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Original Article

Multiplicative Wiener Index of Some Graphs

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Abstract - *The Multiplicative Wiener Index, $\pi(G)$, is equal to the product of distance between all the pairs of vertices of G. In this paper, we investigate the Multiplicative Wiener Index of some standard graphs, which satisfies Harmonic Mean labeling.*

Keywords - *Graph, Harmonic Mean graphs, Path, Triangular snake graph, Comb graph, Hurdle graph, Friendship graph.*

1. Introduction

All graphs in this paper are finite, simple and undirected graphs. Let $G = (V, E)$ be a graph with p vertices and q edges. For a detailed survey of graph labeling, we refer to Gallian [1]. For all other standard terminology and notation, we follow Harary [2]. S. Somasundaram, R. Ponraj and S.S. Sandhya introduced Harmonic Mean labeling of graphs [3].

The Wiener Index is the sole topological Index that has been employed in drug discovery research. In 1947, Chemist Harold Wiener [7] created the Wiener Index of a graph $G\ (V, E)$ denoted by $W(G)$. The sum of the distances between every pair of vertices in a graph G is the Wiener Index $W(G) = \sum_{i=1}^{n} d\left(v_i^k\ v_j^k\right)$, where $d\left(v_i^k\ v_j^k\right)$is the smallest distance between the vertices $v_i\ and\ v_j$ in Graph G

The Multiplicative Wiener Index (π-index), proposed by Gutman et al. in 2000 [4][5] and is defined as $\pi(G) = \prod_{\{v_i,v_j\}\subseteq V(G)} \mathrm{d}_G\left(v_i, v_j\right)$

We provide a summary of definitions which are necessary for the present investigation.

Definition 1.1: A Graph $G = (V, E)$ with p vertices and q edges is said to be a Harmonic Mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ 1,2 ... ... $q + 1$ in such a way that when each edge $e = uv$ is labeled with $f(e = uv) = \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ (or) $\left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$, then the resulting edge labels are all distinct. In this case, f is called Harmonic Mean labeling of G.

Definition 1.2: A walk in which $u_0, u_1, u_2, \dots, u_n$ are distinct is called a Path. A Path on n vertices is denoted by P_n.

Definition 1.3: A Triangle snake T_n is obtained from a Path $u_1, u_2, u_3, u_4 \dots u_n$ by joining $u_1\ and\ u_{i+1}$ to a new vertex v_i from$1 \le i \le n - 1$. That is, every edge of the Path is replaced by a Triangle C_3 .

Definition 1.4: Comb is a graph obtained by joining a single pendant edge K_1 to each vertex of a path P_n , and it is represented by $P_n \odot K_1$.

Definition 1.5: A graph obtained from a path P_n by attaching a pendent edge to every internal vertices of the Path is called a Hurdle graph with $n - 2$ hurdles and is denoted by Hd_n.

Definition 1.6: A friendship graph is a graph which consists of n triangles with a common vertex and is denoted by F_n .

2. Main Results

Theorem 2.1: The Multiplicative Wiener Index of the Path graph is $\pi(P_n) = \prod_{k=1}^{n-1} k^{n-k}$

Proof:

Let $G = (V, E)$ be a Path graph P_n with n vertices and $n-1$ edges which satisfies the Harmonic mean labeling.
The Multiplicative Wiener Index of P_n is given by

$$\pi(G) = \prod_{\{v_i, v_j\} \subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$\pi(P_n) = d(v_1, v_2).d(v_1, v_3).d(v_1, v_4) \ldots d(v_1, v_n).d(v_2, v_3).d(v_2, v_4) \ldots d(v_2, v_n) \ldots d(v_{n-1}, v_1).d(v_{n-1}, v_2) \ldots d(v_{n-1}, v_n)$$

$$= (1.2.3 \ldots .(n-1)) \times (1.2.3 \ldots (n-2)) \times (1.2.3 \ldots (n-3)) \times \ldots \times (1.2.3) \times (1.2) \times 1$$

$$= 1^{n-1} \times 2^{n-2} \times 3^{n-3} \times \ldots \times (n-2)^2 \times (n-1)^1$$

$$\pi(P_n) = \prod_{k=1}^{n-1} k^{n-k}$$

Example 2.1:

The Multiplicative Wiener Index of Path P_5 is given below

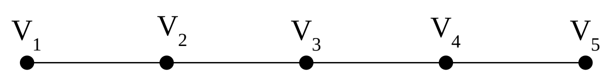

Fig. 1 P_5

$$\pi(G) = \prod_{\{v_i, v_j\} \subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$W(P_5) = d(v_1, v_2).d(v_1, v_3).d(v_1, v_4).d(v_1, v_5). d(v_2, v_3).d(v_2, v_4) \ldots d(v_2, v_5) \ldots d(v_4, v_5)$$

$$= (1.2.3.4) \times (1.2.3) \times (1.2) \times 1$$

$$= 1^4 \times 2^3 \times 3^2 \times 4^1$$

$$= 1^{n-1} \times 2^{n-2} \times 3^{n-3} \times 4^{n-4}$$

$$\pi(P_n) = \prod_{k=1}^{n-1} k^{n-k}$$

Theorem 2.2: The Multiplicative Wiener Index of the Triangular Snake graph is $\pi(T_n) = \prod_{k=2}^{n} k^{4(n-(k-1))}$

