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Original Article

Orthogonal (σ, τ)-Derivations on Semiprime Γ-Semirings

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Abstract - *Assume S is a semiprime Γ-semiring and d: S→S is an additive mapping that obeys $d(u\alpha v) = d(u)\alpha\sigma(v) + \tau(u)\alpha d(v)$ for all $u, v \in S, \alpha \in \Gamma$, then d is termed as (σ, τ)- derivation on S. This paper introduces orthogonal (σ, τ)-derivations within semiprime Γ-semirings and provides several characterizations of these semirings. It also establishes the criteria under which two (σ, τ)-derivations can be deemed orthogonal.*

Keywords - *$(\boldsymbol{\sigma}, \boldsymbol{\tau})$-derivation, Orthogonal $(\boldsymbol{\sigma}, \boldsymbol{\tau})$-derivation, **Γ**–semiring.*

1. Introduction

The idea of a semiring was first proposed by H.S. Vandiver [8]. The concept of a ***Γ***-ring was introduced by Nobusawa [14] as a generalization of a ring. Sen [10] introduced the notion of a Γ-semigroup. Murali Krishna Rao [12,13] further developed this by introducing the concept of Γ-semirings. S.Huang et al. [16] examined orthogonal generalized (σ, τ)-derivations in semiprime near-rings. K.K.Dey et al. [9] investigated orthogonal generalized derivations in semiprime ***Γ***-near-rings. M.A. Javed et al. [11] presented the idea of derivations in prime Γ-semirings. N. Suganthameena et al. [15] introduced the concept of orthogonal derivations on semirings. B. Venkateswarlu et al. [2,3] established some necessary and sufficient conditions for orthogonal derivations and reverse derivations in semiprime ***Γ***-semirings. A.H. Majeed [1] explored orthogonal generalized derivations in semiprime ***Γ***-semirings. Recently, C. Jaya Subba Reddy et al. [4,5,6,7] proved some results on the orthogonality of generalized symmetric reverse bi-(σ, τ)-derivations in semiprime rings and orthogonality of generalized reverse -(σ,τ)-derivations in semiprime Γ-rings.

In this paper, we propose the concept of (σ, τ)-derivations for semiprime **Γ**-semirings. We also establish the necessary and sufficient conditions for the orthogonality of two such derivations in semiprime **Γ**-semirings, expanding upon the results found in [2].

2. Preliminaries

A set S is called a semiring if it is equipped with two associative binary operations denoted by '+' (addition) and '.' (multiplication) and satisfies the following conditions:
(i) The addition operation is commutative.
(ii) The multiplication operation is distributive over addition from both the left and the right.
(iii) There exists an element 0 in S such that , $u + 0 = u$ and $u \cdot 0 = 0. u = 0, \forall u \in S$.

If $(S, +)$ and $(\Gamma, +)$ are two abelian semigroups with identity elements 0 and θ of S and Γ respectively, and there exists a mapping of $S \times \Gamma \times S \to S$ satisfying the following properties for $u, v, w \in S$ and $\alpha, \beta \in \Gamma$

1. $(u + v)\alpha w = u\alpha w + v\alpha w$.
2. $u(\alpha + \beta)v = u\alpha v + u\beta v$.
3. $u\alpha(v + w) = u\alpha v + u\alpha w$.
4. $(u\alpha v)\beta w = u\alpha(v\beta w)$.
5. $u\alpha 0 = 0\alpha u = 0$ and $u\theta v = 0$.

then S is termed a Γ-semiring.

Let S be a Γ- semi-ring. S is said to be a semiprime if $u\Gamma S\Gamma u = 0$ implies $u = 0, \forall u \in S$. A Γ-semiring S is said to be 2 torsion free if $2u = 0$,then $u = 0, \forall u \in S$. A Γ-semiring S is said to be commutative if $u\alpha v = v\alpha u, \forall u, v \in S, \alpha \in \Gamma$.

A Γ-semiring S is said to have a zero element if there exists an element $0 \in S$ such that $0 + u = u = u + 0, \forall u \in S$. An additive mapping $d_1 : S \to S$ is called a derivation if $d_1(u\alpha v) = d_1(u)\alpha v + ud_1(v), \forall u, v \in S$. An additive mapping $d_1 : S \to S$ is called a (σ, τ)-derivation if $d_1(u\alpha v) = d_1(u)\alpha\sigma(v) + \tau(u)\alpha d_1(v), \forall u, v \in S, \alpha \in \Gamma$. Two (σ, τ)-derivations d_1 and d_2 on S are deemed orthogonal if $d_1(u)\Gamma S\Gamma d_2(v) = 0 = d_2(u)\Gamma S\Gamma d_1(v), \forall u, v \in S$.

