

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|B_t\|_{L^2}^2 \lesssim \left(\|u\|_{H^2}^2 + \|B\|_{H^2}^2\right)\left(1+\|u\|_{H^1}^2+\|B\|_{H^1}^2\right). \tag{3.10}$$

Step 3. H^1 estimate of (u, B).

Making the L^2 product of equation $(1.7)_1$ with $-\Delta u$, then it can obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \mu\|\Delta u\|_{L^2}^2 = -\int(\partial_1 B + B\cdot\nabla B - u\cdot\nabla u)\cdot\Delta u dx. \tag{3.11}$$

For B, The inner product of L^2 of $(1.7)_2$ and $-\Delta B$ is obtained

$$\frac{1}{2}\frac{d}{dt}\|\nabla B\|_{L^2}^2 = -\int(\partial_1 u + u\cdot\nabla B - B\cdot\nabla u)\cdot\Delta B dx. \tag{3.12}$$

Along the same line as (3.8) and (3.9) will have

$$\begin{aligned}-\int(\partial_1 B + B\cdot\nabla B - u\cdot\nabla u)\cdot\Delta u dx &\le \|\partial_1 B + B\cdot\nabla B - u\cdot\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\ &\lesssim \left(\|u\|_{H^2}^2+\|B\|_{H^2}^2\right)\left(1+\|u\|_{H^2}^2+\|B\|_{H^2}^2\right),\end{aligned} \tag{3.13}$$

and

$$-\int(\partial_1 u + u\cdot\nabla B - B\cdot\nabla u)\cdot\Delta B dx \le \|\partial_1 u + u\cdot\nabla B - B\cdot\nabla u\|_{L^2}\|\Delta B\|_{L^2}$$

$$\lesssim\left(\|u\|_{H^2}^2+\|B\|_{H^2}^2\right)\left(1+\|u\|_{H^2}^2+\|B\|_{H^2}^2\right). \tag{3.14}$$

Thus together with $(3.11)-(3.14)$, we conclude that

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)+\mu\|\Delta u\|_{L^2}^2\lesssim\left(\|u\|_{H^2}^2+\|B\|_{H^2}^2\right)\left(1+\|u\|_{H^2}^2+\|B\|_{H^2}^2\right). \tag{3.15}$$

Step 4. H^1 estimate of (u_t, B_t).

To estimate the one-order derivative of u_t and B_t, multiplying $(1.7)_1$ by $-\Delta u_t$ to obtain

$$\frac{\mu}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2=-\int(\partial_1 B+B\cdot\nabla B-u\cdot\nabla u)\cdot\Delta u_t dx. \tag{3.16}$$

A direct computation implies

$$\begin{aligned}
&-\int(\partial_1 B+B\cdot\nabla B-u\cdot\nabla u)\cdot\Delta u_t dx\\
&\leq\|\nabla(\partial_1 B+B\cdot\nabla B-u\cdot\nabla u)\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\lesssim\left(\|u\|_{H^2}^2+\|B\|_{H^2}^2\right)\left(1+\|u\|_{H^2}^2+\|B\|_{H^2}^2\right)+\frac{1}{2}\|u_t\|_{L^2}^2.
\end{aligned} \tag{3.17}$$

To estimate the one-order derivative of B_t, applying operator ∇ to $(1.7)_2$ and taking the L^2 inner product will have

$$\begin{aligned}
\|\nabla B_t\|_{L^2}&\leq\|\nabla(\partial_1 u+B\cdot\nabla u-u\cdot\nabla B)\|_{L^2}\\
&\lesssim(\|u\|_{H^2}+\|B\|_{H^2})(1+\|u\|_{H^2}+\|B\|_{H^2}).
\end{aligned} \tag{3.18}$$

Thus plugging (3.17) into (3.16), together with (3.18), ones have

$$\frac{\mu}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2+\|\nabla B_t\|_{L^2}\lesssim(\|u\|_{H^2}+\|B\|_{H^2})(1+\|u\|_{H^2}+\|B\|_{H^2}). \tag{3.19}$$

Step 5. H^3 estimate of (u, B).

By the Stokes estimates, i.e., Lemma 2.3, it gives

$$\begin{aligned}
\|\nabla u\|_{H^2}+\|\nabla \mathrm{p}\|_{H^1}&\lesssim\|\partial_1 B-u\cdot\nabla u+B\cdot\nabla B-u_t\|_{H^1}\\
&\lesssim\|u_t\|_{H^1}+(\|u\|_{H^2}+\|B\|_{H^2})(1+\|u\|_{H^2}+\|B\|_{H^2}).
\end{aligned} \tag{3.20}$$

Then by collecting (3.6), (3.10), (3.15), (3.19)and (3.20) will obtain

$$\frac{d}{dt}\left(\|u\|_{H^2}^2+\|B\|_{H^2}^2\right)+\|\nabla u\|_{H^2}^2+\|u_t\|_{H^1}^2+\|B_t\|_{H^1}^2\lesssim\left(\|u\|_{H^2}^2+\|B\|_{H^2}^2\right)\left(1+\|u\|_{H^2}^2+\|B\|_{H^2}^2\right).$$

So, there are positive constants $T\triangleq min\{1, M^{-1}\}$,

$$\sup_{0\leq t\leq T}\left(1+\|u\|_{H^2}^2+\|B\|_{H^2}^2\right)\leq M,$$

which together with Lemma 2.2 gives (3.1).

