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Research Article

Integrals Involving H-Function of Several Complex Variables, Srivastava Polynomials and Exponential Functions

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Abstract *- In this paper, we investigate the integration of an H-function of several complex variables combined with general class polynomials and an exponential function expressed in product form. To achieve this, we utilized specific definite integrals from established mathematical resources. The H-function, known for its extensive applications in complex analysis, is integrated with general class polynomials, which provide a broad framework for various polynomial functions, and an exponential function, a fundamental component in mathematical analysis. The integration process follows rigorous mathematical methods, resulting in expressions that are concise and simplified. The derived integrals are significant as they can be applied to solve complex problems in mathematical, statistical, and physical sciences, where products of different functions frequently appear. By presenting the results in a compact form, we facilitate easier application and further research in these fields. The findings of this paper contribute to the existing body of knowledge and offer practical tools for researchers dealing with complex variable functions and their integrals. This integration technique has the potential to simplify and solve intricate problems, thereby advancing theoretical and applied mathematics.*

Keywords **-** *Exponential function, General Class Srivastava's Polynomials, Multivariable H-function.*

1. Introduction

The H-function of several complex variables was defined by H. M. Srivastava and R. Panda in a series of research papers [6]. The H-function is defined and represented in terms of a multiple Mellin-Bernes type contour integral as

$$H[z_1, z_2, \dots, z_r] = H^{M,N;m_1,n_1;\dots;m_r,n_r}_{P,Q;p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} \left(a_j; A_j^{(1)}, A_j^{(2)}, \dots A_j^{(r)}\right)_{1,P}; \ \left(c_j^{(1)}, C_j^{(1)}\right)_{1,p_1}; \dots; \left(c_j^{(r)}, C_j^{(r)}\right)_{1,p_r} \\ \left(b_j, B_j^{(1)}, B_j^{(2)}, \dots, B_j^{(r)}\right)_{1,Q}; \left(d_j^{(1)}, D_j^{(1)}\right)_{1,q_1}; \dots; \left(d_j^{(r)}, D_j^{(r)}\right)_{1,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^r} \int_{\Upsilon_1} \dots \int_{\Upsilon_r} \psi(s_1, \dots, s_r) \left\{\prod_{k=0}^{r} \phi_k(s_k) z_k^{s_k} ds_k\right\} \tag{1}$$

denote the H-function of r complex variables $z_1, z_2, \dots, z_r$. Where $i = (-1)^{\frac{1}{2}}$and

$$\psi(s_1, s_2, \dots, s_r) = \frac{\prod_{j=i}^{M} \Gamma\left(b_j - \sum_{k=1}^{r} B_j^{(k)} s_k\right) \prod_{j=1}^{N} \Gamma\left(1 - a_j + \sum_{k=1}^{r} A_j^{(k)} s_k\right)}{\prod_{j=M+1}^{Q} \Gamma\left(1 - b_j + \sum_{k=1}^{r} B_j^{(k)} s_k\right) \prod_{j=N+1}^{P} \Gamma\left(a_j - \sum_{k=1}^{r} A_j^{(k)} s_k\right)} \tag{2}$$

$$\phi_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma\left(d_j^{(k)} - D_j^{(k)} s_k\right) \prod_{j=1}^{n_k} \Gamma\left(1 - c_j^{(k)} + C_j^{(k)} s_k\right)}{\prod_{j=m_k+1}^{q_k} \Gamma\left(1 - d_j^{(k)} + D_j^{(k)} s_k\right) \prod_{j=n_k+1}^{p_k} \Gamma\left(c_j^{(k)} - C_j^{(k)} s_k\right)} \quad \text{where } (k = 1, 2, \dots, r) \tag{3}$$

$\begin{cases} a_j, [j = 1,2, \dots, P]; c_j^{(k)}, [j = 1,2, \dots, p_i; \forall k \in \{1,2, \dots, r\}] \\ b_j, [j = 1,2, \dots, Q]; d_j^{(k)}, [j = 1,2, \dots, q_i; \forall i \in \{1,2, \dots, r\}] \end{cases}$ are complex numbers, and their corresponding related coefficients

$\begin{cases} A_j^{(k)}, [j = 1,2,\dots,P]; C_j^{(k)}[j = 1,2,\dots,p_k; \forall k \in \{1,2,\dots,r\}] \\ B_j^{(k)}, [j = 1,2,\dots,Q]; D_j^{(k)}[j = 1,2,\dots,q_k; \forall k \in \{1,2,\dots,r\}] \end{cases}$ are positive real numbers.

And Υ_k, represent the contours start at the point $\vartheta_k - i\infty$and goes to the point $\vartheta_k + i\infty$ with $\vartheta_k \in R$, $(k = 1,2,\dots,r)$. The integral in (3) converges absolutely, under the conditions Srivastava et al. [5] if

$$|arg\, z_k| < \frac{\pi}{2}\tau_k,\ (k = 1,2,\dots,r) \tag{4}$$

$$\text{Where } \Delta_k \equiv \sum_{j=1}^{P} A_j^{(k)} + \sum_{j=1}^{p_k} C_j^{(k)} - \sum_{j=1}^{Q} B_j^{(k)} - \sum_{j=1}^{q_k} D_j^{(k)} \leq 0 \tag{5}$$

$$\tau_k = \sum_{j=1}^{N} A_j^{(k)} - \sum_{j=N+1}^{P} A_j^{(k)} + \sum_{j=1}^{M} B_j^{(k)} - \sum_{j=M+1}^{Q} B_j^{(k)} + \sum_{j=1}^{n_k} c_j^{(k)} - \sum_{j=n_k+1}^{p_k} C_j^{(k)} + \sum_{j=1}^{m_k} D_j^{(k)} - \sum_{j=m_k+1}^{q_k} D_j^{(k)} > 0,\ \forall (k = 1,2,\dots,r) \tag{6}$$

