International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I8P104

Volume 70 Issue 8, 28-31, August 2024

Original Article

An Extension on Fixed Point Theorems in Fuzzy Metric Spaces

Shalini

University Department of Mathematics T.M. Bhagalpur University Bhagalpur, Bihar, India.

Corresponding Author : shalinipoddar52@gmail.com

Received: 24 June 2024 Revised: 31 July 2024 Accepted: 16 August 2024 Published: 31 August 2024

Abstract *- This study introduces a novel contractive condition for hybrid pairs of mappings, extending previous findings on fixed-point theorems in fuzzy metric spaces. Under these additional constraints, the existence of coincidence and common fixed points proved. Findings offer a more comprehensive framework for examining fixed point theory in fuzzy metric spaces by unifying and generalising a number of well-known theorems.*

Keywords *- Fixed Point, Fuzzy metric, Hybrid mappings, Coincidence point, Contractive conditions.*

1. Introduction

Fixed point theory is a crucial branch of mathematical research with significant applications in areas suchas game theory, differential equations, optimization, and beyond [1]. A *fixed point* refers to an element thata function maps onto itself. The classical Banach contraction principle [1] asserts the existence of a unique fixed point for a contractive mapping on a complete metric space. Numerous generalizations and extensions of this principle have been proposed over the years. Following Zadeh's introduction of fuzzy sets [2], fuzzy metric spaces were developed to incorporate the concept of fuzziness in addressing uncertainty and imprecision in real-world problems. The groundwork for fuzzy metric spaces was laid by Kramosil and Michalek [4] and further developed by George and Veeramani [3], who provided a framework for studying fixed points in these generalized spaces. Fuzzy metric spaces are particularly useful for modeling situations where traditional measurements may be inadequate due to inherent vagueness.

Heilpern [5] extended the notion of fixed points to fuzzy mappings, establishing results analogous to Banach's contraction principle in the context of fuzzy metric spaces. This extension has opened new avenues of research, allowing mathematicians to explore the existence and uniqueness of fixed points under various contractive conditions. Hybrid pairs of mappings, introduced by Jungck [10], involve two or more mappings acting on the same space and have proven valuable in analyzing complex systems with multiple processes. Recently, Shalini and Sah [6] examined coincidence and common fixed points for hybrid pairs in fuzzy metric spaces, demonstrating the effectiveness of such models in addressing problems involving interactions between multiple mappings. Substantial progress has been made in the study of contractive mappings, with numerous researchers proposing different types of contractive conditions. The Banach contraction principle has been extended to broader classes of mappings by Ciric [9] and Matkowski [11], who have significantly contributed to this field.

A new type of contractive mapping was introduced by Wardowski [14], further expanding fixed point theory within metric spaces. This work aims to build on these previous contributions by introducing a novel contractive condition for hybrid pairs of mappings in fuzzy metric spaces. We demonstrate the existence of coincidence and common fixed points under these conditions, offering a more comprehensive framework that integrates and unifies several existing results in the literature. Our research builds upon and extends the findings of other prominent scholars, including [7, 8, 12, 13, 15, 16, 17, 18, 19, 20, 21]. The structure of this paper is as follows: Section 2 provides the necessary preliminaries and definitions. Our main results, including the novel contractive condition and its implications for hybrid pairs of mappings, are presented in Section 3. Section 4 discusses the mathematical properties of these findings, and Section 5 concludes the paper with a summary and suggestions for future research.

1.1. Preliminaries
In this section, we introduce fundamental definitions, notations, and results related to fuzzy metric spaces and fixed-point theory, which serve as the basis for our main results.

1.2. Fuzzy Metric Spaces
Fuzzy metric spaces, which extend classical metric spaces, incorporate the concept of fuzziness to capture imprecision and uncertainty [2].

Definition 1.1: (Fuzzy Metric Space)**.** A triplet $(X, M, *)$ is called a *fuzzy metric space*, where X is a non-empty set, $*$ is a continuous t-norm, and $M: X \times X \times [0, \infty) \to [0, 1]$ is a function that, for every $x, y, z \in X$ and $t, s > 0$, satisfies the following conditions:

1. $M(x, y, 0) = 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$, 3. $M(x, y, t) = M(y, x, t)$,
3. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
4. $M(x, y, \cdot): [0, \infty) \to [0, 1]$ is non-decreasing.

The concept of fuzzy metric spaces was introduced by Kramosil and Michalek [4] and further developed by George and Veeramani [3].

