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Original Article

A Short Note of the Relationship between Loose Tangles and Filters

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Abstract - *Tangle, a concept related to graph width parameters, has been defined and studied in graph theory. It has a dual relationship with branch width. Loose Tangle relaxes the axioms of Tangle and also holds significance in research. Filters, defined based on symmetric submodular functions, are related to Tangles. This short paper aims to establish the cryptomorphism between Loose Tangle and filter.*

Keywords - *Tangle, Loose tangle, Filter.*

1. Introduction

Graph parameters encompass numerical values that serve to depict the characteristics inherent in a graph. These width parameters find diverse applications across an array of disciplines, including but not limited to matroid theory, lattice theory, theoretical computer science, game theory, network theory, artificial intelligence, graph minor theory, graph combinatorics, and various domains within discrete mathematics [19-22]. The notion of Tangle, as introduced in reference [1], exhibits a profound correlation with width parameters, particularly within the realm of graph theory. Reference [1] illuminates the duality existing between Tangle and branch width, a prominent width parameter. Likewise, reference [2] elucidates the concept of Linear Tangle, which establishes a dual relationship with linear branch width, akin to a caterpillar-like variant of branch width. Width parameters such as branch width and linear branch width have left an indelible imprint on the landscape of graph theory research, thereby instigating active investigations into Tangle and its interconnectedness with width parameters (e.g., [3-6, 11-18, 23-26, 28]). Consequently, the exploration of Tangle assumes a position of remarkable significance. Loose Tangle, introduced in reference [7], relaxes the axioms of Tangle. Reference [7] demonstrates the exclusive existence of either an order $k+1$ Loose Tangle or an order k branch decomposition tree on (X, f). Similar to Tangle, Loose Tangle maintains a deep connection with width parameters, leading to the publication of several related articles (e.g., [8, 9]). Tangle has played a crucial role in algorithms across various fields, such as graph minors, width parameters, and graph isomorphism. Filters are well-established concepts in topology and algebra. Recently, a filter on (X, f) has been defined, incorporating the conditions of symmetric submodular functions within Boolean algebras (cf. [10, 27, 28]). This short paper directly establishes the cryptomorphism between order $k+1$ Loose Tangle and order $k+1$ filters.

2. Definitions and Notations in this paper

This section provides mathematical definitions of each concept.

2.1. Symmetric Submodular Function and Connectivity system

Below is the definition of a symmetric submodular function. It is worth noting that while symmetric submodular functions can typically have real values, this paper specifically concentrates on the subset of functions that exclusively deal with natural numbers.

Definition 1: Let X be a finite set. A function $f: X \rightarrow \mathbb{N}$ is called symmetric submodular if it satisfies the following conditions:
1. For all $A \subseteq X$, $f(A) = f(X \backslash A)$.
2. For all $A, B \subseteq X$, $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

Hereafter, in this manuscript, we denote the symmetric submodular function as f, a (finite) domain set as X, and a natural number as k. Additionally, for a (finite) domain set X and a subset $A \subseteq X$, $A = X \backslash A$ is expressed. And in this short paper, a pair (X, f) of a finite set X and a symmetric submodular function f is called a connectivity system.

2.2. Filter on a Connectivity System (X, f)

Filters are fundamental mathematical concepts widely recognized in various fields such as topology and algebra. By introducing constraints based on symmetric submodular functions, one can derive unforeseen outcomes, including dual theorems involving graph parameters.
In the present chapter, we expand the notion of filters in algebra by incorporating symmetric submodular functions. Filters on *(X, f)*, where f represents a symmetric submodular function, are defined as follows. In contrast to filters in algebra, filters on *(X, f)* primarily encompass the constraints of symmetric submodularity.

Definition 2 (cf. [10, 31, 32]): Let X represent a finite set and f denote a symmetric submodular function. $F \subseteq 2^X$ is called an (non-principal) filter of order $k + 1$ on a connectivity system (X, f) if it satisfies the following five axioms:
(F0) $\forall A \in F, f(A) \leq k$.
(F1) For $A, B \in F$, if $f(A \cap B) \leq k$, then $A \cap B \in F$.
(F2) For $A \in F$, if $A \subseteq B \subseteq X$ and $f(B) \leq k$, then $B \in F$.
(F3) $\emptyset \notin F$.
(F4) $\forall e \in X$, if $f(\{e\}) \leq k$, then $\{e\} \notin F$.

2.3. Loose Tangle on a connectivity system (X, f)

The definition of Loose Tangle is presented as follows. Loose Tangle is known to exhibit a dual relationship with branch width.