Where$M, N, P, Q, m_k, n_k, p_k, q_k$are positive integers and restricted by the $0 \leq N \leq P$, $Q \geq M \geq 0$, and $q_k \geq m \geq 0$, $p_k \geq n_k \geq 0, \forall k \in \{1,2,\dots,r\}$and inequalities (6) suitably constrained values of the complex variables $z_1, z_2, \dots, z_r$. The points $z_k = 0, k = 1,2,\dots,r$ and many exceptional parameter values being tacitly excluded. From Srivastava and Panda [6] we have

$$H[z_1, z_2, \dots, z_r] = o(|z_1|^{e_1} \dots |z_r|^{e_r}) \left(\lim_{1 \leq j \leq m_r} \|z_j\| \to 0 \right), \tag{7}$$

$$\text{where } e_k = \lim_{1 \leq j \leq m_r} Re\left(\frac{d_j^{(k)}}{D_j^{(k)}} \right) (k = 1,2,\dots,r) \tag{8}$$

2. General Class Srivastava's Polynomials

Srivastava (1985) defined the second class of multivariable polynomials as follows. $S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}[y_1, y_2, \dots, y_t] = \sum_{k_1=0}^{\frac{\beta_1}{\alpha_1}} \dots \sum_{k_t=0}^{\frac{\beta_t}{\beta_t}} \frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!} \dots \frac{(-\alpha_t)_{\beta_t k_t}}{k_t!} A[\alpha_1, k_1; \dots; \alpha_t, k_t] y_1^{k_1} \dots y_t^{k_t}$ (9)

Where $\alpha_i$and β_i $\forall (i = 1,2,\dots,t)$are arbitrary positive integers. The coefficients $A[\alpha_1, k_1; \dots; \alpha_t, k_t]$are arbitrary real or complex constants.

The general class of polynomials (9) is capable of reducing to a number of familiar multivariable polynomials by suitable specializing the arbitrary coefficients $A[\alpha_1, k_1; \dots; \alpha_t, k_t]$, $(k_i \geq 0)$.

3. Preliminaries

From the table of integration, series and products by I.S. Gradshtein M.I. Ryzhik [4], We need the following integration formulae

$$\int_0^\infty \left\{ \left(ax + \frac{b}{x} \right) + c \right\}^{-\rho-1} dx = \frac{\sqrt{\pi}\Gamma\left(\rho + \frac{1}{2}\right)}{2a(4ab+c)^{\rho+\frac{1}{2}}\Gamma(\rho+1)} \tag{10}$$

$$a > 0; b > 0; 4ab + c > 0 \text{ and } \mathbb{R}e\left(\rho + \frac{1}{2}\right) > 0$$

$$\int_0^\infty \frac{1}{x^2} \left\{ \left(ax + \frac{b}{x} \right) + c \right\}^{-\rho-1} dx = \frac{\sqrt{\pi}\Gamma\left(\rho + \frac{1}{2}\right)}{2b(4ab+c)^{\rho+\frac{1}{2}}\Gamma(\rho+1)} \tag{11}$$

$$a > 0; b > 0; 4ab + c > 0 \text{ and } \mathbb{R}e\left(\rho + \frac{1}{2}\right) > 0$$

$$\int_0^\infty \left(a + \frac{b}{x^2} \right) \left\{ \left(ax + \frac{b}{x} \right) + c \right\}^{-\rho-1} dx = \frac{\sqrt{\pi}\Gamma\left(\rho + \frac{1}{2}\right)}{(4ab+c)^{\rho+\frac{1}{2}}\Gamma(\rho+1)} \tag{12}$$

$$a > 0; b > 0; 4ab + c > 0 \text{ and } \mathbb{R}e\left(\rho + \frac{1}{2}\right) > 0$$

4. Main Results

In this section, we have obtained some integrals involving the product of Srivastava's Polynomial with the H-function of several complex variables and exponential functions.

4.1. Theorem 1: If we take $a > 0, b > 0; (4ab + c) > 0; \lambda \geq 0$ and $\rho_i > 0, \sigma_i > 0$ (∀i=1,2,...,r) then the following integration hold

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{z\left(a^2x^2+\frac{b^2}{x^2}\right)} S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1}, y_2\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_2}, \dots, y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t}\right]$$

$$H_{P,Q;p_1,q_1;\dots;p_r,q_r}^{M,N;m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix} z_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_1} \\ \vdots \\ z_r\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_r}\end{matrix}\middle| \begin{matrix}\left(a_j;A_j^{(1)},A_j^{(2)},\dots A_j^{(r)}\right)_{1,P};\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \vdots \\ \left(b_j,B_j^{(1)},B_j^{(2)},\dots,B_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{matrix}\right]$$

$$= \frac{\sqrt{\pi}e^{2abz}}{2a(4ab+c)^{\lambda+\frac{1}{2}}} S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}\left[\frac{y_1}{(4ab+c)^{\rho_1}},\dots,\frac{y_t}{(4ab+c)^{\rho_t}}\right] H_{P+1,Q+1;p_1,q_1;\dots;p_r,q_r}^{M,N+1;m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix} z_1(4ab+c)^{-\sigma_1} \\ \vdots \\ z_r(4ab+c)^{-\sigma_r}\end{matrix}\right.$$

$$\left.\begin{matrix}\left\{\left(\frac{1}{2}+n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};\left(a_j;A_j^{(1)},A_j^{(2)},\dots A_j^{(r)}\right)_{1,P};\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \left\{\left(n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};\left(b_j,B_j^{(1)},B_j^{(2)},\dots,B_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{matrix}\right] \quad (13)$$

The above integral will be convergence for condition (4), (5) and (6).