5.2. Proof of Theorems 1.1

The above has derived a priori estimate of the high regularity of u and B, so a standard argument can be used to establish the existence of a strong solution: We construct approximate solutions in a strip domains by a semi-discrete Galerkin scheme to derive a uniform bound, thereby obtaining a solution by passing to the limit.

Moreover, we can deduce from (3.1) that the sequence $\left(u^\delta, B^\delta\right)$converges, up to the extraction of subsequences, to some limit (u, B) in the obvious weak sense, that is, as $\delta\to 0$, we obtaion

$$\begin{aligned}
u^\delta\to u\ \ &weakly*\ \ in\ L^\infty(0,T;H^2),\\
B^\delta\to B\ \ &weakly*\ \ in\ L^\infty(0,T;H^2),\\
\nabla u^\delta\to\nabla u\ \ &weakly*\ \ in\ L^2(0,T;H^2),
\end{aligned}$$

$$u_t^\delta \to u_t \quad weakly * \quad in \;\; L^2(0,T;H^1),$$
$$B_t^\delta \to B_t \quad weakly * \quad in \;\; L^2(0,T;H^1)$$

Then by making $\delta \to 0$, it implies that (u, B) is a strong solution of (1.7) on $\Omega \times (0, T]$.

Finally, it's easy to see the uniqueness of the strong solution (u, B) holds, that is, it remains by using the method which is due to Germain [9].

References

[1] Hmidi Abidi, and Taoufik Hmidi, "Results Existence in Critical Spaces for the Inhomogeneous MHD System," *Annales Mathematiques Blaise Pascal*, vol. 14, no. 1, pp. 103-148, 2007. [CrossRef] [Google Scholar] [Publisher Link]

[2] Hammadi Abidi, and Marius Paicu, "Global Existence for the MHD System in Critical Spaces," *Proceedings of the Royal Society of Edinburgh, Section A*, pp. 1-31, 2008. [CrossRef] [Google Scholar] [Publisher Link]

[3] Gui-Qiang Chen, and Dehua Wang, "Global Solutions of Nonlinear Magnetohydrodynamics with Large Initial Data," *Journal of Differential Equations*, vol. 182, no. 2, pp. 344-376, 2002. [CrossRef] [Google Scholar] [Publisher Link]

[4] Gui-Qiang Chen, and Dehua Wang, "Existence and Continuous Dependence of Large Solutions for the Magnetohy-Drodynamic Equations," *Journal of Applied Mathematics and Physics ZAMP*, vol. 54, pp. 608-632, 2003. [CrossRef] [Google Scholar] [Publisher Link]

[5] Bernard Ducomet, and Eduard Feireisl, "The Equations of Magnetohydrodynamics: On the Interaction between Matter and Radiation in the Evolution of Gaseous Stars," *Communications in Mathematical Physics*, vol. 266, pp. 595-629, 2006. [CrossRef] [Google Scholar] [Publisher Link]

[6] B. Desjardins, and C. Le Bris, "Remarks on a Nonhomogeneous Model of Magnetohydrodynamics," *Differential Integral Equations*, vol. 11, no. 3, pp. 377-394, 1998. [CrossRef] [Google Scholar] [Publisher Link]

[7] G. Duvaut, and J.L. Lions, "Inequalities in Thermoelasticity and Magnetohydrodynamics," *Archive for Rational Mechanics and Analysis*, vol. 46, pp. 241-279, 1972. [CrossRef] [Google Scholar] [Publisher Link]

[8] J.F. Gerbeau, and C. Le Bris, "Existence of Solution for a Density-Dependent Magnetohydrodynamic Equation," *Advance Differential Equations*, vol. 2, no. 3, pp. 427-452, 1997. [CrossRef] [Google Scholar] [Publisher Link]

[9] Pierre Germain, "Weak-Strong Uniqueness for the Isentropic Compressible Navier–Stokes System," *Journal of Mathematical Fluid Mechanics*, vol. 13, pp. 137-146, 2011. [CrossRef] [Google Scholar] [Publisher Link]

[10] Fei Jiang, and Song Jiang, "On Magnetic Inhibition Theory in 3D Non-Resistive Magnetohydrodynamic Fluids: Global Existence of Large Solutions," *Archive for Rational Mechanics and Analysis*, vol. 247, no. 96, 2023. [CrossRef] [Google Scholar] [Publisher Link]

[11] Fanghua Lin, and Ping Zhang, "Global Small Solutions to an MHD-Type System: The Three-Dimensional Case," *Communications on Pure and Applied Mathematics*, vol. 67, no. 4, pp. 531-580, 2014. [CrossRef] [Google Scholar] [Publisher Link]

[12] Xiaoxia Ren, Zhaoyin Xiang, and Zhifei Zhang, "Global Well-Posedness for the 2D MHD Equations without Magnetic Diffusion in a Strip Domain," *Nonlinearity*, vol. 29, no. 4, 2016. [CrossRef] [Google Scholar] [Publisher Link]

[13] Michel Sermange, and Roger Temam, "Some Mathematical Questions Related to the MHD Equations," *Communications on Pure and Applied Mathematics*, vol. 36, no. 5, pp. 635-664, 1983. [CrossRef] [Google Scholar] [Publisher Link]

[14] Lawrence C. Evans, *Partial Differential Equations*, 2nd ed., American Mathematical Society, pp. 1-662, 2022. [Google Scholar] [Publisher Link]

[15] Yuan Cai, and Zhen Lei, "Global Well-Posedness of the Incompressible Magnetohydrodynamics," *Archive for Rational Mechanics and Analysis*, vol. 228, pp. 969-993, 2018. [CrossRef] [Google Scholar] [Publisher Link]