Definition 3 [7]: Let X represent a finite set and f denote a symmetric submodular function. $L \subseteq 2^X$ is called a loose tangle of order k + 1 on (X, f) if it satisfies the following conditions:
(L1) $\forall e \in X$, if $f(\{e\}) \leq k$, then $\{e\} \in T$.
(L2) For $A, B \in T$, $C \subseteq A \cup B$, and if $f(C) \leq k$, then $C \in T$.
(L3) $X \notin T$.

3. Cryptomorphism Between Loose Tangle and Filter

In this section, we establish the cryptomorphism between Loose Tangle and Filter. We show that if a (co-)Loose tangle exists on *(X, f)* of order *k+1*, then there exists a free ultrafilter on *(X, f)* of order *k+1*.

Lemma 1: Let X represent a finite set and f denote a symmetric submodular function. Let F be an order $k + 1$ (non-principal) filter on (X, f). If we define $T = \{A \mid X\backslash A \in F\}$, then T is an order $k + 1$ loose tangle on (X, f).

Proof: Firstly, from the definition of T and Axiom (F4), Axiom (L1) is clearly satisfied. Furthermore, due to Axiom (F3), Axiom (L3) is also evidently true. Secondly, we need to prove axiom (L2). Let's consider $A, B \in T$ and $C \subseteq A \cup B$ such that $f(C) \leq k$. Since A and B belong to T, we have $X\backslash A, X\backslash B \in F$. Applying Axiom (F0), it follows that $f(X\backslash A) \leq k$ and $f(X\backslash B) \leq k$. Now, let's define W as the set contained in $X\backslash A \cap X\backslash B$ such that W is a subset of C, and $f(W)$ is minimized. It follows from the minimality of W, for any subset Y containing $X\backslash A \cap X\backslash B$, that $f(W) \leq f(Y)$ and $f(W) \leq f(C) \leq k$. Therefore, we can infer that $f(Y \cup W) \leq f(Y)$ using the symmetric submodular property. Next, for $Y = X\backslash A$, we have $f((X\backslash A) \cup W) = f(X\backslash A) \leq k$. Therefore, by applying Axiom (F2), it follows that $X\backslash A \cup W \in F$. Similarly, for $Y = X\backslash B$, we have $X\backslash B \cup W \in F$. Due to $f(W) \leq k$ and Axiom (F1), we also have $W \in F$.

As W is a subset of C and $f(C) \leq k$, $C \in F$ by Axiom (F2). Since $X\backslash C \in T$ by definition, we conclude that $C \in T$, which proves Axiom (L2). Thus, all three axioms of a loose tangle are satisfied, so T is an order $k + 1$ loose tangle on (X, f). This proof is completed.

Lemma 2: Let X represent a finite set and f denote a symmetric submodular function. Let T be an order $k + 1$ loose tangle on (X, f). If we define $F = \{A \mid X\backslash A \in T\}$, then F is an order $k + 1$ (non-principal) filter on (X, f).

Proof: Axiom (F0) is apparent because if $A \in F$, it means $X\backslash A \in T$, and by Axiom (L2) in Definition 3, $f(X\backslash A) \leq k$, hence $f(A) \leq k$. Axiom (F3) follows from Axiom (L3). $X \in T$ would contradict Axiom (L3), so $X \notin F$. Axiom (F4) is evident from Axiom (L1). If for any $e \in X, f(\{e\}) \leq k$, then $\{e\} \in T$ which means $X\backslash\{e\} \in F$. Next, we prove Axiom (F1). Assume $A, B \in F$ and $f(A \cap B) \leq k$. Then, $A, B \in T$ must hold, and we have $f(X\backslash A) = f(A) \leq k$ and $f(X\backslash B) = f(B) \leq k$. If $f(A \cap B) \leq k$, then by applying the symmetric submodularity, $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ leading to $f(A \cup B) \leq k$.

Applying Axiom (L2), we find that $A \cup B \in T$, which implies $X \backslash (A \cup B) \in F$, thus $A \cap B \in F$. Finally, let's prove Axiom (F2). Assume $A \in F$, $A \subseteq B$, and $f(B) \leq k$. From $A \in F$, we have $X \backslash A \in T$. Given that $A \subseteq B$, we have $X \backslash B \subseteq X \backslash A$.

From the symmetric submodularity, $f(X \backslash B) \leq f(X \backslash A) = f(A) \leq k$. Therefore, by applying Axiom (L2), we conclude that $X \backslash B \in T$, which means $B \in F$. Hence, all five axioms of a filter are satisfied, and F is an order $k + 1$ filter on (X, f). This proof is completed.

From Lemma 1 and Lemma 2, we can establish the following Theorem 3.

Theorem 3: Let X represent a finite set and f denote a symmetric submodular function. In (X, f), the existence of an order $k + 1$ filter is both necessary and sufficient for the existence of an order $k + 1$ loose tangle.

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