International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I5P104
Volume 70 Issue 5, 19-27, May 2024

Original Article

An Invertible Subspace in Clifford Algebras

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Received: 26 March 2024 Revised: 29 April 2024 Accepted: 15 May 2024 Published: 29 May 2024

Abstract *- The goal of this paper is to find a subspace in the Cliiford algebra in which every non-zero element has an invertible element. The paper begins with some basic knowledge in the classical Clifford algebra, then shows that not all non-zero elements are invertible through some specific examples. The construction of the invertible subspace is presented in the third part of the paper.*

Keywords - *Clifford algebras, Hyper complex analysis, Holomorphic function, Invertible subspace.*

1. Introduction

We start the paper by remind readers the definition of Clifford algebras introduced by F. Brackx, R. Delanghe & F. Sommen[29], where the elements in the Clifford algebra can be considered an extension of complex numbers in classical complex analysis. Here, a Clifford number has many imaginary parts instead of just one imaginary unit like in complex numbers. By constructing the structure of the Clifford number it is expected to solve problems that are difficult to solve in classical complex analysis. With the addition of the number of equations, functions and variables, there are many new difficulties appear, one suggested an alternative extension: to contructed the theory of hyper-complex numbers and hyper-complex functions. Started by Moisil (see [11]) in 1931, this theory has been growing steadily and has many important applications using the results of Moisil[11-14], Theodorescu [6,7,8], Angelescu. A[9], Nef [14], Sobrero [15], Fueter[16,17], Iftimie[1,2], Delanghe[3,4,5], Goldschmidt [18-20], Gilbert [21,22], Colton [23,24], Sommen[25], Neldelcu Coroi M.[13], Tutschke. W. [26,27].

In the classical complex analysis, every non-zero element has an inverse. However, this will no longer be true for elements in the Clifford algebra due to the non-commutative nature of this algebra. The goal of this paper is to find a subspace in the Cliiford algebra in which every non-zero element has an invertible element. The paper begins with some basic knowledge in the classical Clifford algebra, then shows that not all non-zero elements are invertible through some specific examples. The construction of the invertible subspace is presented in the third part of the paper. Finally, there is a summary with some possible directions for further research in this subspace.

2. Clifford Algebras

Let $e_1, e_2, \dots, e_n$ be standard basis of the Euclidean space $\mathbb{R}^n$ and let e_0 be unit element satisfying

$$\begin{cases} e_0 e_i = e_i e_0 = e_i; i = 1, \dots, n, \\ e_i e_j + e_j e_i = 0; \forall\, i \neq j, \\ e_j^2 = -1; j = 1, \dots, n. \end{cases} \tag{2.1}$$

Consider the 2^n- dimensional real linear space, denoted by $\mathcal{A}_n$ with the basis

$$\{e_A = e_{h_1} e_{h_2} \dots e_{h_r}, \qquad A = (h_1, \dots, h_r) \in PN, 1 \leq h_1 < \dots < h_r \leq n\}$$

where $PN = \{0,1,2, \dots, n, 12,13, \dots, 1n, 123, \dots, 12 \dots n\}$ and $e_\phi = e_0$.

Let A, B be two sets, a product on $\mathcal{A}_n$ is defined by

$$e_A e_B = (-1)^{n((A\cap B)\setminus S)}.(-1)^{p(A,B)} e_{A\Delta B}, \tag{2.2}$$

where S stands for the set $\{1,2, \dots, s\}, n(A) = \# A$ is cardinality of A,

$$p(A,B) = \sum_{j\in B} p(A,j)\,, p(A,j) = \# \{i \in A : i > j\}, \tag{2.3}$$

and $A\Delta B = (A\backslash B) \cup (B\backslash A)$.

