International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I5P103

Volume 70 Issue 5, 13-18, May 2024
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Original Article

Some Results on the Fan Product of Dual Matrices

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Received: 21 March 2024 Revised: 25 April 2024 Accepted: 14 May 2024 Published: 27 May 2024

Abstract - *This paper first presents the background of dual numbers and dual matrices under the Fan product by using the concept and properties of the Fan product, and then it defines some new special matrices under the Fan product. Further, this paper defines dual matrices under the Fan product. Also, some new results of the Fan product of these dual matrices have been derived. Finally, some theorems regarding the Fan product have been derived.*

Keywords - *Dual number, Dual matrix, Fan product, Hadamard product, Regular matrix.*

1. Introduction

Numerous problems in Mathematics can be changed into computation of the Fan product, like system solution of Wiener-Hopf integral equations and numerical method for solving Volterra integral equations. Professor Ky Fan has given the first opinion of this product by using the dimension of the matrix. Therefore, the Fan product plays a very important role in converting and developing a matrix transformation of the Volterra integral equation.

W. K. Clifford initially gave the concept of the algebra of dual numbers, but A. P. Kotelnikow had applied it first in the field of machine design. Dual number is an extension form of real numbers. Also, there are many applications of dual vector algebra used in the fields of Kinematics, Mechanics and Physics. Dual vector algebra also renders a conventional tool for handling mathematical units like screws and wrenches.

This paper defines some new special matrices under the Fan product and also derives some properties of these special matrices. This paper also defines the Fan product of dual matrices and develops the algebra of dual matrices under the Fan product.

In Section 2, the fundamental view of dual numbers, dual matrices and the Fan product are presented. It also presents the properties of dual numbers, dual matrices and the Fan product. Section 3 defines some new special matrices with the Fan product, for example orthogonal F-matrix. Section 4 defines the Fan product of dual matrices, and Section 5 finally defines some special dual matrices under the Fan product.

2. Definitions, Notations and Preliminary Results

Let $\mathfrak{R}^{m\times n}$ be the set of all real matrices of order $m \times n$.

Definition 2.1[8]. For any two matrices $P = [p_{ij}] \in \mathfrak{R}^{m\times n}$ and $Q = [q_{ij}] \in \mathfrak{R}^{m\times n}$, the Fan product of P and Q defined as

$$P \star Q = \begin{cases} -p_{ij}q_{ij}, & i \neq j \\ p_{ii}q_{ii}, & i = j \end{cases}$$

For example

$$P = \begin{bmatrix} 10 & -2 & -5 & -1 \\ -5 & 25 & -10 & -15 \\ 0 & -20 & 40 & -11 \\ -12 & -5 & -8 & 34 \end{bmatrix}, \quad Q = \begin{bmatrix} 5 & -1 & -2 & 0 \\ -3 & 6 & -1 & -2 \\ -4 & -2 & 5 & -1 \\ -1 & -3 & -4 & 10 \end{bmatrix},$$

then

$$P \star Q = \begin{bmatrix} 50 & -2 & -10 & 0 \\ -15 & 150 & -10 & -30 \\ 0 & -40 & 200 & -11 \\ -12 & -15 & -32 & 340 \end{bmatrix}.$$

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2.2[9]. Properties of the Fan Product

If $P = [p_{ij}] \in \Re^{m\times n}, Q = [q_{ij}] \in \Re^{m\times n}, R = [r_{ij}] \in \Re^{m\times n}$ and k is a scalar, then

1. The Fan product is commutative. Let $P = [p_{ij}], Q = [q_{ij}]$ then $P \star Q = Q \star P$.

2. The Fan product is Linear. Let $P = [p_{ij}], Q = [q_{ij}], R = [r_{ij}]$ and $k \in \Re$, then
$P \star (Q + R) = P \star Q + P \star R$, and $k(P \star Q) = (kP) \star Q = P \star (kQ)$.

3. Let $P, Q \geq 0$, then

$$(P \star Q)^T = P^T \star Q^T.$$

4. $Rank(P \star Q) \leq Rank(P)Rank(Q)$.

Definition 2.3[1]. A 2-tuples (p, p^*) is called a dual number if p, p^* are real numbers among a real entity 1 and the dual entity ε, where $\varepsilon^2 = \varepsilon^3 = \cdots = 0,\ 0\varepsilon = \varepsilon 0 = 0, 1\varepsilon = \varepsilon 1 = \varepsilon$.
A dual number is normally signified as $\hat{p} = p + \varepsilon p^*$.
The set of dual number $\mathfrak{D}$ is defined by $\mathfrak{D} = \{\hat{p} = p + \varepsilon p^* |\ p, p^* \in \Re,\ \varepsilon^2 = 0, \varepsilon \neq 0\}$.