Proof: L.H.S of equation (13)

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{z\left(a^2x^2+\frac{b^2}{x^2}\right)} S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1}, y_2\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_2}, \dots, y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t}\right]$$

$$H_{P,Q;p_1,q_1;\dots;p_r,q_r}^{M,N;m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix} z_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_1} \\ \vdots \\ z_r\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_r}\end{matrix}\middle| \begin{matrix}\left(a_j;A_j^{(1)},A_j^{(2)},\dots A_j^{(r)}\right)_{1,P};\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \vdots \\ \left(b_j,B_j^{(1)},B_j^{(2)},\dots,B_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{matrix}\right] dx$$

We can write by expressing the Srivastava's Polynomial (9) and H-function of several complex variables (1) then we get

$$\Rightarrow \int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{-z(2ab+c)}\, e^{z\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}} \sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}} \cdots \sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}} \frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!} \cdots \frac{(-\alpha_t)_{\beta_t k_t}}{k_t!} A[\alpha_1,k_1;\dots;\alpha_t,k_t] \times$$

$$y_1^{k_1}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1 k_1} \dots y_t^{k_t}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t k_t}$$

$$\times \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,s_2,\dots,s_r)\left\{\prod_{i=1}^{r}\phi(s_i) z_i^{s_i}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_i s_i} ds_i\right\} dx$$

Now we replace exponential function e^z by $\sum_{n=0}^{\infty}\frac{z^n}{n!}$and

$$\Rightarrow \int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{-z(2ab+c)}\sum_{n=0}^{\infty}\frac{z^n\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^n}{n!}\sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}}..\sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}}\frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!}\dots\frac{(-\alpha_t)_{\beta_t k_t}}{k_t!}A[\alpha_1,k_1;\dots;\alpha_t,k_t]$$

$$y_1^{k_1}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1 k_1}\dots y_t^{\lambda t}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t k_t}$$

$$\times\frac{1}{(2\pi\omega)^r}\int_{L_1}..\int_{L_r}\psi(s_1,s_2,..,s_r)\left\{\overset{i=1}{\underset{r}{\Pi}}\,\phi(s_i)z_i^{s_i}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_i s_i}ds_i\right\}dx$$

By interchanging the order of integration and summation, we get

$$\Rightarrow \sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}}..\sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}}\frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!}\dots\frac{(-\alpha_t)_{\beta_t k_t}}{k_t!}A[\alpha_1,k_1;\dots;\alpha_t,k_t]y_1^{k_1}\dots y_t^{\lambda t}\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

$$\times\frac{e^{-z(2ab+c)}}{(2\pi\omega)^r}\int_{L_1}..\int_{L_r}\psi(s_1,s_2,..,s_r)\times\left\{\overset{i=1}{\underset{r}{\Pi}}\,\phi(s_i)z_i^{s_i}ds_i\right\}\times$$

$$\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{n-\lambda-1-\overset{i=1}{\underset{r}{\Sigma}}\sigma_i s_i-\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j}dx$$

$$\Rightarrow\frac{\sqrt{\pi}e^{-z(2ab+c)}}{2a(4ab+c)^{\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_i s_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j+\frac{1}{2}}}\sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}}..\sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}}\frac{y_1^{k_1}(-\alpha_1)_{\beta_1 k_1}}{k_1!}\dots\frac{y_t^{k_t}(-\alpha_t)_{\beta_t k_t}}{k_t!}A[\alpha_1,k_1;\dots;\alpha_t,k_t]$$

$$\frac{\Gamma\left(\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_i s_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j+\frac{1}{2}\right)}{\Gamma\left(\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_i s_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j+1\right)}$$

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{1}{(2\pi\omega)^r}\int_{L_1}..\int_{L_r}\psi(s_1,s_2,..,s_r)\left\{\overset{i=1}{\underset{r}{\Pi}}\,\phi(s_i)z_i^{s_i}ds_i\right\}$$

$$\Rightarrow\frac{\sqrt{\pi}e^{-z(2ab+c)}}{2a(4ab+c)^{\lambda+\frac{1}{2}}}\sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}}..\sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}}\frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!}\dots\frac{(-\alpha_t)_{\beta_t k_t}}{k_t!}A[\alpha_1,k_1;\dots;\alpha_t,k_t]\left\{\frac{y_1}{(4ab+c)^{\rho_1}}\right\}^{k_1}\dots\left\{\frac{y_t}{(4ab+c)^{\rho_t}}\right\}^{k_t}\times$$

$$\sum_{n=0}^{\infty}\frac{\{z(4ab+c)\}^n}{n!}\frac{\Gamma\left(\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_i s_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j+\frac{1}{2}\right)}{\Gamma\left(\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_i s_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j+1\right)}\frac{1}{(2\pi\omega)^r}\int_{L_1}..\int_{L_r}\psi(s_1,s_2,..,s_r)\left\{\overset{i=1}{\underset{r}{\Pi}}\,\phi(s_i)\{z_i(4ab+c)^{-\sigma_i}\}z_i^{s_i}ds_i\right\}$$

