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Original Article

The Accurate Degree Domination Number of a Graph

Moumita K. Chatterjee[1*], Mallikarjun B. Kattimani[1], Pavitra P. Kumbargoudra[1]

[1]*Department of Mathematics, The Oxford College of Engineering, Bengaluru, India.*

[*]*Corresponding Author : kali.moumita@gmail.com*

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Abstract - *Consider a graph $G(V,E)$ and a dominating set D_k, the degree of a dominating set D_k is the sum of the degree of all the vertices in D_k and is written as $deg(D_k)$. The degree of the set is the sum of the degrees of all the vertices of the set. The minimum degree among all the dominating sets is called the degree dominating set and is written as D_i. Now, a dominating set D_i is an accurate degree dominating set if $V - D_i$ has no degree dominating set of the cardinality of D_i. The accurate degree domination number $\gamma_a^{\circ}(G)$ of a given graph G is the minimum number of vertices in an accurate degree dominating set of G. This paper initiates the study of an accurate degree domination number $\Gamma_a^{\circ}(G)$. Further, obtain some bounds for $\gamma_a^{\circ}(G)$, and the numerical value of $\gamma_a^{\circ}(G)$ some standard graphs like path, cycle, wheel, complete graph, complete bipartite graph.*

Keywords - *Dominating set, Degree dominating set, Accurate dominating set, Accurate degree dominating set.*

Math. Subject Classification - *05C69.*

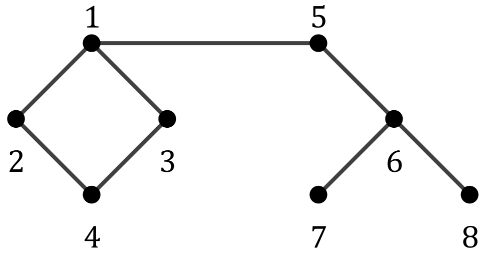
1. Introduction

The graphs considered here are nontrivial, undirected and connected. Undefined terms or notations in this paper may be found in Harary [1].

Let $G(V,E)$ be a graph. A subset D of a vertex set $V(G)$ is a dominating set if every vertex not in D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. The degree of a vertex v_i, where $i = 1,2,\ldots,n$ of a graph G is denoted by d_i or $deg(v_i)$ is the number of edges incident on v_i[6]. For any dominating set $D_k \subseteq V(G)$, the degree of a dominating set D_k is defined as $\forall u_i \in D_k$, where $i = 1,2,3,\ldots,n$, $\deg(D_k) = \sum_{i=1}^{n} \deg(u_i)$. A set $D_i \subseteq V(G)$ is said to be a degree dominating set of given graphs G if $\deg(D_i) = \delta$ is minimum. The degree domination number $\gamma^{\circ}(G)$ is the minimum cardinality of a degree dominating set and an upper degree domination number Γ° is the maximum cardinality of a degree dominating set of G[5].

One of the intriguing topics under domination is the accurate domination of a graph. This parameter was first introduced by V. R. Kulli and M.B. Kattimani in the year 2012[7, 8]. After this idea was formalized, the area of accurate domination was explored in Accurate Total Domination in Graphs, Global Accurate Domination in Graphs, Connected Accurate Domination in Graphs, and Global Connected Accurate and Maximal Domination in Graphs. In the later years, some more concepts based on accurate domination and accurate domination polynomials also materialized. Now, expand this idea of accurate domination by including the degree domination and the derived results; we are interested in undertaking a deep study into this emerging area, especially to empower the unique set of accurate degree domination. The main motivation of this domination invariant lies in its uniqueness in identifying the dominating set such that no other dominating set with the same properties exists.

A dominating set D of G is an accurate dominating set if $V - D$ has no dominating set of size $|D|$. The accurate domination number $\gamma_a(G)$ of G equals the minimum number of vertices in an accurate dominating set of G. Similarly, we define an upper accurate domination number $\Gamma_a(G)$ of G as the maximum number of vertices in an accurate dominating set of G. In this paper, we obtain some bounds for $\gamma_a^{\circ}(G)$ and its exact values for some standard graphs. For instance, consider the following graph.