This paper explores the concept of orthogonal (σ, τ)-derivations within semiprime Γ-semirings. We identify various characterizations of semiprime Γ-semirings using orthogonal (σ, τ)-derivations and provide the necessary and sufficient conditions for two (σ, τ)-derivations to be orthogonal.

In this paper, we work under the assumption that S is a 2-torsion-free semiprime Γ-semiring. Here, σ and τ are automorphisms of S, and d_1, d_2 are are (σ, τ)-derivations on S such that $d_1\tau = \tau d_1$, $d_2\tau = \tau d_2$, $\sigma d_1 = d_1\sigma$, $\sigma d_2 = d_2\sigma$.

Lemma 1:[Lemma 3.1, [2]] Let a and b be two elements of a 2 torsion –free semiprime Γ-semiring S. Then the following statements are equivalent:
(i)$a\Gamma u\Gamma b = 0$.
(ii) $b\Gamma u\Gamma a = 0$.
(iii) $a\Gamma u\Gamma b + b\Gamma u\Gamma a = 0, \forall u \in S$.
If one of these conditions is fulfilled then $a\Gamma b = b\Gamma a = 0$.

Lemma 2: [Lemma 3.2, [2]] Let S be a 2-torsion free semiprime Γ-semiring.If additive mappings d_1 and d_2 of S into itself satisfy $d_1(u)\Gamma S\Gamma d_2(u) = 0, \forall u \in S$, then $d_1(u)\Gamma S\Gamma d_2(v) = 0, \forall u, v \in S$.

3. Main Results

Theorem 1 : Given S is a 2-torsion-free semiprime Γ-semiring and d_1, d_2 are (σ, τ)-derivations on S, then d_1 and d_2 are orthogonal if and only if $d_1(u)\alpha d_2(v) + d_2(u)\alpha d_1(v) = 0, \forall u, v \in S, \alpha \in \Gamma$.
Proof: Suppose that $d_1(u)\alpha d_2(v) + d_2(u)\alpha d_1(v) = 0, \forall u, v \in S, \alpha \in \Gamma$. (3.1)
Replacing v by $v\beta u, \forall u \in S. \beta \in \Gamma$ in (3.1) and using (3.1) we obtain
$d_1(u)\alpha\tau(v)\beta d_2(u) + d_2(u)\alpha\tau(v)\beta d_1(u) = 0$.
Since τ is an automorphism on S and using Lemma 1
$d_1(u)\alpha\tau(v)\beta d_2(u) = 0 = d_2(u)\alpha\tau(v)\beta d_1(u)$.
Since τ is an automorphism on S and using Lemma 2,
$d_1(u)\alpha\tau(v)\beta d_2(v) = 0 = d_2(u)\alpha\tau(v)\beta d_1(v)$.
Using Lemma 1, we get $d_1(u)\alpha d_2(u) = 0 = d_2(u)\alpha d_1(v), \forall u, v \in S, \alpha \in \Gamma$.
Thus d_1 and d_2 are orthogonal.
Conversely, Suppose that d_1 and d_2 are orthogonal.
Then, $d_1(u)\Gamma S\Gamma d_2(v) = 0, \forall u, v \in S$.
By Lemma 1, we can write $d_1(u)\Gamma d_2(v) = 0 = d_2(u)\Gamma d_1(v), \forall u, v \in S$.
Hence proved.