Proof:

Let T_n Triangular Snake graph with $2n+1$ vertices
Here T_n is a Harmonic mean labeled graph.
The Multiplicative Wiener Index of T_n is given by

$$\pi(G) = \prod_{\{v_i, v_j\} \subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$\pi(T_n) = d(v_1, v_2).d(v_1, v_3).d(v_1, v_4) \ldots d(v_1, v_n).d(v_2, v_3).d(v_2, v_4) \ldots d(v_2, v_n) \ldots d(v_{n-1}, v_n).$$

$$= (1.1.2.2.3.3 \ldots n.n) \times (1.2.2.3.3 \ldots n.n) \times ((1.1.2.2.3.3 \ldots (n-1)(n-1)) \times ((1.2.2.3.3 \ldots (n-1)(n-1)) \ldots (1.1.2.2) \times (1.2.2) \times (1.1) \times 1$$

$$= 1^{3n} \times 2^{4(n-1)} \times 3^{4(n-2)} \times 4^{4(n-3)} \times \ldots \times (n-1)^1$$

$$W(T_n) = 1^{3n} \times \prod_{k=2}^{n} k^{4(n-(k-1))}$$

Example 2.2:

The Multiplicative Wiener Index of the Triangular Snake Graph T_5 is given below

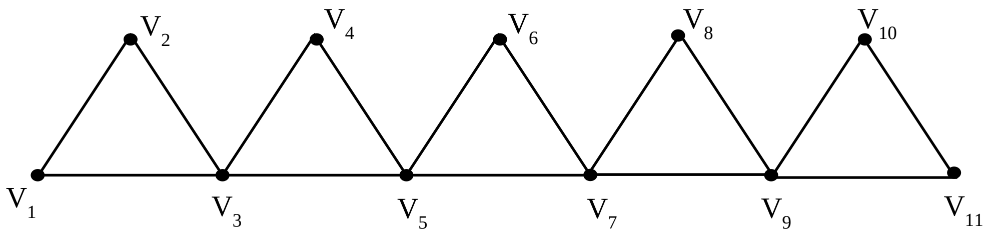

Fig. 2 $\boldsymbol{T_5}$

Here,

$$\pi(G) = \prod_{\{v_i,v_j\}\subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$W(T_5) = d(v_1, v_2).d(v_1, v_3).d(v_1, v_4) \dots d(v_1, v_{11}).d(v_2, v_3)\, d(v_2, v_4) \dots$$
$$d(v_2, v_{11}) \dots d(v_{10}, v_1) + d(v_{10}, v_2) \dots d(v_{10}, v_{11})$$
$$= (1.1.2.2.3.3.4.4.5.5) \times (1.2.2.3.3.4.4.5.5) \times (1.1.2.2.3.3.4.4) \times (1.2.2.3.3.44) \times (1.1.2.2.3.3) \times$$
$$(1.2.2.3.3) \times (1.1.2.2) \times (1.2.2) \times (1.1) \times 1$$
$$= 1^{15} \times 2^{16} \times 3^{12} \times 4^8 = 1^{3n} \times 2^{4(n-1)} \times 3^{4(n-2)} \times 4^{4(n-3)}$$
$$W(T_n) = 1^{3n} \times \prod_{k=2}^{n} k^{4(n-(k-1))}$$

Theorem 2.3: The Multiplicative Wiener Index of the Comb graph is
$\pi(P_n \odot K_1) = 2^{3n-4} \times \prod_{k=3}^{n} k^{4(n-k+1)} \times (n+1)$

Proof:

Let $P_n \odot K_1$ be a Comb graph with $2n$ vertices $2n-1$ edges

Also, $P_n \odot K_1$ is a Harmonic mean labeled graph.