Example 2.1. Consider $e_A = e_2e_5e_9, e_B = e_4e_5e_8e_{10}$, we have

$$A \cap B = \{5\}, Card(A \cap B) = 1,$$

$$p(A,B) = 2 + 1 + 1 = 4,$$

$$A\,\Delta B = \{2,4,8,9,10\}.$$

Thus

$$e_Ae_B = (-1)^1(-1)^4 e_2e_4e_9e_9e_{10}.$$

It is not difficult to check that, by this structure $\mathcal{A}_n$ is a linear, associate, but non-commutative algebra over $\mathbb{R}^{n+1}$. It is called the universal Clifford algebra[…]. For the unit element vector e_0, sometime we denote by 1, and we see that $\dim \mathcal{A}_n = 2^n$.

Each element $a \in \mathcal{A}_n$ can presented by

$$a = \sum_A a_Ae_A\,, a_A \in \mathbb{R}, \tag{2.4}$$

and so, the multiplication for two elements in $\mathcal{A}_n$ as follows: suppose $a = \sum_A a_Ae_A\,, b = \sum_B a_Be_B$. Then

$$ab = \sum_A \sum_B a_Ab_B\, e_Ae_B. \tag{2.5}$$

This multiplication is combinedly, distributary for addition with unit element e_0, but not commutatatively.

Example 2.2. Suppose $a = e_1 + e_2e_3e_5, b = 2e_2e_4, c = e_2 - 3e_1$. Calculate for $a(b+c), ab + ac$?

We have

$$b + c = 2e_2e_4 + e_2 - 3e_1,$$

$$\begin{aligned} a(b+c) &= (e_1 + e_2e_3e_5)(2e_2e_4 + e_2 - 3e_1) \\ &= 2e_1e_2e_4 + e_1e_2 + 3e_0 + 2e_3e_4e_5 - e_3e_5 + 3e_1e_2e_3e_5. \end{aligned}$$

$$\begin{aligned} ab &= 2e_1e_2e_4 + 2e_3e_4e_5, \\ ac &= e_1e_2 + 3e_0 - e_3e_5 + 3e_1e_2e_3e_5. \end{aligned}$$

A conjugate element of $e_A = e_{k_1}e_{k_2} \dots e_{k_t}$ can be presented by

$$\bar{e}_A = (-1)^t e_{k_t} \dots e_{k_2}e_{k_1},$$

this mean

$$\bar{e}_A = (-1)^{t(t+1)/2} e_A.$$

In specical case, we have

$$\bar{e}_0 = e_0,$$

$$\bar{e}_i = -e_i,$$

$$e_A\bar{e}_A = e_0,$$

$$\bar{e}_Ae_A = e_0,$$

$$\bar{\bar{e}}_A = e_A,$$

$$\overline{e_Ae_B} = \bar{e}_B\bar{e}_A.$$

In deed, for instance, we will prove that $\bar{e}_A e_A = e_0$. If $e_A = e_{k_1} e_{k_2} \dots e_{k_t}, \bar{e}_A = (-1)^t e_{k_t} \dots e_{k_2} e_{k_1}$, then

$$\begin{aligned}\bar{e}_A e_A &= \left((-1)^t e_{k_t} \dots e_{k_2} e_{k_1}\right)\left(e_{k_1} e_{k_2} \dots e_{k_t}\right)\\ &= (-1)^t e_{k_t} \dots e_{k_2}\left(e_{k_1} e_{k_1}\right) e_{k_2} \dots e_{k_t}\\ &= (-1)^{t+1} e_{k_t} \dots \left(e_{k_2} e_{k_2}\right) \dots e_{k_t}\\ &\dots\\ &= (-1)^{t+(t-1)}\left(e_{k_t} e_{k_t}\right)\\ &= (-1)^{t+t} e_0\\ &= e_0.\end{aligned}$$