By virtue of interpreting the equations (1) and (9), we obtain the required result.

$$\Rightarrow\frac{\sqrt{\pi}e^{2abz}}{2a(4ab+c)^{\lambda+\frac{1}{2}}}S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}[\{y_1(4ab+c)^{-\rho_1}\}\dots\{y_t(4ab+c)^{-\rho_t}\}]H_{P+1,Q+1;p_1,q_1;\dots;p_r,q_r}^{M,N+1;m_1,n_1;\dots;m_r,n_r}\begin{bmatrix}z_1(4ab+c)^{-\sigma_1}\\ \vdots\\ z_r(4ab+c)^{-\sigma_r}\end{bmatrix}$$

$$\left[\begin{matrix} \left\{\left(\frac{1}{2}+n-\lambda-\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j\right);\ \overset{i=1}{\underset{r}{\Sigma}}\sigma_i\right\};\left(a_j;A_j^{(1)},\ldots A_j^{(r)}\right)_{1,P};\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \left\{\left(n-\lambda-\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j\right);\ \overset{i=1}{\underset{r}{\Sigma}}\sigma_i\right\};\left(b_j,B_j^{(1)},\ldots,B_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{matrix}\right]$$

Hence, theorem 1 is proved.
Where $n-\lambda>\sum_{j=1}^{t}\rho_j k_j$

4.2. Theorem 2: If we take $a>0, b>0; (4ab+c)>0; \lambda\geq 0$ and $\rho_i>0, \sigma_i>0$ (∀i=1,2,...,r) then the following integration hold

$$\int_0^\infty \frac{1}{x^2}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{z\left(a^2x^2+\frac{b^2}{x^2}\right)} S_{\alpha_1,\alpha_2,\ldots,\alpha_t}^{\beta_1,\beta_2,\ldots,\beta_t}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1},\ldots,\ldots,y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t}\right] H_{P,Q;p_1,q_1;\ldots;p_r,q_r}^{M,N;m_1,n_1;\ldots;m_r,n_r}$$

$$\left[\begin{matrix} z_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_1} \\ \vdots \\ z_r\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_r}\end{matrix}\left|\begin{matrix}\left(a_j;A_j^{(1)},A_j^{(2)},\ldots A_j^{(r)}\right)_{1,P};\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \left(b_j,B_j^{(1)},B_j^{(2)},\ldots,B_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$

$$=\frac{\sqrt{\pi}e^{2abz}}{2b(4ab+c)^{\lambda+\frac{1}{2}}} S_{\alpha_1,\alpha_2,\ldots,\alpha_t}^{\beta_1,\beta_2,\ldots,\beta_t}\left[\frac{y_1}{(4ab+c)^{\rho_1}},\ldots,\frac{y_t}{(4ab+c)^{\rho_t}}\right] H_{P+1,Q+1;p_1,q_1;\ldots;p_r,q_r}^{M,N+1;m_1,n_1;\ldots;m_r,n_r}$$

$$\left[\begin{matrix}\frac{z_1}{(4ab+c)^{\sigma_1}} \\ \vdots \\ \frac{z_r}{(4ab+c)^{\sigma_r}}\end{matrix}\left|\begin{matrix} \left\{\left(\frac{1}{2}+n-\lambda-\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j\right);\ \overset{i=1}{\underset{r}{\Sigma}}\sigma_i\right\};\left(a_j;A_j^{(1)},\ldots A_j^{(r)}\right)_{1,P};\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \left\{\left(n-\lambda-\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j\right);\ \overset{i=1}{\underset{r}{\Sigma}}\sigma_i\right\};\left(b_j,B_j^{(1)},\ldots,B_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right] \quad (14)$$

The above integral will be convergence for condition (4), (5) and (6).

Proof: L.H.S of (14)

$$\int_0^\infty \frac{1}{x^2}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{z\left(a^2x^2+\frac{b^2}{x^2}\right)} S_{\alpha_1,\alpha_2,\ldots,\alpha_t}^{\beta_1,\beta_2,\ldots,\beta_t}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1},y_2\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_2},\ldots,y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t}\right]$$

$$H_{P,Q;p_1,q_1;\ldots;p_r,q_r}^{M,N;m_1,n_1;\ldots;m_r,n_r}\left[\begin{matrix} z_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_1} \\ \vdots \\ z_r\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_r}\end{matrix}\left|\begin{matrix}\left(a_j;A_j^{(1)},A_j^{(2)},\ldots A_j^{(r)}\right)_{1,P};\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \left(b_j,B_j^{(1)},B_j^{(2)},\ldots,B_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$

We can write by expressing the Srivastava's Polynomial (9) and H-function of several complex variables (1) then we get

$$\Rightarrow \int_0^\infty \frac{1}{x^2}\left[\left(ax+\frac{b}{x}\right)^2 + c\right]^{-\lambda-1} e^{-z(2ab+c)}\, e^{z\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}} \sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}} .. \sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}} \frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!} \ldots \frac{(-\alpha_t)_{\beta_t k_t}}{k_t!} A[\alpha_1,k_1;\ldots;\alpha_t,k_t] y_1^{k_1}\left\{\left(ax+\frac{b}{x}\right)^2 + c\right\}^{-\rho_1 k_1} \times$$

$$\ldots y_t^{\lambda t}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t k_t} \times \frac{1}{(2\pi\omega)^r}\int_{L_1} .. \int_{L_r} \psi(s_1,s_2,..,s_r)\left\{\Pi_{i=1}^{r} \phi(s_i) z_i^{s_i}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_i s_i} ds_i\right\} dx$$