Theorem 2: Suppose S is a 2-torsion-free semiprime Γ-semiring and d_1,d_2 be (σ, τ)derivations on S. Then the necessary and sufficient condition that d_1 and d_2 are orthogonal is $d_1 d_2 = 0$.
Proof: Suppose $d_1 d_2 = 0$.
Then $d_1 d_2(u\alpha v) = 0$
Using the fact that σ, τ are automorphism of S and $\tau d_2 = d_2\tau, \sigma d_2 = d_2\sigma, \tau d_1 = d_1\tau, \sigma d_1 = d_1\sigma$,
we obtain $d_2(u)\alpha d_1(v) + d_1(u)\alpha d_2(v) = 0, \forall u, v \in S, \alpha \in \Gamma$. (since $d_1 d_2 = 0$)
Therefore d_1 and d_2 are orthogonal. (By Theorem 1)
Conversely, Suppose that d_1 and d_2 are orthogonal.
Then $d_1(u)\Gamma S\Gamma d_2(v) = 0, \forall u, v \in S$.
$d_1(u)\alpha w\beta d_2(v) = 0, \forall u, v, w \in S, \alpha, \beta \in \Gamma$.
Then, we have $d_1(d_1(u)\alpha w\beta d_2(v)) = 0, \forall u, v, w \in S, \alpha, \beta \in \Gamma$.
$d_1(d_1(u))\alpha\sigma(w)\beta\sigma(d_2(v)) + \tau(d_1(u))\alpha(d_1(w)\beta\sigma(d_2(v)) + \tau(w)\beta d_1 d_2(v)) = 0$.
Since σ, τ are automorphism of S and using $\sigma d_2 = d_2\sigma, \tau d_1 = d_1\tau$, we get
$d_1(d_1(u))\alpha\sigma(w)\beta d_2(v) + d_1(u)\alpha d_1(w)\beta d_2(v) + d_1(u)\alpha\tau(w)\beta d_1 d_2(v) = 0$.
The first and second summands are zeros as d_1 and d_2 are orthogonal.
Hence, we get $d_1(u)\beta\tau(w)\alpha d_1 d_2(v) = 0$.

Replace u by $d_2(v)$ in the above equation, we get
$d_1d_2(v)\alpha\tau(w)\beta d_1d_2(v) = 0$.
Since τ is an automorphism of a semiprime Γ-semiring S, $d_1d_2(v)=0$. Hence the result .

Corollary 1: Given that S is a 2-torsion-free semiprime **Γ**-semiring, and d_1, d_2 are (σ,τ)-derivations on S, then the necessary and sufficient condition that d_1 and d_2 are orthogonal is $d_2d_1 = 0$.

Theorem 3: In a 2-torsion-free semiprime Γ-semiring **S**, let d_1 and d_2 be (σ,τ)-derivations of S into S. Then the necessary and sufficient condition that d_1 and d_2 are orthogonal is that $d_1d_2 + d_2d_1 = 0$.
Proof: Suppose that $d_1d_2 + d_2d_1 = 0$ (3.2)
$(d_1d_2 + d_2d_1)(u\alpha v) = 0$
$d_1\big(d_2(u)\big)\alpha\sigma^2(v) + \tau(d_2(u)\alpha d_1(\sigma(v)) + d_1(\tau(u))\alpha\sigma(d_2(v))\big) + \tau^2(u)\alpha d_1d_2(v) + +d_2(d_1(u)\alpha\sigma^2(v) + \tau\big(d_1(u)\big)\alpha d_2(\sigma(v))$
$+ d_2(\tau(u))\alpha\sigma(d_1(v)) + \tau^2(u)\alpha d_2d_1(v) = 0$
Since σ,τ are automorphisms of S , $\tau d_2 = d_2\tau$, $\sigma d_1 = d_1\sigma$, $\tau d_1 = d_1\tau$, $\sigma d_2 = d_2\sigma$, we get
$(d_1d_2 + d_2d_1)(u)\alpha\sigma(v) + \tau(u)\alpha(d_1d_2 + d_2d_1)(v)$+2$(d_2(u)\alpha d_1(v) + d_1(u)\alpha d_2(v)) = 0$.
Since S is 2 torsion free and using (3.2)
$d_2(u)\alpha d_1(v) + d_1(u)\alpha d_2(v)$ =0.
Hence, d_1 and d_2 are orthogonal. (By Theorem 1).
Conversely, suppose that d_1 and d_2 are orthogonal, then by Theorem 2 and corollary 3,
it is evident that $d_1d_2 + d_2d_1 = 0$.

Theorem 4: Given S is a 2-torsion-free semiprime Γ-semiring and d_1, d_2 are (σ,τ)-derivations on S , then the necessary and sufficient condition that d_1 and d_2 are orthogonal is d_1d_2 is a (σ,τ)-derivation.
Proof: Assume that d_1 and d_2 are orthogonal.
We have $d_1d_2 = 0$ (By Theorem 2) and can be written as
$d_1d_2(u\alpha v) = d_1d_2(u)\alpha\sigma(v) + \tau(u)\alpha d_1d_2(v) = 0$.
Therefore d_1d_2 is a (σ,τ)-derivation.
Conversely, Suppose that d_1d_2 is a (σ,τ)-derivation.
Now, $d_1d_2(u\alpha v) = d_1\big(d_2(u\alpha v)\big)$
Since σ,τ are automorphisms of M, $\tau d_2 = d_2\tau$, $\sigma d_1 = d_1\sigma$, $\tau d_1 = d_1\tau$, $\sigma d_2 = d_2\sigma$ we get
$d_1d_2(u\alpha v) = d_1d_2(u)\alpha\sigma(v) + d_2(u)\alpha d_1(v) + d_1(u)\alpha d_2(v) + \tau(u)\alpha d_1d_2(v)$. (3.3)
But $d_1d_2(u\alpha v) = d_1d_2(u)\alpha\sigma(v) + \tau(u)\alpha d_1d_2$ (v) as d_1d_2 is a (σ,τ)derivation. (3.4)
Comparing the equations (3.3) and (3.4), we get
$d_2(u)\alpha d_1(v) + d_1(u)\alpha d_2(v) = 0$.
By Theorem 1, we can conclude that d_1 and d_2 are orthogonal.