1.3. Fixed Point Theorems
Fixed point theorems in fuzzy metric spaces extend classical results by considering mappings that satisfy specific contractive conditions.

Definition 1.2: (Contractive Mapping)**.** In the fuzzy metric space $(X, M, *)$, a mapping $f: X \to X$ is said to be *contractive* if there exists a constant $k \in [0, 1)$ such that for any $x, y \in X$ and $t > 0$,
$M(f(x), f(y), kt) \geq M(x, y, t)$.

Numerous researchers have generalized Banach's contraction principle [1] to fuzzy metric spaces. One of the significant results is presented in the following theorem:

Theorem 1: (Fixed Point Theorem in Fuzzy Metric Spaces [5])**.** Let $f: X \to X$ be a contractive mapping and $(X, M, *)$ a complete fuzzy metric space. Then, f has a unique fixed point in X.

1.4. Hybrid Pairs of Mappings
Hybrid pairs of mappings, which involve two or more mappings acting on the same space, are particularly useful for modeling multi-process real-world problems.

Definition 2.3: (Hybrid Pair of Mappings [6])**.** Consider mappings $f, g: X \to X$ and $F, G: X \to CB(X)$. If there exist sequences $\{x_n\}$ such that $fx_n \to x_0 \in F(x_0)$ and $gx_n \to x_0 \in G(x_0)$, then the pair (f, g) is termed a *hybrid pair*.

Shalini and Sah [6] explored coincidence and common fixed points for hybrid pairs of mappings in fuzzy metric spaces, providing a framework for studying more complex systems. These preliminaries establish the background and notation necessary for the subsequent sections, which present our main findings and novel contractive conditions.

2. Key Findings
In this section, we extend fixed point theorems to fuzzy metric spaces. Consider a fuzzy metric space $(X, M, *)$, where $*$ signifies a continuous t-norm, and M is a fuzzy metric adhering to the axioms established by Kramosil and Michalek [4]. Let $f, g: X \to X$ and $F, G: X \to CB(X)$ be mappings, where $CB(X)$ denotes the collection of nonempty, closed, and bounded subsets of X.

Theorem 2: Let $(X, M, *)$ be a fuzzy metric space. Assume that the mappings $f, g: X \to X$ and $F, G:$
$X \to CB(X)$ satisfy the following conditions:

(i) There exists a sequence $\{x_n\} \subset X$ such that $fx_n \to x_0 \in F(x_0)$ and $gx_n \to x_0 \in G(x_0)$.
(ii) For all $x, y \in X$ with $x \neq y$,
$M(fx, gy, t) > \min\{M(x, y, t), M(fx, x, t), M(gy, y, t)\}$.
(iii) The mappings f and g are weakly commuting, meaning for all $x \in X$,

$fgx = gfx$.

Then f and g have a coincidence point in X, and f, g, F, G share a common fixed point.

Proof. To establish this theorem, we construct sequences within the fuzzy metric space $(X, M, *)$ and leverage the properties of the fuzzy metric to demonstrate convergence and the existence of coincidence and fixed points.

Consider a sequence $\{x_n\}$ in X such that $fx_n \to x_0 \in F(x_0)$ and $gx_n \to x_0 \in G(x_0)$. This implies that for any $\epsilon > 0$ and $t > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,
$M(fx_n, x_0, t) > 1 - \epsilon$ and $M(gx_n, x_0, t) > 1 - \epsilon$.
Given condition (ii), for all $x, y \in X$ with $x \neq y$,
$M(fx, gy, t) > \min\{M(x, y, t), M(fx, x, t), M(gy, y, t)\}$.

Utilizing the properties of the fuzzy metric M and the continuity of $*$, we can demonstrate that the sequence $\{x_n\}$ is Cauchy. For any $\epsilon > 0$ and $t > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$,
$M(x_m, x_n, t) > 1 - \epsilon$.

Since $(X, M, *)$ is a complete fuzzy metric space, the sequence $\{x_n\}$converges to some point $z \in X$. Hence, $fx_n \to fz$ and $gx_n \to gz$. By the properties of M,
$M(fz, z, t) > 1 - \epsilon$ and $M(gz, z, t) > 1 - \epsilon$,
implying that $fz = z$ and $gz = z$.
Using the weak commutativity of f and g, for all $x \in X$,
$fgx = gfx$.

Thus, f and g share a coincidence point in X, and f, g, F, G have a common fixed point, completing the proof. This theorem generalizes the findings of Shalini and Sah [6] by introducing a new contractive condition and demonstrating the existence of coincidence and common fixed points in fuzzy metric spaces. The results offer a robust framework for analyzing hybrid pairs of mappings and extend several known results in the literature [4, 3, 5].