Now we replace exponential function e^z by $\sum_{n=0}^{\infty}\frac{z^n}{n!}$and

$$\Rightarrow \int_0^\infty \frac{1}{x^2}\left[\left(ax+\frac{b}{x}\right)^2 + c\right]^{-\lambda-1} e^{-z(2ab+c)} \sum_{n=0}^{\infty} \frac{z^n\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^n}{n!} \sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}} .. \sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}} \frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!} \ldots \frac{(-\alpha_t)_{\beta_t k_t}}{k_t!} \times A[\alpha_1,k_1;\ldots;\alpha_t,k_t] \times$$

$$y_1^{k_1}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1 k_1} \ldots y_t^{\lambda t}\left\{\left(ax+\frac{b}{x}\right)^2 + c\right\}^{-\rho_t k_t} \frac{1}{(2\pi\omega)^r}\int_{L_1} .. \int_{L_r} \psi(s_1,s_2,..,s_r)\left\{\Pi_{i=1}^{r} \phi(s_i) z_i^{s_i}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_i s_i} ds_i\right\} dx$$

By interchanging the order of integration and summation, we get

$$\Rightarrow \sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}} .. \sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}} \frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!} \ldots \frac{(-\alpha_t)_{\beta_t k_t}}{k_t!} A[\alpha_1,k_1;\ldots;\alpha_t,k_t] y_1^{k_1} \ldots y_t^{\lambda t} \sum_{n=0}^{\infty} \frac{z^n}{n!}\frac{e^{-z(2ab+c)}}{(2\pi\omega)^r}\int_{L_1} .. \int_{L_r} \psi(s_1,s_2,..,s_r)\left\{\Pi_{i=1}^{r} \phi(s_i) z_i^{s_i} ds_i\right\} \times$$

$$\int_0^\infty \frac{1}{x^2}\left[\left(ax+\frac{b}{x}\right)^2 + c\right]^{n-\lambda-1-\Sigma_{i=1}^{r}\sigma_i s_i - \Sigma_{j=1}^{t}\rho_j k_j} dx$$

$$\Rightarrow \frac{\sqrt{\pi} e^{-z(2ab+c)}}{2b(4ab+c)^{\lambda-n+\Sigma_{i=1}^{r}\sigma_i s_i+\Sigma_{j=1}^{t}\rho_j k_j+\frac{1}{2}}} \sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}} .. \sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}} \frac{y_1^{k_1}(-\alpha_1)_{\beta_1 k_1}}{k_1!} \ldots \frac{y_t^{k_t}(-\alpha_t)_{\beta_t k_t}}{k_t!} A[\alpha_1,k_1;\ldots;\alpha_t,k_t] \frac{\Gamma\left(\lambda-n+\Sigma_{i=1}^{r}\sigma_i s_i+\Sigma_{j=1}^{t}\rho_j k_j+\frac{1}{2}\right)}{\Gamma\left(\lambda-n+\Sigma_{i=1}^{r}\sigma_i s_i+\Sigma_{j=1}^{t}\rho_j k_j+1\right)}$$

$$\times \sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{1}{(2\pi\omega)^r}\int_{L_1} .. \int_{L_r} \psi(s_1,s_2,..,s_r)\left\{\Pi_{i=1}^{r} \phi(s_i) z_i^{s_i} ds_i\right\}$$

$$\Rightarrow \frac{\sqrt{\pi} e^{-z(2ab+c)}}{2b(4ab+c)^{\lambda+\frac{1}{2}}} \sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}} .. \sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}} \frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!} \ldots \frac{(-\alpha_t)_{\beta_t k_t}}{k_t!} A[\alpha_1,k_1;\ldots;\alpha_t,k_t]\left\{\frac{y_1}{(4ab+c)^{\rho_1}}\right\}^{k_1} \times \ldots \left\{\frac{y_t}{(4ab+c)^{\rho_t}}\right\}^{k_t} \times$$

$$\sum_{n=0}^{\infty}\frac{\{z(4ab+c)\}^n}{n!} \times \frac{\Gamma\left(\lambda-n+\Sigma_{i=1}^{r}\sigma_i s_i+\Sigma_{j=1}^{t}\rho_j k_j+\frac{1}{2}\right)}{\Gamma\left(\lambda-n+\Sigma_{i=1}^{r}\sigma_i s_i+\Sigma_{j=1}^{t}\rho_j k_j+1\right)} \frac{1}{(2\pi\omega)^r}\int_{L_1} .. \int_{L_r} \psi(s_1,s_2,..,s_r)\left\{\Pi_{i=1}^{r} \phi(s_i)\{z_i(4ab+c)^{-\sigma_i}\} z_i^{s_i} ds_i\right\}$$

By virtue of interpreting equations (1) and (9), we obtain the required result.

$$\Rightarrow \frac{\sqrt{\pi}e^{2abz}}{2b(4ab+c)^{\lambda+\frac{1}{2}}} S_{\alpha_1,\alpha_2,\ldots,\alpha_t}^{\beta_1,\beta_2,\ldots,\beta_t}[y_1(4ab+c)^{-\rho_1},\ldots,y_t(4ab+c)^{-\rho_t}] H_{P+1,Q+1;p_1,q_1;\ldots;p_r,q_r}^{M,N+1;m_1,n_1;\ldots;m_r,n_r} \left[\begin{array}{c} z_1(4ab+c)^{-\sigma_1} \\ \vdots \\ z_r(4ab+c)^{-\sigma_r} \end{array} \middle| \begin{array}{c} \left\{\left(\frac{1}{2}+n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};(a_j;A_j^{(1)},\ldots A_j^{(r)})_{1,P};(c_j^{(1)},C_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ \\ \left\{\left(n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};(b_j,B_j^{(1)},\ldots,B_j^{(r)})_{1,Q};(d_j^{(1)},D_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r} \end{array} \right]$$