In the above figure, the dominating sets are, $D_1 = \{2,3,6\}, D_2 = \{1,2,6\}, D_3 = \{1,4,6\}, D_4 = \{4,5,7,8\}, D_5 = \{1,4,7,8\}, D_6 = \{2,3,5,7,8\}$ with degree of the dominating sets are $7, 8, 8, 6, 7, 8$ respectively. Here D_4 is the degree dominating set with cardinality 4 and minimum degree is 6, D_1 and D_2 are the accurate dominating set with minimum cardinality. Now D_4 is the accurate degree dominating set with degree 6 and cardinality 4. Hence $\gamma(G) = 3$, $\gamma^{\circ}(G) = 4$ and $\gamma_a^{\circ}(G) = 4$.

This paper initiates the study of accurate degree domination numbers. The accurate degree dominating set is the degree dominating set. D_I such that $V - D_i$ is not having any degree domination set of the cardinality D_i of graphs. The minimum cardinality of the accurate degree dominating set is called the accurate degree domination number and is denoted by $\gamma_a^{\circ}(G)$. The maximum cardinality of the accurate degree dominating set is called the accurate upper degree domination number and is denoted by $\Gamma_a^{\circ}(G)$. We also extend the degree domination number of a graph with an accuracy in nature. Found some results on standard graphs and their relation with other known parameters.

This paper uses the symbol $A_{\gamma^{\circ}}(G)$, which denotes the set of all minimum degree dominating sets and $A_{\gamma_a^{\circ}}(G)$ to denote the set of all minimum accurate degree dominating sets of G. Similarly, we denote $\delta_{\gamma^{\circ}}$ and $\delta_{\gamma_a^{\circ}\circ}$ to represent the minimum degree of degree dominating set and accurate degree dominating set, respectively [3].

2. Results

The following are the numerical values of degree domination and accurate degree domination numbers of some of the standard graphs.

Observation 2.1. For some standard graphs, the following results are observed,

1. For any complete graph $K_n, n \geq 2$ vertices, $\gamma^{\circ}(K_n) = 1,\ \delta_{\gamma^{\circ}} = n-1,\ \gamma_a^{\circ}(K_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1,\ \delta_{\gamma_a^{\circ}} = n-1$.
2. For any cycle $Cn, n \geq 3$ vertices, $\gamma^{\circ}(C_n) = \left\lceil \frac{n}{3} \right\rceil,\ \delta_{\gamma^{\circ}} = \left\lceil \frac{n}{2} \right\rceil \times 2,\ \gamma_a^{\circ}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1,\ \delta_{\gamma_a^{\circ}} = \left\lceil \frac{n}{2} \right\rceil \times 2$.
3. For any path $Pn, n \geq 2$ vertices, $\gamma^{\circ}(P_n) = \left\lceil \frac{n}{3} \right\rceil,\ \delta_{\gamma^{\circ}} = \left\lceil \frac{2n}{3} \right\rceil,\ \gamma_a^{\circ}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ where $n \cong 0(mod 3)$ and $\delta_{\gamma_a^{\circ}}$ is equal to $(n-1)$ and $(n+1)$.
4. For any wheel $Wn, n \geq 4$ vertices, $\gamma^{\circ}(W_n) = 1,\ \delta_{\gamma^{\circ}} = n-1,\ \gamma_a^{\circ}(W_n) = 1,\ \delta_{\gamma_a^{\circ}} = n-1$.
5. For any star $K_{1,n}, n \geq 2$ vertices, $\gamma^{\circ}(K_{1,n}) = 1,\ \delta_{\gamma^{\circ}} = n-1,\ \gamma_a^{\circ}(K_{1,n}) = 1,\ \delta_{\gamma_a^{\circ}} = n$.
6. For any double star $S_{1,n,n}, n \geq 3$ vertices, $\gamma^{\circ}(S_{1,n,n}) = n,\ \delta_{\gamma^{\circ}} = n+1,\ \gamma_a^{\circ}(S_{1,n,n}) = n, \delta_{\gamma_a^{\circ}} = n+1$.
7. For any complete bipartite graph $K_{m,n}, m \geq 2, n \geq 2$ vertices and $m \leq n, \gamma^{\circ}(K_{m,n}) = 2, \delta_{\gamma^{\circ}} = m+n,\ \gamma_a^{\circ}(K_{m,n}) = m+1, \delta_{\gamma_a^{\circ}} = m(n+1)$.