Corollary 2: Given S is a 2-torsion-free semiprime Γ-semiring and d_1, d_2 are (σ,τ)-derivations on S , then necessary and sufficient condition that d_1 and d_2 are orthogonal is that d_2d_1 is a (σ,τ)-derivation.

Corollary 3 : Assume that S is a semiprime Γ-semiring free from 2-torsion. Let $\boldsymbol{d_1},\boldsymbol{d_2}$ be (σ,τ)-derivations of on S . Then, the under mentioned conclusions are identical:
(i) d_1 and d_2 are orthogonal.
(ii) $d_1d_2 = 0$.
(iii) $d_2d_1 = 0$.
(iv) $d_1d_2 + d_2d_1 = 0$.
(v) d_1d_2 is a (σ,τ)-derivation.
(vi) d_2d_1 is a (σ,τ)-derivation.
The proof the above corollary is evident from the Theorems 2,3,4 and corollaries 1,2.

Corollary 4: Consider S as a 2-torsion-free semiprime **Γ**-semiring. If d_1 and d_2 are orthogonal (σ,τ)-derivations on **S**, it follows that either d_1 is zero or d_2 is zero.

Theorem 5: In the context of a 2-torsion-free semiprime **Γ**-semiring **S**, suppose d_1 is a (σ,τ)-derivations on S and d_1^2 is also a (σ,τ)-derivation, then d_1 is necessarily zero.
Proof: Suppose that d_1^2 is a (σ,τ)-derivation. Then, we can have

$d_1^2(u\alpha v) = d_1\big(d_1(\mathrm{u\alpha v})\big)$
$= d_1\big(d_1(\mathrm{u}))\alpha\sigma^2(\mathrm{v})\big) + \tau\big(d_1(\mathrm{u})\big)\alpha d_1\big(\sigma(\mathrm{v})\big) + d_1\big(\tau(\mathrm{u})\big)\alpha\sigma\big(d_1(\mathrm{v})\big) + \tau^2(\mathrm{u})\alpha d_1(d_1(\mathrm{v}))$.
Since σ, τ are automorphisms of S, $\sigma d_1 = d_1\sigma$, $\tau d_1 = d_1\tau$, we get
$= d_1^2(\mathrm{u})\,\alpha\sigma(\mathrm{v}) + \tau(\mathrm{u})\alpha d_1^2(\mathrm{v}) + d_1(\mathrm{u})\alpha d_1(\mathrm{v}) + d_1(\mathrm{u})\alpha d_1(\mathrm{v})$
$= d_1^2(\mathrm{u}\alpha v)) + d_1(\mathrm{u})\alpha d_1(\mathrm{v}) + d_1(\mathrm{u})\alpha d_1(\mathrm{v})$

Therefore , $d_1^2(u\alpha v) = d_1^2(\mathrm{u}\alpha v) + d_1(\mathrm{u})\alpha d_1(\mathrm{v}) + d_1(\mathrm{u})\alpha d_1(\mathrm{v})$.
Since S is 2 torsion free, we get $d_1(\mathrm{u})\alpha d_1(\mathrm{v}) = 0$. (3.5)

Replacing u by $u\beta w, \forall\, w \in \mathrm{S}, \beta \in \Gamma$ in (3.5) and using the same equation, we get
$\mathrm{d}_1(u)\beta\sigma(w)\alpha d_1(\mathrm{v}) = 0$. (3.6)

Replace v by $u + v$ in the equation (3.6) and using the same equation, we get
$\mathrm{d}_1(u)\beta\sigma(w)\alpha d_1(\mathrm{u}) = 0$.

Since σ is an automorphism on a semiprime Γ-semiring M, we get $\mathrm{d}_1 = 0$. Hence Proved.