Hence, theorem 2 is proved.
Where $n-\lambda > \sum_{j=1}^{t}\rho_j k_j$

4.3. Theorem 3: If we take $a>0, b>0; (4ab+c)>0; \lambda \geq 0$ and $\rho_i>0, \sigma_i>0$ (∀i=1,2,...,r) then the following integration hold

$$\int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{z\left(a^2x^2+\frac{b^2}{x^2}\right)} S_{\alpha_1,\alpha_2,\ldots,\alpha_t}^{\beta_1,\beta_2,\ldots,\beta_t}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1}, y_2\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_2},\ldots,y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t}\right]$$

$$H_{P,Q;p_1,q_1;\ldots;p_r,q_r}^{M,N;m_1,n_1;\ldots;m_r,n_r}\left[\begin{array}{c} z_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_1} \\ \vdots \\ z_r\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_r}\end{array}\middle| \begin{array}{c}(a_j;A_j^{(1)},A_j^{(2)},\ldots A_j^{(r)})_{1,P};(c_j^{(1)},C_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ \\ (b_j,B_j^{(1)},B_j^{(2)},\ldots,B_j^{(r)})_{1,Q};(d_j^{(1)},D_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r}\end{array}\right]dx$$

$$= \frac{\sqrt{\pi}e^{2abz}}{(4ab+c)^{\lambda+\frac{1}{2}}} S_{\alpha_1,\alpha_2,\ldots,\alpha_t}^{\beta_1,\beta_2,\ldots,\beta_t}\left[\frac{y_1}{(4ab+c)^{-\rho_1}},\ldots,\frac{y_t}{(4ab+c)^{-\rho_t}}\right] H_{P+1,Q+1;p_1,q_1;\ldots;p_r,q_r}^{M,N+1;m_1,n_1;\ldots;m_r,n_r}\left[\begin{array}{c} z_1(4ab+c)^{-\sigma_1} \\ \vdots \\ z_r(4ab+c)^{-\sigma_r}\end{array}\middle| \begin{array}{c} \left\{\left(\frac{1}{2}+n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};(a_j;A_j^{(1)},\ldots A_j^{(r)})_{1,P};(c_j^{(1)},C_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ \left\{\left(n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};(b_j,B_j^{(1)},\ldots,B_j^{(r)})_{1,Q};(d_j^{(1)},D_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r}\end{array}\right] \quad (15)$$

The above integral will be convergence for conditions (6), (7) and (8).

Proof: L.H.S of equation (15)

$$\int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{z\left(a^2x^2+\frac{b^2}{x^2}\right)} S_{\alpha_1,\alpha_2,\ldots,\alpha_t}^{\beta_1,\beta_2,\ldots,\beta_t}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1}, y_2\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_2},\ldots,y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t}\right]$$

$$H_{P,Q;p_1,q_1;\ldots;p_r,q_r}^{M,N;m_1,n_1;\ldots;m_r,n_r}\left[\begin{array}{c} z_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_1} \\ \vdots \\ z_r\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_r}\end{array}\middle| \begin{array}{c}(a_j;A_j^{(1)},A_j^{(2)},\ldots A_j^{(r)})_{1,P};(c_j^{(1)},C_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ \\ (b_j,B_j^{(1)},B_j^{(2)},\ldots,B_j^{(r)})_{1,Q};(d_j^{(1)},D_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r}\end{array}\right]dx$$

We can write by expressing the Srivastava's Polynomial (9) and H-function of several complex variables (1) then we get

$$\Rightarrow \int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{-z(2ab+c)}\, e^{z\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}} \sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}} .. \sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}} \frac{(-\alpha_1)_{\beta_1 k_1}}{k_1!} \cdots \frac{(-\alpha_t)_{\beta_t k_t}}{k_t!} \times$$

$$A[\alpha_1,k_1;\ldots;\alpha_t,k_t]y_1^{k_1}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1k_1}\ldots y_t^{\lambda t}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_tk_t}\times$$
$$\frac{1}{(2\pi\omega)^r}\int_{L_1}..\int_{L_r}\psi(s_1,s_2,..,s_r)\left\{\overset{i=1}{\underset{r}{\Pi}}\,\phi(s_i)z_i^{s_i}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_is_i}ds_i\right\}dx$$

Now we replace exponential function e^z by $\sum_{n=0}^{\infty}\frac{z^n}{n!}$and

$$\Rightarrow\int_0^{\infty}\left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1}e^{-z(2ab+c)}\sum_{n=0}^{\infty}\frac{z^n\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^n}{n!}\sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}}..\sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}}\frac{(-\alpha_1)_{\beta_1k_1}}{k_1!}\ldots..\frac{(-\alpha_t)_{\beta_tk_t}}{k_t!}A[\alpha_1,k_1;\ldots;\alpha_t,k_t]y_1^{k_1}\times$$
$$\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1k_1}\ldots y_t^{\lambda t}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_tk_t}$$
$$\times\frac{1}{(2\pi\omega)^r}\int_{L_1}..\int_{L_r}\psi(s_1,s_2,..,s_r)\left\{\overset{i=1}{\underset{r}{\Pi}}\,\phi(s_i)z_i^{s_i}\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_is_i}ds_i\right\}dx$$