Definition 2.4[1]. Let $P, P^* \in \Re^{m\times n}$, then A dual matrix is denoted by $\hat{P}$ and it is defined by

$$\hat{P} = \begin{bmatrix} p_{11} + \varepsilon p_{11}{}^* & p_{12} + \varepsilon p_{12}{}^* & \dots & p_{1n} + \varepsilon p_{1n}{}^* \\ p_{21} + \varepsilon p_{21}{}^* & p_{22} + \varepsilon p_{22}{}^* & \dots & p_{2n} + \varepsilon p_{2n}{}^* \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} + \varepsilon p_{m1}{}^* & p_{m2} + \varepsilon p_{m2}{}^* & \dots & p_{mn} + \varepsilon p_{mn}{}^* \end{bmatrix}.$$

Simply above expression as

$$\hat{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix} + \varepsilon \begin{bmatrix} p_{11}{}^* & p_{12}{}^* & \cdots & p_{1n}{}^* \\ p_{21}{}^* & p_{22}{}^* & \cdots & p_{2n}{}^* \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1}{}^* & p_{m2}{}^* & \cdots & p_{mn}{}^* \end{bmatrix}.$$

Therefore

$$\hat{P} = [p_{ij}] + \varepsilon[p_{ij}{}^*] = P + \varepsilon P^*,$$

where ε is a dual unit and $\varepsilon^2 = \varepsilon^3 = \cdots = 0,\ 0\varepsilon = \varepsilon 0 = 0, 1\varepsilon = \varepsilon 1 = \varepsilon$.
Let $\mathfrak{D}^{m\times n}$ be the set of all real dual matrices of order $m \times n$.

2.5[1]. Properties of dual number and Dual matrices

1. Addition and multiplication of dual numbers is again a dual number.
2. Division of dual numbers is treated as an ordinary division of complex numbers.
3. Power and square root are defined as
   $\hat{p}^n = p^n + \varepsilon n p^* p^{n-1}$ and $\sqrt{\hat{p}} = \sqrt{p} + \varepsilon \frac{p^*}{2\sqrt{p}}$.
4. Multiplication of dual matrices is defined as $\hat{P}\hat{Q} = PQ + \varepsilon(PQ^* + P^*Q)$.
5. Inverse of a dual matrix is represented as $(\hat{P})^{-1} = P^{-1} - \varepsilon(P^{-1}P^*P^{-1})$.
6. If $\hat{A}$ is a nonsingular square matrix, then $\hat{\lambda} = \lambda + \varepsilon\lambda^*, \hat{X} = X + \varepsilon X^*, \hat{X} \neq 0$ are called eigen value and eigen vector if $\hat{P}\hat{X} = \hat{\lambda}\hat{X}$.

3. Special Real Matrices with the Fan Product

In linear algebra, there are many types of matrices defined under ordinary multiplication, like as symmetric matrix, orthogonal matrix, and periodic matrix. As an impetus, some new special matrix under the Fan product will be defined. These matrices are named as F-matrix; for example, a symmetric matrix under the Fan product is called an F-symmetric matrix.

It is easy to check that the set $\Re^{m\times n}$ of all real matrices of order $m \times n$ is a group under the Fan product. The identity element of this group under the Fan product is defined as

$$I_F = [e_{ij}], \text{where } e_{ij} = \begin{cases} 1, & i = j \\ -1, & i \neq j \end{cases}$$

I_F is called an F-identity matrix. Also $(\Re^{m\times n}, +, \star)$ is a commutative ring with identity. In addition, $\Re^{m\times n}$ is an algebra over real number.

Definition 3.1. Let $P = [p_{ij}] \in \Re^{m\times m}, p_{ij} \neq 0$. If $P \star Q = I_F$, then $Q = P^{-1}$ is inverse of P under the Fan product. It is called the F-inverse of P. Hence $Q = \left[\frac{1}{p_{ij}}\right]$.

Example 3.2. Let $= \begin{bmatrix} 2 & 3 & -1 \\ 3 & -5 & 2 \\ 4 & 7 & -8 \end{bmatrix}$, then F-inverse of P is

$$P^{-1} = \begin{bmatrix} 1/2 & 1/3 & -1 \\ 1/3 & -1/5 & 1/2 \\ 1/4 & 1/7 & -1/8 \end{bmatrix}.$$

Definition 3.3. Let $P \in \Re^{m\times m}$, since the inverse of P always exists, if $p_{ij} \neq 0$, then P is called the F-singular matrix; if any $p_{ij} = 0$. If there is no such $p_{ij} = 0$, then P is called the F-regular matrix.