From the conjugate element of e_A we can define a conjugate of factor $a = \sum_A a_A e_A$ by

$$\bar{a} = \sum_A a_A \bar{e}_A. \tag{2.6}$$

An inverse element of element $a \in \mathcal{A}_n$ denoted by a^{-1}, which satisfies the following:

$$\begin{aligned}a^{-1}.\, a &= e_0,\\ a.\, a^{-1} &= e_0.\end{aligned}$$

In the case $n = 2$ (i.e $\mathcal{A}_n = \mathbb{C}$ $(complex\ numbers)$), for any $z \in \mathbb{C}, z \neq 0$, there exist the inverse element as form

$$z^{-1} = \frac{\bar{z}}{|z|^2}.$$

We will show that, when $\dim \mathcal{A}_n = 4, (n = 3, \mathcal{A}_n\ is\ Quaternion\ algrbra\ \mathbb{H})$, one has the same result, that means, for any $a \in \mathbb{H}, a \neq 0$, there exist the inverse element as form

$$a^{-1} = \frac{\bar{a}}{|a|^2}.$$

In deed, suppose $a = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3, e_3 = e_1 e_2, |a|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2 > 0$. We have $\bar{e}_3 = \overline{e_1 e_2} = \bar{e}_2 \bar{e}_1 = (-e_2)(-e_1) = e_2 e_1 = -e_3$. Thus

$$\bar{a} = a_0 e_0 - a_1 e_1 - a_2 e_2 - a_3 e_3$$

and

$$\begin{aligned}a\bar{a} &= \left(\sum_{i=0}^{3} a_i e_i\right)\left(\sum_{i=0}^{3} a_j \bar{e}_j\right)\\ &= \sum_{i=0}^{3} a_i^2 e_i \bar{e}_i + \sum_{i<j} a_i a_j \left(e_i \bar{e}_j + e_j \bar{e}_i\right)\\ &= \sum_{i=0}^{3} a_i^2 e_0\\ &= |a|^2 e_0.\end{aligned}$$

And so $a\bar{a} = |a|^2 e_0$, therefore

$$a.\frac{\bar{a}}{|a|^2} = e_0.$$

On the other hand

$$\bar{a}a = \left(\sum_{i=0}^{3} a_j \bar{e}_j\right)\left(\sum_{i=0}^{3} a_i e_i\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{3} a_i^2 \bar{e}_i e_i + \sum_{j<i} a_j a_i (\bar{e}_j e_i + \bar{e}_i e_j) \\
&= \sum_{i=0}^{3} a_i^2 e_0 \\
&= |a|^2 e_0.
\end{aligned}
$$

Thus

$$\frac{\bar{a}}{|a|^2}.a = e_0.$$

To sum up

$$a^{-1} = \frac{\bar{a}}{|a|^2}.$$

However, the problem becomes more difficult in order $\dim \mathcal{A}_n > 4$. The following proposition shows that, there exist a element which different from zero, such that, it has no inverse element.

Proposition 2.1. If $a, b \in \mathcal{A}_n$, such that $a \neq 0, b \neq 0$ and $a.b = 0$, then neither a nor b has no inverse element.

Proof. Suppose otherwise, i.e. there exist the inverse element a^{-1} of a. Then

$$a^{-1}(ab) = 0 \Leftrightarrow (a^{-1}a)b = 0 \Leftrightarrow b = 0,$$

This is contrary to the assumption. Similarly, if there exist b^{-1} of b then we also have

$$(ab)b^{-1} = 0 \Leftrightarrow a = 0.$$

The proposition is proved.

Example 2.3. Consider $a = e_0 - e_1 e_2 e_3$, $b = e_0 + e_1 e_2 e_3$, we have

$$
\begin{aligned}
a.b &= (e_0 - e_1 e_2 e_3)(e_0 + e_1 e_2 e_3) \\
&= e_0 + e_1 e_2 e_3 - e_1 e_2 e_3 - e_1 e_2 e_3 (e_1 e_2 e_3) \\
&= e_0 + e_2 e_3 e_2 e_3 \\
&= e_0 - e_0 \\
&= 0.
\end{aligned}
$$

Thus, neither a nor b has no inverse element. Therefore, if $\dim \mathcal{A}_n > 4$, then there exist a element which different from zero, such that, it has no inverse element.

