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Original Article

Inverse Semigroup with Source of Semiprimeness

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Abstract – *In this study, by using the set of the source of semiprimeness, we give the definitions of $|S_S| -$ inverse semigroup and $|S_S| -$ inverse monoid which are generalizations of semigroup and monoid concepts. Then, we investigate these structures and determine their basic properties. Then, we define $|S_S| -$ group using these definitions. Using the definitions given, we examine the relationships between the new structures.*

Keywords *- Semigroup, Unit element, Inverse element, Semiprime semigroup, Source of semiprimeness.*

1. Introduction

Studies on the commutativity of rings have an important place in ring theory. Over time, these studies were adapted to semigroups. In the studies carried out, new results regarding algebraic structures are achieved by using the concepts of primeness and semiprimeness. Recently, it has been discussed how to generalize the results of the concept of semiprimeness in studies on algebraic structures. Obtaining the set of The Source of Semiprimeness emerged as a result of these discussions. With the discovery of elements that provide semiprimeness in the algebraic structure, various authors have created new types of rings, semigroups and groups and contributed to the literature.

In this study, firstly, we investigate some properties of the source of semiprimeness of semigroups. Then, by using inverse semigroup, inverse monoid and group concepts, we give three new semigroups, monoid and group definitions, which are the main purpose of the study. With these new definitions, the semigroups without unit elements, inverse elements or unit elements, inverse semigroup, inverse monoid and group have been obtained by using the source of semiprimeness.

The set of the source of semiprimeness is defined in [3] for rings and is used to obtain new ring, domain and field definitions. The properties of the source of semiprimeness of rings are examined, and various conclusions are reached. Then in [2], the set is defined for semigroups and their properties related to semigroups are examined. With the help of this set, new generalizations about semiprimeness can be reached.

Now, let us take a quick look at the results we have achieved. In the second section of the study, the set of the source of semiprimeness $S_S = \{a \in S | aSa = 0\}$ is given for a semigroup S as defined in [2]. The basic properties of the set are given, and their properties are investigated for different types of semigroups. In the third section, the definitions of S_S inverse semigroup, S_S inverse monoid and S_S group are given. In this part, the relations of newly defined semigroups with each other and different types of semigroups are studied. Examples are given for each of the different semigroups and, specifically, properties of the monoid $(\mathbb{Z}_n, \cdot)$ are investigated according to the new definitions. Finally, we give important theorems about these new concepts, and we define equivalent definitions that may be useful in future studies.

2. Preliminaries

Before getting down to the main results, let us give the well-known definitions of semiprime theory that we will use in our paper.

First, let us give the definitions of special elements and semigroup types used in our study. Definitions are referenced by [4] and [5]. Let $(S,\cdot)$ be a semigroup. An identity element of a semigroup S is an element $1_S \in S$ such that $1_S x = x 1_S = x$ for all $x \in S$. A semigroup that does have an identity element is called a monoid. Similarly, a zero element of a semigroup S is an element $0_S \in S$ such that $0_S x = x 0_S = 0_S$ for all $x \in S$. In this study, semigroups containing zero were studied. An element x

of a semigroup S is called an inverse element if there exists at least one $y \in S$such that $xyx = x$ and $yxy = y$. If there is an inverse element with uniqueness for each element of the S semigroup, the S semigroup is called the inverse semigroup. An element x of a semigroup S with identity element is called a unit element if there exists $y \in S$ such that $xy = yx = 1_S$. We donate y by x^{-1}. If $xy = yx$ is satisfied for all $x, y \in S$, then the S semigroup is called the commutative semigroup. A semigroup S is a cancellative semigroup such that $xz = yz$ $(zx = zy)$ implies $x = y$ for all $x, y, z \in S$.

Now, let us define the semiprime semigroup that forms the basis of this paper. According to [1] and [6], the ideal I is called a semiprime ideal if $xSx \in I$ with $x \in S$ implies $x \in I$. A semigroup S is called semiprime if the zero ideal is a semiprime ideal of S. Thus, the equivalent definition can be given as follows: if $xSx = 0$ with $x \in S$ implies $x = 0$, then S is called semiprime semigroup.

3. Results of the Set of the Source of Semiprimeness

For similar results obtained for the ring and semigroup structures from the results given in this section, see [1] and [2].