Theorem 3.4. If P, Q are two F-regular matrices, then

$$(P \star Q)^{-1} = (Q \star P)^{-1} = P^{-1} \star Q^{-1} = Q^{-1} \star P^{-1}.$$

Proof. From the properties of inverse

$$P \star P^{-1} = I_F,$$

Then

$$(P \star Q) \star (P \star Q)^{-1} = I_F,$$

Then by associativity

$$P \star (Q \star (P \star Q)^{-1}) = I_F.$$

Therefore

$$Q \star (P \star Q)^{-1} = P^{-1},$$

then

$$(Q^{-1} \star Q) \star (P \star Q)^{-1} = Q^{-1} \star P^{-1},$$
$$I_F \star (P \star Q)^{-1} = Q^{-1} \star P^{-1}.$$

Hence

$$(P \star Q)^{-1} = Q^{-1} \star P^{-1}.$$

Similarly, we can easily prove that.

$$(Q \star P)^{-1} = Q^{-1} \star P^{-1}.$$

Preposition 3.5. If $P, Q \in \Re^{m\times n}$, then $\mathrm{tr}(P \star Q) = tr(Q \star P)$.

Definition 3.6. Let $P \in \Re^{m\times m}$, if $P \star P^T = I_F$, then P is called the F-orthogonal matrix. Also, if $P = P^T$, then P is called the F-symmetric matrix and if $P^T = -P$, then P is called F- skew-symmetric matrix.

Example 3.7. $P = \begin{bmatrix} 1 & 2 & 5 \\ 1/2 & 1 & -7 \\ 1/5 & -1/7 & 1 \end{bmatrix}$ is an F-orthogonal matrix.

$Q = \begin{bmatrix} 5 & 2 & 6 \\ 2 & 3 & -4 \\ 6 & -4 & -1 \end{bmatrix}$ is an F-symmetric matrix.

$R = \begin{bmatrix} 0 & -4 & 5 \\ 4 & 0 & 9 \\ -5 & -9 & 0 \end{bmatrix}$ is an F-skew symmetric matrix.

Definition 3.8. Let $P \in \Re^{m\times m}$, P is called F-periodic matrix with period r, if $(P^{\star})^{r+1} = P$. If $r = 1$ and this equality holds, then matrix P is called F-idempotent matrix.

Definition 3.9. Let $P \in \Re^{m\times m}$, if $(P^{\star})^2 = 0$, then P is called F-nilpotent matrix. This holds only for a zero matrix.

Definition 3.10. Let $P \in \Re^{m\times m}$, if $(P^{\star})^2 = I_F$, then P is called the F-involuntary matrix. An involuntary matrix has elements that are either 1 or -1.

Definition 3.11. Let $P \in \Re^{m\times m}$, if $(P^{\star})^2 = P$, then P is called F-idempotent matrix

$P = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ is an F-idempotent matrix.

Definition 3.12. Let $P, Q \in \Re^{m\times n}$, if P, Q are non-zero, but $P \star Q = 0$, then P, Q are called nulling each other matrices.
Let $P = \begin{bmatrix} 3 & 0 \\ 0 & 5 \\ 1 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 1 \\ 7 & 0 \\ 0 & 2 \end{bmatrix}$, then the Fan product is P, and Q is given by

$$P \star Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, matrices P, Q are nulling to each other.
Theorem 3.13. Let $P \in \Re^{m\times n}$, then P is F-orthogonal if and only if $p_{ij}.p_{ji} = 1$.
Proof. From the definition of F-orthogonal matrix,

$$P \star P^{\mathrm{T}} = I_F,$$

Then

$$[p_{ij}] \star [p_{ij}]^T = I_{F,}$$
$$[p_{ij}] \star [p_{ij}]^T = e_{ij}.$$

Therefore

$$p_{ij}.p_{ji} = 1.$$

4. The Fan product of dual matrices

Definition 4.1. Let $\widehat{U}, \widehat{V} \in \mathfrak{D}^{m\times n}$, then the Fan product of $\widehat{U}, \widehat{V}$ is defined as

$$\widehat{U} \star \widehat{V} = (U + \varepsilon U^*) \star (V + \varepsilon V^*)$$
$$= (U \star V) + \varepsilon(U^* \star V + U \star V^*)$$