Proposition 2.1. Every accurate degree dominating set is an accurate dominating set.
Proof. Let D be an accurate degree dominating set, which implies that $V - D$ has no degree dominating set with $|D|$. Let us assume D is not an accurate dominating set; then, there exists a dominating set. $D' \in V - D$ such that $|D| = |D'|$. However, clearly D' is an accurate degree dominating set, which contradicts our assumption. Hence D is an accurate dominating set.

Proposition 2.2. For any graph G, $\delta_{\gamma^{\circ}} \leq \delta_{\gamma_a^{\circ}}$.
Theorem 2.3. If G be an r-regular graph, then $\gamma^{\circ}(G) \leq \gamma_a^{\circ}(G) - 1$.
Proof. Let G is a regular graph. Let D is γ-set, and each vertex in D has the same degree, and the degree of D is $r.|D|$ Moreover, that will be the minimum. Hence, D is also γ ∘ -set and therefore $\gamma^{\circ}(G) = \gamma(G)$.

Theorem 2.4. Let G be connected graph with $\gamma^{\circ}(G) \geq 2$ then,

$$\gamma_a^{\circ}(\overline{G}) \geq \left\lceil \frac{\delta(G)}{\gamma_a^{\circ}(G) - 1} \right\rceil + 1.$$

Proof. Let δ be the minimum degree of the graph G, and let v be a vertex of degree δ. Now let $A = N(v) = v_i : 1 \leq i \leq \delta$. Let $A = \left\lceil \frac{\delta}{\gamma_a^{\circ} - 1} \right\rceil$ and partition A into k sets $A_1, A_2, \ldots A_k$ each contains at most γ_a°. Also, we observe that no A_i is an accurate degree dominating set. For each set A_i, select one vertex which is not formed an accurate dominating set, i.e., $a_i \in V \setminus A_i$ such that $N(a_i) \notin A_i$. Now let $A' = \cup a_i$. Here, $| A' \leq k$ and A' is an accurate degree dominating set. Therefore, the set $A' \cup v$ is an accurate degree dominating set of $\overline{G}$, and so $\gamma_a^{\circ}(\overline{G}) \geq |A'| + 1 = 1 + \left\lceil \frac{\delta}{\gamma_a^{\circ} - 1} \right\rceil$.

Theorem 2.5. A dominating set D of a graph G is an accurate degree dominating set, and u ∈ D then,

1. u is an isolated vertex
2. u is a pendent vertex
3. There does not exist $v \in N(u)$ such that $d(u) > d(v)$.

Proposition 2.6. Let G be any graph. Then $\gamma^{\circ} \leq \Delta(G) + 1$.

Proposition 2.7. For any graph G,

$$\gamma^{\circ}(G) \leq \gamma_a^{\circ}(G).$$

Proof. Clearly, every accurate degree dominating set is a dominating set of G is a degree dominating set of G, thus the result is true,

Remark: The path P_{11} achieves this bound.

Theorem 2.8. If G contains any isolated vertex, then a minimum degree dominating set of G is an accurate degree dominating set.

Proof. Suppose the graph G contains an isolated vertex v. Thus v is in every dominating set of G. Thus $V - D$ has no dominating set of cardinality $|D|$ and hence the theorem.

Theorem 2.9. For any Graph G,

$$\gamma^{\circ}(G) \leq \gamma_a(G) \leq \gamma_a^{\circ}(G)$$

Proof. Let us first prove the first part of the inequality,
We know that,

$$\gamma^{\circ} \leq P - \gamma(G) \tag{2}$$

Also, we have $\gamma_a(G) \leq P - \gamma(G) + 1$

$$\gamma_a(G) - 1 \leq P - \gamma(G) \tag{3}$$

From (1) and (3), we conclude that $\gamma^{\circ}(G) \leq \gamma_a(G)$ Since every accurate degree dominating set is an accurate dominating set hence.

$$\gamma_a(G) \leq \gamma_a^{\circ}(G) \tag{4}$$

Combining (4) and (5), we obtain the result.