Definition 1: [2] The definition of the source of semipirimeness of the A in S is made as follows. In this definition, A is a nonempty subset of S. As can be seen, this set is a subset of S.

$$S_S(A) = \{a \in S \mid aAa = 0\}$$

We will use the notation S_S instead of $S_S(S)$ for semigroup S. In this case, the definition given above is written as follows.

$$S_S = \{a \in S \mid aSa = 0\}$$

First, let us investigate the special cases of the semigroup S and the source of semiprimeness.

1. Let be $S_S = 0$. In this case, $aSa = 0$ implies $a = 0$ for $a \in S$. Thus, S is semiprime semigroup. Conversely, let S be semiprime semigroup and $\in S_S$. By using the definition of the set S_S, we get $aSa = 0$. Since S is a semiprime, we get $a = 0$. This means that $S_S = \{0\}$.

2. Let A be a subsemigroup of S and $S_S(A) = \{0\}$. In this case, $aAa = 0$ implies $a = 0$ for $a \in S$. Then especially, $aAa = 0$ implies $a = 0$ also for $a \in A$.Hence we have A is semiprime subsemigroup.

3. It is clear that $S_A \subseteq S_S(A)$ for any subsemigroup A of S.

4. Let A be a subsemigroup of S and $a, b \in S_A$. Then, $a, b \in A$ and $aAa = bAb = 0$. Since A is a subsemigroup, we obtain $bx \in A$ for any $x \in A$. Thus, we get $(ab)\,x\,(ab) = (a\,(bx)\,a)\,b = 0b = 0$. This means that $ab \in S_A$.

Using the above results, we give the following proposition.

Proposition 1: *For a semigroup S and its subsemigroup A, the following holds true:*

1. $S_S = \{0\}$ *if and only if S is semiprime.*

2. $S_S(A) = \{0\}$ *implies A is semiprime.*

3. S_A *is a subsemigroup of S. Specially,* S_S *is a subsemigroup of S.*

Now, let us examine some different properties of the source of semiprimeness for subsets and subsemigroups of semigroup S.

Proposition 2: *For a semigroup S and subsets A, B, of S, the following holds true:*

1. *If A is subsemigroup, then* $S_A = A \cap S_S(A)$.

2. *If A, B, and AB are subsemigroups and* $A \subseteq B$*, then* $S_A.S_B \subseteq S_{AB}$.

3. *If $A \subseteq B$, then $S_S(B) \subseteq S_S(A)$. Specially, $S_S \subseteq S_S(A)$ is provided.*

Proof 1. Let $a \in A \cap S_S(A)$. Then, $a \in A$ and $a \in S_S(A)$. Hence, we get $a \in A$ and $aAa = 0$. So, we obtain $a \in S_A$ and $A \cap S_S(A) \subseteq SA$. Conversely, since $SA \subseteq S_S(A)$ and $S_A \subseteq A$, the expression $S_A \subseteq A \cap S_S(A)$is provided. Thus, we get $S_A = A \cap S_S(A)$.

2. If $b \in S_A.S_B$, then $aAa = 0$ and $bBb = 0$ for $a \in A, b \in B$. Since $A \subseteq B$ and B is closed under multiplication, we get $ABa \subseteq B$. By using this expression, we obtain $ab\ (AB)\ ab = ab\ (ABa)\ b \subseteq a\ (bBb) = \{0\}$. This means that $b \in S_{AB}$. So, we get $S_A.S_B \subseteq S_{AB}$.

3. It is easily proved by the same method in [3] Proposition 2.2. □

At this point, we want to mention semigroups with inverse elements and semigroups with unit elements. These semigroups form the basis of the structures that we will construct in the next section. Now, let us give some simple observations about the source of the semiprimeness of these semigroups.

1. Let S be an inverse semigroup and $a \in S_S$. Since a is an inverse element, there exists $b \in S$ such that $aba = a$ and $bab = b$. Also, since $a \in S_S$, $aba = 0$ is provided for $b \in S$. This means that $a = 0$. So, we get $S_S = 0$.

2. Let S be a monoid with each element other than zero as a unit element. If $0 \neq a \in S_S$, then $axa = 0$ for all $x \in S$. From this, $aa^{-1}a = 0$ is satisfied for $a^{-1} \in S$. This result leads us to the contradiction $a = 0$. So, we get $S_S = \{0\}$.

From the above results, it is easy to see that proof of Corollary 3. This result is the basis for the new structures in the next section.

Corollary 3: *If S be a semigroup, then there is no inverse element or unit element in S_S.*

4. Results of $|S_S|$ −Inverse Semigroups, $|S_S|$ −Inverse Monoids and $|S_S|$ −Groups

Definition 2: Let S be semigroup with zero such that $S \neq S_S$.