3. Invertible Subspaces

The goal of this section is introduce a subspace of $\mathcal{A}_n$ ($\dim \mathcal{A}_n > 4$), in which, this subspace is invertible (i.e. for any element in this subspace which different from zero has inverse element).

3.1. In Special Case

It is easy to see that, the subspace generated by basis elements $\{e_0, e_1, e_2, \ldots, e_n\}$ is invertible. Indeed, for any element $a = \sum_{i=0}^{n} a_i e_i$, $\bar{a} = \sum_{j=0}^{n} a_j \bar{e}_j$, we have

$$
\begin{aligned}
\bar{a}a &= \left(\sum_{i=0}^{n} a_j \bar{e}_j\right)\left(\sum_{i=0}^{n} a_i e_i\right) \\
&= \sum_{j=0}^{n} a_j^2 \bar{e}_j e_j + \sum_{j<i} a_j a_i (\bar{e}_j e_i + \bar{e}_i e_j) \\
&= \sum_{j=0}^{n} a_j^2 e_0 \\
&= |a|^2 e_0.
\end{aligned}
$$

Thus

$$\frac{\bar{a}}{|a|^2}.a = e_0.$$

By the same way, we have

$$\begin{aligned} a\bar{a} &= \left(\sum_{i=0}^{n} a_i e_i\right)\left(\sum_{i=0}^{n} a_j \bar{e}_j\right) \\ &= \sum_{i=0}^{n} a_i^2 e_i \bar{e}_i + \sum_{i<j} a_i a_j \left(e_i \bar{e}_j + e_j \bar{e}_i\right) \\ &= \sum_{i=0}^{n} a_i^2 e_0 \\ &= |a|^2 e_0. \end{aligned}$$

and so

$$a.\frac{\bar{a}}{|a|^2} = e_0.$$

Therefore

$$a^{-1} = \frac{\bar{a}}{|a|^2}.$$

3.2. In General Case

From basis element system $\{e_A\}$ of the Clifford algebras, we choice m vectors denoted by $\{e_{A_1}, e_{A_2}, \dots, e_{A_m}\}$ and these generate a subspace denoted by $\mathcal{C}_m$. Then we have following theorem

Theorem 3.1. Suppose elements system $\{e_{A_i}\}, i = 1, \dots, m$ satisfies following "canonical" condition

$$\begin{aligned} &e_{A_i}\bar{e}_{A_j} + e_{A_j}\bar{e}_{A_i} = 0, \\ &i \neq j, i, j = 1, \dots, m. \end{aligned} \tag{3.1}$$

Then the subspace $\mathcal{C}_m$ is invertible.

Proof. Firstly, we prove that, form condition (3.1), leads to

$$\bar{e}_{A_i} e_{A_j} + \bar{e}_{A_j} e_{A_i} = 0. \tag{3.2}$$

To achieve this, multiplication (from the left-hand side) of (3.1) with $\bar{e}_{A_i}$, we have

$$\begin{aligned} &\bar{e}_{A_i}(e_{A_i}\bar{e}_{A_j}) + \bar{e}_{A_i}(e_{A_j}\bar{e}_{A_i}) = 0, \\ &\bar{e}_{A_j} + (\bar{e}_{A_i} e_{A_j})\bar{e}_{A_i} = 0. \end{aligned} \tag{3.3}$$

After that, , multiplication (from the right-hand side) of (3.3) with e_{A_i}, we obtain

$$\begin{aligned} &\bar{e}_{A_j} e_{A_i} + \bar{e}_{A_i} e_{A_j} (\bar{e}_{A_i} e_{A_i}) = 0, \\ &\bar{e}_{A_j} e_{A_i} + \bar{e}_{A_i} e_{A_j} = 0. \end{aligned}$$

Interchange i and j ech other in (3.3), we have (3.2).