Theorem 6: Assume S is a 2-torsion-free semiprime Γ-semiring and d_1, d_2 are (σ, τ)-derivations on S. Then, the necessary and sufficient condition that d_1 and d_2 are orthogonal is $d_1 d_2(\mathrm{u}) = \mathrm{s\alpha u} + \mathrm{u\alpha t}$, $\forall u \in \mathrm{S}, \alpha \in \Gamma$ for $s, t \in S$.
Proof: Suppose that $d_1 d_2(\mathrm{u}) = \mathrm{s\alpha u} + \mathrm{u\alpha t}, \forall\, u \in \mathrm{S}, \alpha \in \Gamma$. (3.7)
Replace u by $u\beta v$, $\forall\, v \in M$, $\beta \in \Gamma$ in (3.7), we get
$d_1 d_2(u\beta v) = \mathrm{s}\alpha(u\beta v) + (u\beta v)\alpha\mathrm{t}$, $\forall\, s, t \in M$, $\alpha, \beta \in \Gamma$
$d_1(d_2(u)\beta\sigma(v) + \tau(\mathrm{u})\beta \mathrm{d}_2(\mathrm{v})) = \mathrm{s}\alpha(u\beta v) + (u\beta v)\alpha\mathrm{t}$
$d_1(d_2(u))\beta\sigma^2(v) + \tau(d_2(\mathrm{u})\beta \mathrm{d}_1\big(\sigma(v)\big) + \mathrm{d}_1\big(\tau(\mathrm{u})\big)\beta\sigma\big(\mathrm{d}_2(\mathrm{v})\big) + \tau^2(\mathrm{u})\beta \mathrm{d}_1(\mathrm{d}_2(\mathrm{v})) = \mathrm{s}\alpha(u\beta v) + (u\beta v)\alpha\mathrm{t}$.
Since σ, τ are automorphisms of S, $d_1\sigma = \sigma\, d_1, \tau\, d_1 = d_1\tau$, $\tau d_2 = d_2\tau$, $\sigma d_2 = d_2\sigma$, we get
$d_1 d_2(\mathrm{u})\beta\mathrm{v} + d_2(\mathrm{u})\beta\mathrm{d}_1(\mathrm{v}) + \mathrm{d}_1(\mathrm{u})\beta d_2(\mathrm{v}) + \mathrm{u}\beta\mathrm{d}_1\mathrm{d}_2(\mathrm{v}) = \mathrm{s}\alpha(u\beta v) + (u\beta v)\alpha\mathrm{t}$.
Using (3.7), we get $s\alpha\mathrm{u}\beta v + \mathrm{u\alpha t}\,\beta v + d_2(\mathrm{u})\beta\mathrm{d}_1(\mathrm{v}) + \mathrm{d}_1(\mathrm{u})\beta d_2(\mathrm{v}) + \mathrm{u}\beta s\alpha\mathrm{v} + \mathrm{u\beta\, v\alpha t} = \mathrm{s}\alpha(u\beta v) + (u\beta v)\alpha\mathrm{t}$
$\mathrm{u\alpha t}\,\beta v + d_2(\mathrm{u})\beta\mathrm{d}_1(\mathrm{v}) + \mathrm{d}_1(\mathrm{u})\beta d_2(\mathrm{v}) + \mathrm{u}\beta s\alpha\mathrm{v} = 0$. (3.8)
Replacing v by $v\gamma u, \forall\, u \in \mathrm{S}, \gamma \in \Gamma$ in the equation (3.8), we get
$\mathrm{u\alpha t}\,\beta v\gamma u + d_2(\mathrm{u})\beta\mathrm{d}_1(v\gamma u) + \mathrm{d}_1(\mathrm{u})\beta d_2(v\gamma u) + \mathrm{u}\beta s\alpha v\gamma u = 0$
$\mathrm{u\alpha t}\,\beta v\gamma u + \mathrm{u}\beta s\alpha v\gamma u + d_2(\mathrm{u})\beta\big(\mathrm{d}_1(v)\gamma\sigma(u) + \tau(\mathrm{v})\gamma\mathrm{d}_1(\mathrm{u})\big) + \mathrm{d}_1(\mathrm{u})\beta(d_2(v)\gamma\sigma(u) + \tau(\mathrm{v})\gamma\mathrm{d}_2(\mathrm{u}) = 0$.
Since σ, τ are automorphisms of S and using the equation (3.8), we get
$(\mathrm{u\alpha t}\,\beta v + \mathrm{u}\beta s\alpha v + d_2(\mathrm{u})\beta\mathrm{d}_1(\mathrm{v}) + \mathrm{d}_1(\mathrm{u})\beta d_2(v))\gamma\mathrm{u} + d_2(\mathrm{u})\beta\tau(\mathrm{v})\gamma\mathrm{d}_1(\mathrm{u}) + \mathrm{d}_1(\mathrm{u})\beta\tau(\mathrm{v})\gamma\mathrm{d}_2(\mathrm{u}) = 0$
$d_2(\mathrm{u})\beta\tau(\mathrm{v})\gamma\mathrm{d}_1(\mathrm{u}) + +\mathrm{d}_1(\mathrm{u})\beta\tau(\mathrm{v})\gamma\mathrm{d}_2(\mathrm{u}) = 0, \forall\, u \in \mathrm{S}, \beta \in \Gamma$.
Since σ, τ are automorphisms of S and using Lemma 1, we get
$d_2(\mathrm{u})\beta\tau(\mathrm{v})\gamma\mathrm{d}_1(\mathrm{u}) = 0 = \mathrm{d}_1(\mathrm{u})\beta\tau(\mathrm{v})\gamma\mathrm{d}_2(\mathrm{u})$.
By Lemma 2, we can have $d_2(\mathrm{u})\beta\tau(\mathrm{v})\gamma\mathrm{d}_1(\mathrm{v}) = 0 = \mathrm{d}_1(\mathrm{u})\beta\tau(\mathrm{v})\gamma\mathrm{d}_2(\mathrm{v})$.
By Lemma 1, we can have $d_2(\mathrm{u})\beta\mathrm{d}_1(\mathrm{v}) = 0 = \mathrm{d}_1(\mathrm{u})\beta\, d_2(\mathrm{v})$ and so
$\mathrm{d}_1(\mathrm{u})\beta\, d_2(\mathrm{v}) + d_2(\mathrm{u})\beta\mathrm{d}_1(\mathrm{v}) = 0$ and so the conclusion is arrived.
Conversely, suppose that d_1 and d_2 are orthogonal, then $\mathrm{d}_1\mathrm{d}_2 = 0$. (By Theorem 2)
Then we can choose $s = 0, t = 0$, so that $\mathrm{d}_1\mathrm{d}_2(\mathrm{u}) = \mathrm{s\alpha u} + \mathrm{u\alpha t}$, $\forall\, u \in \mathrm{S}, \alpha \in \Gamma$.