1. S is called $|S_S|$ −inverse semigroup if every element of $S - S_S$ is inverse with uniqueness in S. If $1_S \in S$, then S is called $|S_S|$ −inverse monoid.

2. S is called $|S_S|$ −group if $1_S \in S$ and every element of $S - S_S$ is unit.

These definitions are defined by using the definitions in classical semigroup and group theory. They coincide in some places with inverse semigroup, inverse monoid and group structures, as well as generalization of these structures in terms of some properties. Before proceeding to the examples, we will give some observations about the different properties of these newly identified semigroups.

1. If S = {0}, then $S_S = \{0\} = S$. In this situation, the definitions made have no meaning for zero semigroups for $S - S_S = \emptyset$.

   In case $S = S_S$, $S - S_S = \emptyset$ feature is provided, similar to the feature above. In this situation, too, the definitions made have no meaning.

2. We note that any inverse semigroup (monoid) S is an inverse semigroup (monoid).

3. "if a is a unit element, then a is an inverse element" condition is always satisfied for the elements of a semigroup. Using this condition, the following feature can be easily seen.

   If S is a $|S_S|$ −group, then S is a $|S_S|$ −inverse monoid.

Example 1: Let the first table below be the operation table of the semigroup S.

From the definitions

$$S_S = \{0\}$$

and thus

$$S - S_S = \{a, b\}$$

If the elements of $S - S_S$ are examined, since $aba = a$ and $bab = b$, it is seen that a and b are inverse elements. Then, S is a $|S_S|$ −inverse semigroup. If unit element $1_S \in S$ to this semigroup, then the second table is the operation table of the semigroup S.

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	0	a
b	0	b	a	1

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	b	a
b	0	a	b	b
1	0	a	b	1

In this case, since $1_S \in S$, S is a $|S_S|$ −inverse monoid but not a $|S_S|$ −group because a and b are not a unit element.

Example 2: Let the table below be the operation table of the semigroup S.

From the definitions,

$$S_S = \{0, a\}$$

and thus

$$S - S_S = \{1, b\}$$

If the elements of $S - S_S$ are examined, since $1 \cdot 1 = 1$ ve $b \cdot b = 1$, it is seen that a and b are unit elements. So, S is a $|S_S|$ − group. Thus, it is also $|S_S|$ −inverse monoid.

Example 3: Consider the semigroup $(\mathbb{Q},\cdot)$. Since $S_{\mathbb{Q}} = \{0\}$, every element in $\mathbb{Q} - S_{\mathbb{Q}}$ are unit. So, $\mathbb{Q}$ is $\left|S_{\mathbb{Q}}\right|$ −group. It is also

$\cdot$	0	a	b
0	0	0	0
a	0	a	b
b	0	a	b

$\left|S_{\mathbb{Q}}\right|$ −inverse monoid.

Now, let us take semigroup $(\mathbb{Z}_n, \cdot)$and examine the properties of the source of semiprimeness. We will start with two examples before giving generalizations on $\left|S_{\mathbb{Z}_n}\right|$ −group. In the first example, n is the square of a prime number. In the second example, n is a prime number. In both cases, the given semigroup $\mathbb{Z}_n$ is a $\left|S_{\mathbb{Z}_n}\right|$ −group. Next, we will give a generalization of these two examples.

Example 4: Let $(\mathbb{Z}_9 ,\cdot)$ be semigroup of integers modulo 9. It is easy to see that

$$S_{\mathbb{Z}_9} = \{\bar{0}, \bar{3}, \bar{6}\}$$

and thus

$$\mathbb{Z}_9 - S_{\mathbb{Z}_9} = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}.$$

Since $1 \cdot 1 = 1, 2 \cdot 5 = 1, 4 \cdot 7 = 1, 5 \cdot 2 = 1, 7 \cdot 4 = 1, 8 \cdot 8 = 1$, every element in $\mathbb{Z}_9 - S_{\mathbb{Z}_9}$ are unit. So, $\mathbb{Z}_9$ is $|S_{\mathbb{Z}_9}|-$group. It is also $|S_{\mathbb{Z}_9}|-$inverse monoid.