Theorem 2.10. For any tree with m cut vertices, the degree accurate domination number of the graph is,

$$\gamma_a^{\circ}(T) \leq m + 3 \tag{5}$$

Proof. Suppose M is the set of all cut vertices of T then $|M| = m$. If v is any end vertex of T then $M \cup v$ is an accurate dominating set with the cardinality $m + 1$.

Further, if every cut vertex is adjacent to at least one end vertex, then the cardinality of the accurate degree dominating set is $m + 1 + 1$.
If the cut vertex is adjacent to at least two end vertices, then the cardinality of the accurate degree dominating set is $m + 1 + 2$. Hence, $\gamma_a^{\circ}(T) \leq m + 3$.

Corollary 2.11. For any tree with n end vertices, the degree accurate domination number of the graph is,

$$\gamma_a^{\circ}(T) \leq p - n + 3$$

Theorem 2.12. Let G be a graph, then $\gamma_a^{\circ}(G) = \gamma^{\circ}(G)$ if and only if there exists a set $D \in A_{\gamma^{\circ}}(G)$ where $A_{\gamma^{\circ}}(G)$ denotes the minimum degree dominating sets.
Proof. Suppose that $\gamma_a^{\circ}(G) = \gamma^{\circ}(G)$ and we have D as the minimum accurate degree dominating set of G. Since D is also a degree dominating set of G and $|D| = \gamma_a^{\circ}(G) = \gamma^{\circ}(G)$. Here, we note that $D \in A_{\gamma^{\circ}}(G)$.
Let us assume that D' is an arbitrary minimum degree dominating set of G. If $D \cap D' = \emptyset$ then $D' \subseteq V_G \backslash D$, which implies that D' is $|D|$ same dominating set of G, which contradicts the fact that D is the minimum accurate degree dominating set of G. Hence $D \cap D' \neq \emptyset$.
Conversely, let us assume that there exists a set D which is a minimum degree dominating set, i.e. $D \in A_{\gamma^{\circ}}(G)$ such that $D \cap D' \neq \emptyset$ for every set $D' \in A_{\gamma^{\circ}}(G)$. Then D is an accurate degree dominating set of G, implying that $\gamma_a^{\circ}(G) \leq |D| = \gamma^{\circ}(G) \leq \gamma_a^{\circ}(G)$. Consequently $\gamma_a^{\circ}(G) = \gamma^{\circ}(G)$.

Definition 2.1. For a given graph G, the corona graph $G \circ K_1$ is obtained by adding a path of length 1 to every vertex. The resultant graph is called the corona graph.

Theorem 2.13. If G is a graph, then $\gamma_a^{\circ}(G \circ K_1) \geq \gamma^{\circ}(G \circ K_1)$.
Proof : Let $G' = G \circ K_1$ be a corona graph of G and $L(G') = v'1, v'2, \dots, v'_n$ be a set of all pendent vertices of G'. If $G = K_1$ then $\gamma^{\circ}(G) = 1$ and $\gamma_a^{\circ}(G') = 2$. Now assume that G is any connected graph of order $n \geq 2$. Now let D be an arbitrary minimum degree dominating set of G', which implies $D \in A_{\gamma^{\circ}}(G')$ and $|D \cap \subset \{v, v'\}| = 1 \forall v \in G$. Further D and the $\overline{D} = V(G') - D$ are the minimum degree dominating sets of G. Hence, D is not an accurate degree dominating set of G'. Thus $\gamma_a^{\circ}(G \circ K_1) \geq \gamma^{\circ}(G \circ K_1)$.

3. Conclusion

This article obtained some bounds for accurate degree domination numbers. Also, the accurate degree domination number of some classes of graphs has been calculated. In future, one can obtain the accurate degree domination number for other classes of graphs like generalized Peter son graph, generalized corona graph, join of two graphs, product of two graphs, etc.

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