Second, Let $a \in \mathcal{C}_m$, $a = \sum_{i=1}^{m} a_i e_{A_i}$, $\bar{a} = \sum_{i=1}^{m} a_i \bar{e}_{A_i}$, we have

$$\bar{a}a = \left(\sum_{j=1}^{m} a_j \bar{e}_{A_j}\right)\left(\sum_{i=1}^{m} a_i e_{A_i}\right)$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} a_j^2 (\bar{e}_{A_j} e_{A_j}) \ + \sum_{j<i} a_j a_i \left(\bar{e}_{A_j} e_{A_i} + \bar{e}_{A_i} e_{A_j}\right) \\
&= \sum_{j=1}^{m} a_j^2 e_0 \\
&= |a|^2 e_0 \, .
\end{aligned}
$$

Then

$$\frac{\bar{a}}{|a|^2}.\, a = e_0.$$

Similarly

$$
\begin{aligned}
a\bar{a} &= \left(\sum_{i=1}^{m} a_i e_{A_i}\right)\left(\sum_{j=1}^{m} a_j \bar{e}_{A_j}\right) \\
&= \sum_{i=1}^{m} a_i^2 e_{A_i} \bar{e}_{A_i} \ + \sum_{i<j} a_i a_j \left(e_{A_i} \bar{e}_{A_j} + e_{A_j} \bar{e}_{A_i}\right) \\
&= \sum_{i=1}^{m} a_i^2 e_0 \\
&= |a|^2 e_0 \, ,
\end{aligned}
$$

thus

$$a.\frac{\bar{a}}{|a|^2} = e_0.$$

Therefore

$$a^{-1} = \frac{\bar{a}}{|a|^2}.$$

To sum up, if $a \in \mathcal{C}_m, a \neq 0$ with "canonical" basis then a is invertible and $a^{-1} = \frac{\bar{a}}{|a|^2}$. The theorem is proved.

In the following, we consider a particularly example about a invertible subspace $\mathcal{C}_m$. Suppose dim $\mathcal{A}_n \ = 2^n$, we consider system (3.1) as form

$$\{e_0, e_1, \dots, e_n, e_1 e_2 \dots e_n\}. \tag{3.4}$$

This basis elements generates a subspace, denoted by $\mathcal{C}_{n+2}$.

Here we have a problem: How many n, such that, the system (3.4) is "canonical" system. To be short, we denote $e_A = e_1 e_2 \dots e_n$. In order to find the values of n, we only calculate for the following conditions

$$e_0 \bar{e}_A + e_A \bar{e}_0 = 0, \tag{3.5}$$

$$e_i \bar{e}_A + e_A \bar{e}_i = 0. \tag{3.6}$$

Consider (3.5), we have

$$e_0 \bar{e}_A + e_A \bar{e}_0 = 0,$$

$$\bar{e}_A + e_A = 0,$$

$$(-1)^{\frac{n(n+1)}{2}} e_A + e_A = 0,$$

$$(-1)^{\frac{n(n+1)}{2}} = -1,$$

$$t = \frac{n(n+1)}{2} = odd\ number.$$

We have 4 situations:

- $n = 4k$, then $t = \frac{4k(4k+1)}{2}$ = even number (not satisfied),
- $n = 4k + 1$, then $t = \frac{(4k+1)(4k+2)}{2}$ = odd number,
- $n = 4k + 2$, then $t = \frac{(4k+2)(4k+3)}{2}$ = odd number,
- $n = 4k + 3$, then $t = \frac{(4k+3)(4k+4)}{2}$ = even number (not satisfied).

Therefore, one has $n = 4k + 1$ or $n = 4k + 2$ which satisfies condition (3.5).