Example 5: Let $(\mathbb{Z}_7, \cdot)$ be semigroup of integers modulo 7. Since

$$S_{\mathbb{Z}_7} = \{\bar{0}\}$$

we get

$$\mathbb{Z}_7 - S_{\mathbb{Z}_7} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}.$$

If the elements of $\mathbb{Z}_7 - S_{\mathbb{Z}_7}$ are examined, since $1 \cdot 1 = 1, 2 \cdot 4 = 1, 3 \cdot 5 = 1, 4 \cdot 2 = 1, 5 \cdot 3 = 1, 6 \cdot 6 = 1$, every element in $\mathbb{Z}_7 - S_{\mathbb{Z}_7}$ are unit. So, $\mathbb{Z}_7$ is $|S_{\mathbb{Z}_7}|-$group. It is also $|S_{\mathbb{Z}_7}|-$inverse monoid.

Lemma 4: Let $(\mathbb{Z}_n, \cdot)$ be semigroup of integers modulo n. If one of the following conditions is provided, then $S_{\mathbb{Z}_n} = \{\bar{0}\}$

1. n is a prime p.
2. n is written as the product of different primes $p, q, \dots, r$.

Proof 1: Let $n = p$ be a prime number. We recall that if $gcd\,(p, a) = 1$, then the element $\bar{a}$ of $\mathbb{Z}_n$ is a unit element. Since p is prime, all $\bar{0} \neq \bar{a} \in \mathbb{Z}_n$ is a unit element. Using Corollary 3, we get $S_{\mathbb{Z}_n} = \{\bar{0}\}$.

2. Let $n = p.q\dots r$ for different primes $p, q, \dots, r$. For an arbitrary element $\bar{a}$ of $S_{\mathbb{Z}_n}$, we write $\overline{a\,x\,a} = \bar{0}$ for all $\bar{x} \in \mathbb{Z}_n$. Also, this equation is provided for $q, \dots, r$. So, we get $\overline{aq\dots ra} = \bar{0}$. From this equation, we write $p.q\dots r \mid a\,(q\dots r)\,a$ and thus $p \mid a^2$. Since p is prime, we obtain $p \mid a$.

Similarly, for each prime number in the product, we get $p \mid a\,, q \mid a, \dots, r \mid a$. Then $p.q\dots r \mid a$, and so $a = (p.q\dots r)\,k = nk$ for $k \in \mathbb{Z}$. Using this equation, we have $\bar{a} = \overline{nk} = \bar{0}$. So, $S_{\mathbb{Z}_n} = \{\bar{0}\}$ is provided.

Theorem 5: Let $(\mathbb{Z}_n, \cdot)$ be semigroup of integers modulo n. "n is either a prime p or $p^2 \Leftrightarrow \mathbb{Z}_n$ is a $|S_{\mathbb{Z}_n}|-$group" is satisfied.

Proof: Let $n = p$ be a prime. From Lemma 4, we get $S_{\mathbb{Z}_n} = \{\bar{0}\}$ Since every element is a unit element. Hence $\mathbb{Z}_n$ is a $|S_{\mathbb{Z}_n}|-$group.

Now, let $n = p^2$ for prime p. For an arbitrary element $\bar{a}$ of $S_{\mathbb{Z}_n}$, we write $\overline{a\,x\,a} = \bar{0}$ for all $\bar{x} \in \mathbb{Z}_n$. Specially, $\overline{a\,p\,a} = \bar{0}$ is provided, too. From this equation, we write $p^2 \mid apa$. This means that $p \mid a^2$ and since p is prime, we get $p \mid a$. Then, $a = pk$ for $k \in \mathbb{Z}$. So, we obtain

$$\mathbb{Z}_n - S_{\mathbb{Z}_n} = \{\bar{0}, \bar{p}, \overline{2p}, \overline{3p}, \dots, \overline{(p-1)\,p}\}$$

In this case, for every $\bar{x}$ element of $\mathbb{Z}_n - S_{\mathbb{Z}_n}$, $gcd\,(p, x) = 1$ is provided, and thus these elements are unit elements. So, $\mathbb{Z}_n$ is a $|S_{\mathbb{Z}_n}|-$group.

Conversely, let $\mathbb{Z}_n$ is a $|S_{\mathbb{Z}_n}|-$group. Then, every element of $\mathbb{Z}_n - S_{\mathbb{Z}_n}$ is a unit element. We assume that p is a prime number and $n = pk$ for some integer $1 \leq k < n$. Since $gcd\ (p,n) \neq 1$, $\bar{p}$ is a non-unit element. Hence $\bar{p}$ must be in $S_{\mathbb{Z}_n}$, and so, $\overline{p\ x\ p} = \overline{0}$ for all $\bar{x} \in \mathbb{Z}_n$. Using this equation and the properties of the semigroup $\mathbb{Z}_n$, we get $n \mid p^2x$. Since $n = pk$, we write $pk \mid p^2x$. This equation gives us $k \mid px$. Specially, we write $k \mid p$ for $x = 1$. Since p is prime, we obtain $k = 1$ or $k = p$. So, we get $n = p$ or $n = p^2$.