The Multiplicative Wiener Index of $P_n \odot K_1$ is given by

$$\pi(G) = \prod_{\{v_i,v_j\}\subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$\pi(P_n \odot K_1) = d(v_1, v_2).d(v_1, v_3).d(v_1, v_4) \dots d(v_1, v_n).d(v_2, v_3).d(v_2, v_4)$$
$$\dots d(v_2, v_n) \dots d(v_{n-1}, v_n)$$
$$= \big(1.2.2.3.3 \dots n.(n+1)\big) \times (1.2.2.3.3.n) \times \big((1.2.2.3.3 \dots (n-1).n\big) \times \big((1.2.2.3.3 \dots (n-1)\big)$$
$$\times \dots \times (1.2.3) \times (1.2) \times 1$$
$$= 1^{(2n-1)} \times 2^{(3n-4)} \times 3^{(4n-8)} \times 4^{(4n-12)} \times \dots \times (n+1)^1$$
$$\pi(P_n \odot K_1) = 1^{2n-1} \times 2^{3n-4} \times \prod_{k=3}^{n} k^{4(n-k+1)} \times (n+1)$$

Example 2.3:

The Multiplicative Wiener Index of the Comb graph $P_5 \odot K_1$ is given below

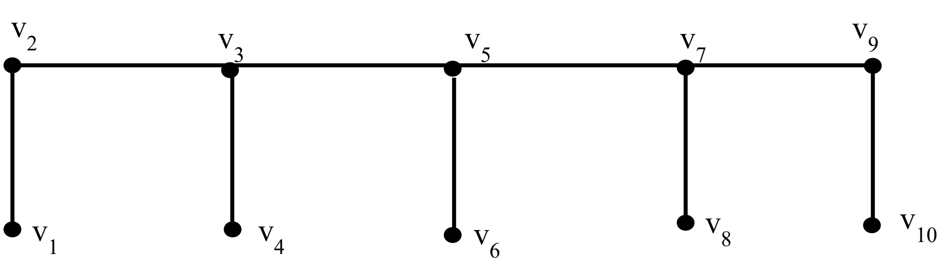

Fig. $P_5 \odot K_1$

Here,

$$\pi(G) = \prod_{\{v_i,v_j\}\subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$\pi(P_5 \odot K_1) = d(v_1, v_2).d(v_1, v_3).d(v_1, v_4) \dots d(v_1, v_{10}).d(v_2, v_3)$$
$$.d(v_2, v_4) \dots d(v_2, v_{10}) \dots d(v_9, v_{10})$$
$$= (1.2.2.3.3.4.4.5.5.6) \times (1.2.2.3.3.4.4.5) \times (1.2.3.3.4.4.5) \times (1.2.2.3.3.4) \times (1.2.3.3.4) \times (1.2.2.3) \times (1.2.3) \times (1.2) \times 1$$
$$= 1^9 \times 2^{11} \times 3^{12} \times 4^8 \times 5^4 \times 6^1$$
$$= 1^{(2n-1)} \times 2^{(3n-4)} \times 3^{(4n-8)} \times 4^{(4n-12)} \times \dots \times (n+1)^1$$
$$\pi(P_n \odot K_1) = 1^{2n-1} \times 2^{3n-4} \times \prod_{k=3}^{n} k^{4(n-k+1)} \times (n+1)$$

Theorem 2.4: The Multiplicative Wiener Index of the Hurdle graph is $\pi(Hd_n) = 2^{3n-6} \times \prod_{k=3}^{n-1} k^{4(n-k)}$

Proof:

Let Hd_n be a Hurdle graph with $2n-2$ vertices and $2n-3$ edges

Also, Hd_n is a Harmonic mean labeled graph.

The Multiplicative Wiener Index of Hd_n is given by

$$\pi(G) = \prod_{\{v_i,v_j\}\subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$\pi(Hd_n) = d(v_1, v_2).d(v_1, v_3).d(v_1, v_4) \dots d(v_1, v_n).d(v_2, v_3).d(v_2, v_4)$$
$$\dots d(v_2, v_n) \dots d(v_{n-1}, v_n)$$
$$= \big(1.2.2.3.3 \dots (n-1).(n-1)\big) \times \big(1.1.2.2.3.3\dots(n-2).(n-2)\big) \times \big((2.3.3 \dots (n-1).(n-1)\big)$$
$$\times \big((1.1.2.2.3.3 \dots (n-3)\big) \times \dots \times (1.1.2.2) \times (2.3.3) \times (1.1) \times 2$$
$$= 1^{(2n-3)} \times 2^{(3n-6)} \times 3^{(4n-12)} \times 4^{(4n-16)} \times \dots \times (n-1)^4$$
$$\pi(Hd_n) = 2^{3n-6} \times \prod_{k=3}^{n-1} k^{4(n-k)}$$

Example 2.4:

The Multiplicative Wiener Index of the Hurdle graph Hd_6 is given below

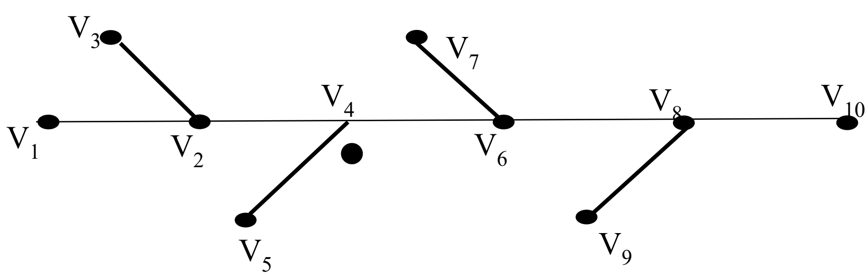

Fig. 4 Hd_6

Here,

$$\pi(G) = \prod_{\{v_i,v_j\}\subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$\pi(Hd_6) = d(v_1, v_2).d(v_1, v_3).d(v_1, v_4) \dots d(v_1, v_{10}).d(v_2, v_3)$$
$$.d(v_2, v_4) \dots d(v_2, v_{10}) \dots d(v_9, v_{10})$$
$$= (1.2.2.3.3.4.4.5.5) \times (1.2.2.3.3.4.4) \times (2.3.3.4.4.5.5) \times (1.1.2.2.3.3) \times (2.3.3.4.4) \times (1.1.2.2) \times (2.3.3) \times (1.1) \times 2$$
$$= 1^9 \times 2^{12} \times 3^{12} \times 4^8 \times 5^4$$
$$= 1^{(2n-3)} \times 2^{(3n-6)} \times 3^{(4n-12)} \times 4^{(4n-16)} \times \dots \times (n-1)^4$$
$$\pi(Hd_n) = 2^{3n-6} \times \prod_{k=3}^{n-1} k^{4(n-k)}$$

Theorem 2.5: The Multiplicative Wiener Index of the Friendship graph is $\pi(F_n) = 2^{2n(n-1)}$
Proof:

Let F_n be a Friendship graph which satisfies the Harmonic mean labeled graph.
The Multiplicative Wiener Index of F_n is given by

$$\pi(G) = \prod_{\{v_i,v_j\}\subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$\pi(F_n) = d(v_0, v_1).d(v_0, v_2).d(v_0, v_3) \dots d(v_0, v_{2n}).d(v_1, v_2).d(v_1, v_3).$$
$$d(v_1, v_4) \dots d(v_1, v_{2n}).d(v_2, v_3).d(v_2, v_4) \dots d(v_2, v_{2n}) \dots d(v_{2n-1}, v_{2n})$$
$$= (1.1.1 \dots 1) \times (1.2.2.2.\dots.2.2) \times \big((2.2.2 \dots 2.2)\big) \times \big((1.2.2 \dots 2.2)\big) \times \dots \times (1.2.2) \times (2.2) \times 1$$
$$= 1^{3n} \times 2^{2n(n-1)}$$
$$\pi(F_n) = 2^{2n(n-1)}$$

Example 2.5:

The Multiplicative Wiener Index of Friendship graph F_4 is given below

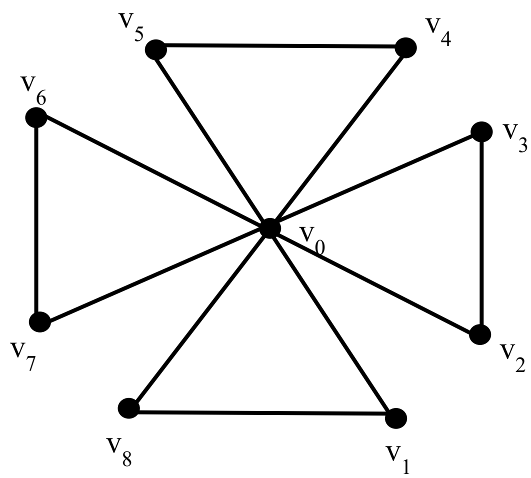

Fig. 5 F_4

Here,

$$\pi(G) = \prod_{\{v_i,v_j\}\subseteq V(G)} \mathrm{d}_G(v_i, v_j)$$

$$\pi(F_4) = d(v_0, v_1).d(v_0, v_2).d(v_0, v_3) \dots d(v_0, v_8).d(v_1, v_2).d(v_1, v_3).$$
$$d(v_1, v_4) \dots d(v_1, v_8).d(v_2, v_3).d(v_2, v_4) \dots d(v_2, v_8) \dots d(v_7, v_8)$$
$$= (1.1.1.1.1.1.1.1) \times (1.2.2.2.2.2.2) \times (2.2.2.2.2.2) \times (1.2.2.2.2) \times (2.2.2.2) \times (1.2.2) \times (2.2) \times 1$$
$$= 1^{12} \times 2^{24} = 1^{3n} \times 2^{2n(n-1)}$$
$$\pi(F_n) = 2^{2n(n-1)}$$

3. Conclusion

We have studied the Multiplicative Wiener Index for various graph structures derived using graph operators. In future a study will be carried out with Multiplicative Wiener Index for various compounds.

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