Consider (3.6), we have

$$\begin{aligned}
&e_i\bar{e}_A + e_A\bar{e}_i = 0,\\
&\Leftrightarrow -e_ie_A + e_A(-e_i) = 0,\\
&\Leftrightarrow e_ie_A + e_Ae_i = 0,\\
&\Leftrightarrow e_i(e_1e_2 \dots e_{i-1}e_ie_{i+1} \dots e_n) + (e_1e_2 \dots e_{i-1}e_ie_{i+1} \dots e_n)e_i = 0,\\
&\Leftrightarrow (-1)^i(e_1e_2 \dots e_{i-1}e_{i+1} \dots e_n) + (-1)^{n-i+1}(e_1e_2 \dots e_{i-1}e_{i+1} \dots e_n) = 0,\\
&\Leftrightarrow (-1)^i + (-1)^{n-i+1} = 0,\\
&\Leftrightarrow i + (n - i + 1)\ is\ odd\ number\\
&\Leftrightarrow n\ is\ even\ number.
\end{aligned}$$

Thus, we only have $n = 4k + 2$ is satisfied.

To sum up, the system $\{e_0, e_1, \dots, e_n, e_1e_2 \dots e_n\}$ generrates a invertible subspace $\mathcal{C}_{n+2}$ if and only if $n = 4k + 2$, and so, for any $a \in \mathcal{C}_{n+2}, a \neq 0$, one has

$$a^{-1} = \frac{\bar{a}}{|a|^2}.$$

Example 3.1. For $n = 6$, we have the invertible $\mathcal{C}_8$ with basis elements $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_1e_2e_3e_4e_5e_6\}$. We check the following conditions

$$\begin{aligned}
&e_0\bar{e}_A + e_A\bar{e}_0 = 0,\\
&e_i\bar{e}_A + e_A\bar{e}_i = 0, i = 0,1, \dots, 6,
\end{aligned}$$

where $e_A = e_1e_2e_3e_4e_5e_6$.

We have

$$\begin{aligned}
&\bar{e}_A = (-1)^{\frac{6.7}{2}}e_A = -e_A,\\
&e_0\bar{e}_A + e_A\bar{e}_0 = -e_0e_A + e_Ae_0 = 0.
\end{aligned}$$

Since

$$\begin{aligned}
&e_i\bar{e}_A + e_A\bar{e}_i = 0\\
&\Leftrightarrow e_i(-e_A) + e_A(-e_i) = 0\\
&\Leftrightarrow e_ie_A + e_Ae_i = 0
\end{aligned}$$

and

$$\begin{aligned}
&e_i(e_1e_2 \dots e_{i-1}e_ie_{i+1} \dots e_6) + (e_1e_2 \dots e_{i-1}e_ie_{i+1} \dots e_6)e_i\\
&= (-1)^i(e_1e_2 \dots e_{i-1}e_{i+1} \dots e_6) + (-1)^{6-i+1}(e_1e_2 \dots e_{i-1}e_ie_{i+1} \dots e_6),
\end{aligned}$$

Then $e_i\bar{e}_A + e_A\bar{e}_i = 0$ leads to $(-1)^i + (-1)^{6-i+1} = 0 \Leftrightarrow i + (6 - i + 1)$= odd number, this always true.

4. Conclusion

In this paper we construct an invertible subspace in classical Clifford algebra. Our goal is that from this subspace we can introduce better results of regular functions than results in complex analysis because of the commutative property of multiplication. Here the Cauchy-Riemann operator in usual Clifford can be replace by the canonical operator as form

$$T = \sum_{i=1}^{m} \alpha_i e_{A_i} \frac{\partial}{\partial x_i},$$

and conjugate operator of operator T can be defined

$$T^* = \sum_{i=1}^{m} \alpha_i \bar{e}_{A_i} \frac{\partial}{\partial x_i},$$

where e_{A_i} satisfying condition (3.1) and $\alpha_i = \pm 1$ (i.e. for each i we can choice arbitrarily: $\alpha_i = 1$ or $\alpha_i = -1$. From this, some results: Cauchy formula integral and Cauchy-Pompeiu formula intagral for Holder continous function can be represented.

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