Now, we will give properties of $|S_S|-$inverse monoid and $|S_S|-$group. Also, we will investigate the relations between these two constructions. Let us start with the below lemma, which gives us some properties of the source of semiprimeness of semigroups. Next, we will give a theorem showing when a $|S_S|-$inverse monoid is a $|S_S|-$group.

Lemma 6 Let S be a $|S_S|-$group. The following holds true:

1. $S_S = \{a \in S \mid a^2 = 0\}$.

2. If A is a nonzero subset of S, then $S_S(A) = S_S$.

Proof 1: Let $B = \{a \in S \mid a^2 = 0\}$. If $\in S_S$, then $aSa = 0$. Using this equation, we write $a1_Sa = a^2 = 0$ for $1_S \in S$. This means that $a \in B$. So, we get $S_S \subseteq B$.

On the other hand, if $b \in B$, then $b^2 = 0$. We suppose that $\notin S_S$. Thus, since $\in S - S_S$, b is a unit element. Right multiplication of equation $b^2 = 0$ by b^{-1} , we obtain $b = 0$. But this result contradicts $\notin S_S$. This means that $\in S_S$. So, we get $B \subseteq S_S$ and $B = S_S$.

2. For the set $S_S\ (A) = \{a \in S \mid aAa = 0\}$, the inclusion $S_S \subseteq S_S\ A)$ follows from Proposition 2. Let us take $t \in S_S\ (A)$ and we suppose $\in S - S_S$. In this case, t is a unit element. Using the equation $tAt = 0$, we lead to the contradiction $A = \{0\}$. This means that $t \in S_S$ and $S_S(A) \subseteq S_S$. So, we get $S_S(A) = S_S$.

Theorem 7: Let S be a semigroup. The following holds true:

1. If S is commutative $|S_S|-$ inverse monoid, then S is a $|S_S|-$group.
2. If S is cancellative $|S_S|-$inverse monoid, then S is a $|S_S|-$group.
3. S is a $|S_S|-$group if and only if the set $S - S_S$ is a group.

Proof 1: Let S be commutative $|S_S|-$inverse monoid. Then, the semigroup S has an identity element 1_S. If $a \in S - S_S$, since a is inverse, then there exists $b \in S$ such that $aba = a$ and $bab = b$. Using these equations and S is commutative, we write $(ab)\ a = a$ and $a\ (ab) = a$, $(ba)\ b = b$ and $b\ (ba) = b$. So, since $ab = ba = 1_S$, a is a unit element. This gives us that S is a $|S_S|-$group.

2. Let S be cancellative $|S_S|-$inverse monoid. Then, the semigroup S has an identity element 1_S. If $\in S - S_S$, since a is inverse, then there exists $b \in S$ such that $aba = a$ and $bab = b$. From these equations, we write $aba = a = 1_Sa$ and using cancellative property, we get $b = 1_S$.

Similarly, it is easy to see that $a = 1_S$. So, since $ab = ba = 1_S$, a is a unit element. This gives us that S is a $|S_S|-$group.

3. Let S be a $|S_S|-$group and $a, x \in S - S_S$. Since a and x are unit elements, ax is also a unit element. Hence $ax \in S - S_S$. Also, element 1_S is in the $S - S_S$.

On the other hand, since a^{-1} is also a unit element, a^{-1} is in the $S - S_S$ for all $a \in S - S_S$. So, the set $S - S_S$ is a group. Conversely, if $S - S_S$ is a group since every element in $S - S_S$ are unit and $1_S \in S - S_S \subseteq S$, then S is a $|S_S|-$group.

5. Conclusion

In this study, previously unexamined features of the source of the semiprimeness were examined, and previous studies were looked at from a different perspective. By making use of group, monoid and semigroup structures, new algebraic structures were obtained with the help of the source of the semiprimeness. $|S_S|-$inverse semigroup, $|S_S|-$inverse monoid and $|S_S|-$ group structures were defined, and their properties were examined using the unit element and inverse element, which are very important in semigroup theory. Additionally, what kind of generalization is made is mentioned and new algebraic structures are explained with examples. With the help of what has been found, cluster $|S_S|-$ subgroup structure can be defined, and its properties can be examined as a continuation of the studies carried out.

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