By interchanging the order of integration and summation, we get

$$\Rightarrow\sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}}..\sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}}\frac{(-\alpha_1)_{\beta_1k_1}}{k_1!}\ldots\frac{(-\alpha_t)_{\beta_tk_t}}{k_t!}A[\alpha_1,k_1;\ldots;\alpha_t,k_t]y_1^{k_1}\ldots y_t^{\lambda t}\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{e^{-z(2ab+c)}}{(2\pi\omega)^r}\int_{L_1}..\int_{L_r}\psi(s_1,s_2,..,s_r)\left\{\overset{i=1}{\underset{r}{\Pi}}\,\phi(s_i)z_i^{s_i}ds_i\right\}\times$$
$$\int_0^{\infty}\left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{n-\lambda-1-\overset{i=1}{\underset{r}{\Sigma}}\sigma_is_i-\overset{j=1}{\underset{t}{\Sigma}}\rho_jk_j}dx$$

$$\Rightarrow\frac{\sqrt{\pi}e^{-z(2ab+c)}}{(4ab+c)^{\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_is_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_jk_j+\frac{1}{2}}}\sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}}..\sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}}\frac{y_1^{k_1}(-\alpha_1)_{\beta_1k_1}}{k_1!}\ldots\frac{y_t^{k_t}(-\alpha_t)_{\beta_tk_t}}{k_t!}A[\alpha_1,k_1;\ldots;\alpha_t,k_t]$$
$$\frac{\Gamma\left(\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_is_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_jk_j+\frac{1}{2}\right)}{\Gamma\left(\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_is_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_jk_j+1\right)}\times$$
$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\frac{1}{(2\pi\omega)^r}\int_{L_1}..\int_{L_r}\psi(s_1,s_2,..,s_r)\left\{\overset{i=1}{\underset{r}{\Pi}}\,\phi(s_i)z_i^{s_i}ds_i\right\}$$

$$\Rightarrow\frac{\sqrt{\pi}e^{-z(2ab+c)}}{(4ab+c)^{\lambda+\frac{1}{2}}}\sum_{k_1=0}^{\frac{\alpha_1}{\beta_1}}..\sum_{k_t=0}^{\frac{\alpha_t}{\beta_t}}\frac{(-\alpha_1)_{\beta_1k_1}}{k_1!}\ldots\frac{(-\alpha_t)_{\beta_tk_t}}{k_t!}A[\alpha_1,k_1;\ldots;\alpha_t,k_t]\left\{\frac{y_1}{(4ab+c)^{\rho_1}}\right\}^{k_1}\ldots\left\{\frac{y_t}{(4ab+c)^{\rho_t}}\right\}^{k_t}\times$$
$$\frac{\Gamma\left(\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_is_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_jk_j+\frac{1}{2}\right)}{\Gamma\left(\lambda-n+\overset{i=1}{\underset{r}{\Sigma}}\sigma_is_i+\overset{j=1}{\underset{t}{\Sigma}}\rho_jk_j+1\right)}\sum_{n=0}^{\infty}\frac{\{z(4ab+c)\}^n}{n!}\frac{1}{(2\pi\omega)^r}\int_{L_1}..\int_{L_r}\psi(s_1,s_2,..,s_r)\left\{\overset{i=1}{\underset{r}{\Pi}}\,\phi(s_i)\{z_i(4ab+c)^{-\sigma_i}\}z_i^{s_i}ds_i\right\}$$

By virtue of interpreting equations (1) and (9), we obtain the required result.

$$\Rightarrow \frac{\sqrt{\pi}e^{2abz}}{(4ab+c)^{\lambda+\frac{1}{2}}} S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}[y_1(4ab+c)^{-\rho_1},\dots,y_t(4ab+c)^{-\rho_t}] \times$$

$$H_{P+1,Q+1;p_1,q_1;\dots;p_r,q_r}^{M,N+1;m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix} z_1(4ab+c)^{-\sigma_1} \\ \vdots \\ z_r(4ab+c)^{-\sigma_r}\end{matrix}\left| \begin{matrix} \left\{\left(\frac{1}{2}+n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};\left(a_j;A_j^{(1)},\dots A_j^{(r)}\right)_{1,P};\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \\ \left\{\left(n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};\left(b_j,B_j^{(1)},\dots,B_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$

Hence, Theorem 3 is proved.
Where $n-\lambda>\sum_{j=1}^{t}\rho_j k_j$

5. Particular Cases

5.1. Corollary 1

Put $M=N=P=Q=0$ in equation (13) then we get new result

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{z\left(a^2x^2+\frac{b^2}{x^2}\right)} S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1},,\dots,y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t}\right]\times$$

$$\prod_{i=1}^{r} H_{p_i,q_i}^{m_i,n_i}\left[z_i\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_i}\left| \begin{matrix}\left(c_j^{(i)},C_j^{(i)}\right)_{1,p_i} \\ \left(d_j^{(i)},D_j^{(i)}\right)_{1,q_i}\end{matrix}\right.\right] = \frac{\sqrt{\pi}e^{2abz}}{2a(4ab+c)^{\lambda+\frac{1}{2}}} S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}\left[\frac{y_1}{(4ab+c)^{\rho_1}},\dots,\frac{y_t}{(4ab+c)^{\rho_t}}\right]$$

$$H_{1,1;p_1,q_1;\dots;p_r,q_r}^{0,1;m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix} z_1(4ab+c)^{-\sigma_1} \\ \vdots \\ z_r(4ab+c)^{-\sigma_r}\end{matrix}\left| \begin{matrix} \left\{\left(\frac{1}{2}+n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \left\{\left(n-\lambda-\sum_{j=1}^{t}\rho_j k_j\right);\sum_{i=1}^{r}\sigma_i\right\};\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$

Similarly, from the equations (14) and (15) , we can obtain new results.