Theorem 7: If S is a 2-torsion-free semiprime Γ-semiring and d_1,d_2 are (σ, τ)-derivations of S such that $d_1^2 = d_2^2$, then the subsequent assertions are true:
(i) $(d_1 + d_2)$ and $(d_1 - d_2)$ are orthogonal. (ii) either $d_1 = -d_2$ or $d_1 = d_2$.
Proof: Suppose $d_1^2 = d_2^2$
To Prove (i): Consider $[(d_1 + d_2)\,(d_1 - d_2) + (d_1 - d_2)(d_1 + d_2)](\mathrm{u})$
= $d_1^2(u) + d_2 d_1(u) - d_1 d_2(u) - d_2^2(u) + d_1^2(u) - d_2 d_1(u) + d_1 d_2(u) - d_2^2(u)$
$= 0$ (Since $d_1^2 = d_2^2$)
Therefore $[(d_1 + d_2)\,(d_1 - d_2) + (d_1 - d_2)(d_1 + d_2) = 0$.
Hence, by the Theorem 3, we can conclude that $(d_1 + d_2)$ and $(d_1 - d_2)$ are orthogonal.
To Prove (ii):
From the result (i), we have $(d_1 + d_2)$ and $(d_1 - d_2)$ are orthogonal.

Then by the corollary 4, we have $(d_1 + d_2) = 0$ or $(d_1 - d_2) = 0$
and hence $d_1 = -d_2$ or $d_1 = d_2$.
Hence Proved.

4. Conclusion

This paper mainly deals with the study of (σ,τ)- derivations within the framework of semiprime $\boldsymbol{\Gamma}$-semirings by introducing the concept of orthogonal (σ,τ)- derivations. During this work, we proved some necessary and sufficient conditions for the orthogonality of two (σ,τ)- derivations in semiprime $\boldsymbol{\Gamma}$-semirings which gives an advanced theoretical approach to this area of algebra. These results enhances the theory of (σ,τ)- derivations which have many applications in algebraic geometry

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