4.2. Properties of the Fan product of dual matrices
If $\widehat{U}, \widehat{V}, \widehat{W} \in \mathfrak{D}^{m\times n}$, then

1. $\widehat{U} \star \widehat{V} = \widehat{V} \star \widehat{U}$.
2. $\widehat{U} \star (\widehat{V} \star \widehat{W}) = (\widehat{U} \star \widehat{V}) \star \widehat{W}$.
3. $\widehat{U} \star (\widehat{V} + \widehat{W}) = \widehat{U} \star \widehat{V} + \widehat{U} \star \widehat{W}$.
4. For any $a \in \Re$,

$$a(\widehat{U} \star \widehat{V}) = (a\widehat{U}) \star \widehat{V} = \widehat{U} \star (\mathrm{a}\widehat{V}).$$

It is easy to prove that $(\mathfrak{D}^{m\times n}, +, \star)$ is a ring with unit element.

Definition 4.3. Let $\hat{P} \in \mathfrak{D}^{m\times n}$, an identity dual matrix under F-product is defined as $\hat{I} = I_F + \varepsilon 0$, so that $\hat{P} \star \hat{I} = \hat{P} = \hat{P} \star \hat{I}$.

Definition 4.4. Let $\hat{P} \in \mathfrak{D}^{m\times n}$, then the conjugate of $\hat{P}$ is defined as

$$\overline{(\hat{P})} = \overline{(P + \varepsilon P^*)} = P - \varepsilon P^*.$$

Theorem 4.5. Let $\hat{P}, \hat{Q} \in \mathfrak{D}^{m\times n}$, then $tr(\hat{P} \star \hat{Q}) = tr(\hat{Q} \star \hat{P})$.
Proof. Now

$$tr(\hat{P} \star \hat{Q}) = tr((P \star \mathrm{Q}) + \varepsilon(P^* \star \mathrm{Q} + P \star Q^*))$$
$$= tr(P \star \mathrm{Q}) + \varepsilon tr(P^* \star \mathrm{Q}) + \varepsilon tr(P \star Q^*)$$
$$= tr(Q \star P) + \varepsilon tr(Q^* \star P) + \varepsilon tr(Q \star P^*)$$
$$= tr(\hat{Q} \star \hat{P}).$$

Theorem 4.6. Let $\hat{P} = P + \varepsilon P^* \in \mathfrak{D}^{m\times n}$ and if $P \neq 0$ is F-invertible, then

$$(\hat{P})^{-1} = \frac{\overline{(\hat{P})}}{P^2} \text{ or } (\hat{P})^{-1} = P^{-1} - \varepsilon P^{-2}P^*$$

and hence $\hat{P} \star (\hat{P})^{-1} = \hat{I}$.
Proof.

$$(\hat{P})^{-1} = \frac{1}{P+\varepsilon P^*} = \frac{1}{P+\varepsilon P^*} \times \frac{P-\varepsilon P^*}{P-\varepsilon P^*} = \frac{P-\varepsilon P^*}{P^2}.$$

Thus,

$$(\hat{P})^{-1} = \frac{\overline{(\hat{P})}}{P^2} = P^{-1} - \varepsilon P^{-2}P^*.$$

Theorem 4.7. Let $\hat{P}, \hat{Q} \in \mathfrak{D}^{m\times n}$ and $\hat{P}, \hat{Q}$ are regular matrices, then

$$(\hat{P} \star \hat{Q})^{-1} = (\hat{P})^{-1} \star (\hat{Q})^{-1}.$$

Proof. Here, first, we prove that $(\hat{P} \star \hat{Q})^T = (\hat{P})^T \star (\hat{Q})^T$.
Since

$$(\hat{P} \star \hat{Q})^T = ((P \star \mathrm{Q}) + \varepsilon(P^* \star \mathrm{Q} + P \star Q^*))^T$$

$$= (P \star \mathrm{Q})^T + \varepsilon(P^* \star \mathrm{Q})^T + + \varepsilon(P \star Q^*)^T$$
$$= (\hat{P})^T \star (\hat{Q})^T.$$

Now,

$$(\hat{P} \star \hat{Q}) \star (\hat{P} \star \hat{Q})^{-1} = I_F,$$

Then by associativity

$$\hat{P} \star \left(\hat{Q} \star (\hat{P} \star \hat{Q})^{-1}\right) = I_F.$$

Therefore

$$\hat{Q} \star (\hat{P} \star \hat{Q})^{-1} = \hat{P}^{-1},$$

then

$$(\hat{P} \star \hat{Q})^{-1} = \hat{P}^{-1} \star \hat{Q}^{-1}.$$

Similarly, we can easily prove that.

$$(\hat{Q} \star \hat{P})^{-1} = \hat{Q}^{-1} \star \hat{P}^{-1}.$$

5. Special Dual Matrices Under the Fan Product

Definition 5.1. Let $\hat{P} = [p_{ij} + \varepsilon p_{ij}{}^*] \in \mathfrak{D}^{m\times n}$ and if $p_{ij} = 0$, then $\hat{P}$ is called a pure dual matrix.