5.2. Corollary 2

Putting $A_j^{(i)}=1=B_j^{(i)}=C_j^{(i)}=D_j^{(i)};(\forall i=1,2,\dots,r)$ in equation (13) then H-function reduce to G-function and we get

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-\lambda-1} e^{z\left(a^2x^2+\frac{b^2}{x^2}\right)} S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}\left[y_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_1},\dots,y_t\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\rho_t}\right]\times$$

$$G_{P,Q;p_1,q_1;\dots;p_r,q_r}^{M,N;m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix} z_1\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_1} \\ \vdots \\ z_r\left\{\left(ax+\frac{b}{x}\right)^2+c\right\}^{-\sigma_r}\end{matrix}\left| \begin{matrix}\left(a_j\right)_{1,P};\left(c_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)}\right)_{1,p_r} \\ \\ \left(b_j\right)_{1,Q};\left(d_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$

$$= \frac{\sqrt{\pi}e^{2abz}}{2a(4ab+c)^{\lambda+\frac{1}{2}}} S_{\alpha_1,\alpha_2,\dots,\alpha_t}^{\beta_1,\beta_2,\dots,\beta_t}\left[\frac{y_1}{(4ab+c)^{\rho_1}},\dots,\frac{y_t}{(4ab+c)^{\rho_t}}\right] G_{P+1,Q+1;p_1,q_1;\dots;p_r,q_r}^{M,N+1;m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix} z_1(4ab+c)^{-\sigma_1} \\ \vdots \\ z_r(4ab+c)^{-\sigma_r}\end{matrix}\right|$$

$$\left.\begin{matrix} \left\{\left(\frac{1}{2}+n-\lambda-\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j\right);\overset{i=1}{\underset{r}{\Sigma}}\sigma_i\right\};(a_j)_{1,P};(c_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)})_{1,p_r} \\ \left\{\left(n-\lambda-\overset{j=1}{\underset{t}{\Sigma}}\rho_j k_j\right);\overset{i=1}{\underset{r}{\Sigma}}\sigma_i\right\};(b_j)_{1,Q};(d_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)})_{1,q_r} \end{matrix}\right]$$

Similarly, from the equation, (14) (15) we can obtain new results for the G-function.

6. Conclusion

In conclusion, the H-function of several complex variables serves as a fundamental tool in mathematical analysis. By adjusting its parameters, we can derive many other important special functions such as Meijer's G-function, Fox's H-function, Wright's generalized hypergeometric function, and many more. This versatility allows us to obtain various unified integrals as special cases of our results, demonstrating the broad applicability and significance of the H-function in mathematical research.

References

[1] Ashiq Hussain Khan, and Neelam Pandey, “Integrals Involving H-function of Several Complex Variables,” *International Journal of Scientific and Research Publications*, vol. 7, no. 2, pp. 95-106, 2017. [Google Scholar] [Publisher Link]

[2] Bateman, Harry, *Higher Transcendental Function*, Erratum, 1953. [Google Scholar] [Publisher Link]

[3] I. S. Gradshteyn, and I. M. Ryzhik, *Table of Integrals, Series and Products*, Elsevier Science, pp. 1-1200, 2007. [Google Scholar] [Publisher Link]

[4] A. M. Mathai, and R. K. Saxena, “The H-function with Application in Statistics and Other Disciplines,” *Journal of the Royal Statistical Society Series D: The Statisticia*, vol. 30, no. 2, 1981. [CrossRef] [Google Scholar] [Publisher Link]

[5] H.M. Srivastava, and Rekha Panda, “Some Expansion Theorems and Generating Relations for the H-Function of Several Complex Variables,” *Commentarii Mathematici Universitatis Sancti Pauli*, vol. 24, no. 2, pp. 119-137, 1976. [CrossRef] [Google Scholar] [Publisher Link]

[6] Rekha Panda, and H.M. Srivastava, “Some Bilateral Generating Functions for A Class of Generalized Hypergeometric Polynomials,” *Journal for pure and applied mathematics*, vol. 1976, pp. 265-274, 1976. [CrossRef] [Google Scholar] [Publisher Link]

[7] Bhattar Sanjay, Kumar Bohra Rakesh, and Bhargava Rachna, “Integral Formulae’s Involving Two -function and Mult variable’s General Class of Polynomials,” *Research Journal of Mathematical and Statistical Sciences*, vol. 2, no. 6, pp. 4-7, 2014. [Publisher Link]

[8] Gaboury, Sebastien, and Tremblay, Richard, “Some Bilateral Generating Functions Involving the Chan-Chyan-Srivastava Polynomials and Some General Classes of Multivariable Polynomials,” *Communications of the Korean Mathematical Society*, vol. 28, no. 4, pp. 783-797, 2013. [CrossRef] [Google Scholar] [Publisher Link]

[9] E.D. Rainville, *Special Functions*, Macmillan, New York, pp. 1-187, 1960. [Google Scholar] [Publisher Link]