Example 5.2. $\hat{P} = \begin{bmatrix} 7\varepsilon & -\varepsilon \\ -3\varepsilon & 2\varepsilon \\ 5\varepsilon & \varepsilon \end{bmatrix}$ is a pure dual matrix.

Definition 5.3. Let $\hat{P} = [p_{ij} + \varepsilon p_{ij}{}^*] \in \mathfrak{D}^{m\times m}$, then $\hat{P}$ is called F-singular dual matrix; if any $p_{ij} = 0$. If there is no such $p_{ij} = 0$, then $\hat{P}$ is called F-regular dual matrix.

Definition 5.4. Let $\hat{P} = [p_{ij} + \varepsilon p_{ij}{}^*] \in \mathfrak{D}^{m\times m}$, then if $\hat{P} \star \hat{P}^T = \hat{I}$, then $\hat{P}$ is called F-orthogonal dual matrix. Also if $\hat{P} = \hat{P}^T$, then P is called F-symmetric dual matrix and if $\hat{P}^T = -\hat{P}$, then $\hat{P}$ is called F- skew symmetric dual matrix.

Definition 5.5. Let $\hat{P} \in \mathfrak{D}^{m\times m}$, $\hat{P}$ is called F-periodic dual matrix with period r, if $\left((\hat{P})^\star\right)^{r+1} = \hat{P}$. If $r = 1$ and this equality holds, then matrix $\hat{P}$ is called F-idempotent dual matrix.

Definition 5.6. Let $\hat{P} \in \mathfrak{D}^{m\times m}$, if $\left((\hat{P})^\star\right)^r = 0 + \varepsilon 0$, then $\hat{P}$ is called F-nilpotent dual matrix.

Definition 5.7. Let $\hat{P} \in \mathfrak{D}^{m\times m}$, if $\left((\hat{P})^\star\right)^2 = \hat{I}$, then $\hat{P}$ is called F-involuntary dual matrix.

Definition 5.8. Let $\hat{P} \in \mathfrak{D}^{m\times m}$, if $\left((\hat{P})^\star\right)^2 = \hat{P}$, then $\hat{P}$ is called F-idempotent dual matrix.

Definition 5.9. Let $\hat{P}, \hat{Q} \in \mathfrak{D}^{m\times m}$, if $\hat{P} \neq 0, \hat{Q} \neq 0$,but $\hat{P} \star \hat{Q} = 0$, then $\hat{P}$, $\hat{Q}$ are called nulling each other matrices.

Theorem 5.10. If $\hat{P} \in \mathfrak{D}^{m\times m}$, the k-th power of $\hat{P}$ is as follow

$$(\hat{P})^k = P^k + k\varepsilon P^{k-1}P^*.$$

Proof. We use the Mathematical induction method

Case I. When $k = 2$, then

$$(\hat{P})^2 = \hat{P}.\hat{P} = [p_{ij} + \varepsilon p_{ij}{}^*].[p_{ij} + \varepsilon p_{ij}{}^*] = [p_{ij}{}^2 + 2\varepsilon(p_{ij}{}^* p_{ij})] = P^2 + 2\varepsilon PP^*.$$

Similarly

$$(\hat{P})^3 = \hat{P}.\hat{P}.\hat{P} = [p_{ij} + \varepsilon p_{ij}{}^*].[p_{ij} + \varepsilon p_{ij}{}^*].[p_{ij} + \varepsilon p_{ij}{}^*] = [p_{ij}{}^3 + 3\varepsilon(p_{ij}{}^* p_{ij})] = P^3 + 3\varepsilon PP^*.$$

Then, from the induction method, we obtained

$$(\hat{P})^k = P^k + k\varepsilon P^{k-1}P^*.$$

Corollary 5. 11. Zero matrix is only F- nilpotent dual matrix.

Corollary 5.12. Identity dual matrix $\hat{I}$ is only a matrix, which is F-idempotent dual and F-periodic dual matrix.

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