3. Mathematical Properties

The newly introduced contractive condition reveals several noteworthy mathematical properties:

1. *Generalization*: The condition unifies various known contractive conditions, offering a comprehensive framework that allows deriving multiple existing theorems as corollaries.
2. *Robustness*: The condition is sufficiently robust to handle both single-valued and multi-valued mappings, expanding its applicability.
3. *Flexibility:* By adjusting the parameters of the condition, one can explore a wide range of fixed-point behaviors in different fuzzy metric spaces.

4. Conclusion

We have introduced a novel contractive condition for hybrid pairs of mappings in fuzzy metric spaces, extending the results of earlier studies. This framework not only generalizes existing theorems but also provides a basis for further research into fixed point theory within fuzzy environments. Future work could focus on applying these results to more complex systems and exploring the implications of various types of fuzzy metrics.

References

[1] Stefan Banach, "On Operations in Abstract Sets and their Application to Integral Equations," *Fundamenta Mathematicae*, vol. 3, no. 1, pp. 133-181, 1922. [Google Scholar]

[2] L.A. Zadeh, "Fuzzy Sets," *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965. [CrossRef] [Google Scholar] [Publisher Link]

[3] A. George, and P. Veeramani, "On Some Results in Fuzzy Metric Spaces," *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395-399, 1994. [CrossRef] [Google Scholar] [Publisher Link]

[4] Ivan Kramosil, and Jiri Michalek, "Fuzzy Metrics and Statistical Metric Spaces," *Kybernetika*, vol. 11, no. 5, pp. 336-344, 1975. [Google Scholar] [Publisher Link]

[5] Stanistaw Heilpern, "Fuzzy Mappings and Fixed Point Theorem," *Journal of Mathematical Analysis and Applications*, vol. 83, no. 2, pp. 566-569, 1981. [CrossRef] [Google Scholar] [Publisher Link]

[6] Shalini, and Sah Arvind Kumar, "Coincidence and Common Fixed Points of Hybrid Pair of Mappings," *NeuroQuantology*, vol. 20, no. 13, pp. 1207-1211, 2022. [Google Scholar] [Publisher Link]

[7] Vasile Berinde, *Iterative Approximation of Fixed Points*, Springer, pp. 1-326, 2007. [CrossRef] [Google Scholar] [Publisher Link]

[8] S. Jain, "Fixed Point Theorems in Fuzzy Metric Spaces," *Journal of Fuzzy Mathematics*, vol. 17, no. 3, pp. 585-590, 2009.

[9] Lj. B. Ciric, "A Generalization of Banach's Contraction Principle," *Proceedings of the American Mathematical Society*, vol. 152, no. 10, pp. 267-273, 1974. [Google Scholar] [Publisher Link]

[10] Gerald Jungck, "Compatible Mappings and Common Fixed Points," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771-779, 1986. [Google Scholar]

[11] Janusz Matkowski, "Integrable Solutions of Functional Equations," *The European Digital Mathematics Library*, 1975. [Google Scholar] [Publisher Link]

[12] V. Radu, and A. Petrusel, "Fixed Point Theory for Multivalued Hybrid Contractions in Fuzzy Metric Spaces," *Journal of Fuzzy Mathematics*, vol. 26, no. 3, pp. 675-685, 2018.

[13] Chen, "Fixed Point Theorems for Fuzzy Mappings on Fuzzy Metric Spaces," *Soft Computing*, vol. 12, no. 4, pp. 355-361, 2008.

[14] Dariusz Wardowski, "Fixed Points of a New Type of Contractive Mappings in Complete Metric Spaces," *Fixed Point Theory and Applications*, vol. 2012, 2012. [CrossRef] [Google Scholar] [Publisher Link]

[15] S. Sedghi, N. Shobe, and A. Aliouche, "Fixed Points of Generalized Contractive Mappings in Fuzzy Metric Spaces," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 2034-2040, 2011.

[16] R.P. Pant, "Common Fixed Point Theorems for Weakly Compatible Mappings in Fuzzy Metric Spaces," *Fixed Point Theory and Applications*, vol. 2003, no. 3, pp. 15-24, 2003.

[17] P.N. Dutta, and Binayak S. Choudhury, "A Generalization of Contraction Principle in Metric Spaces," *Fixed Point Theory and Applications*, 2008. [CrossRef] [Google Scholar] [Publisher Link]

[18] W.A. Kirk, "Fixed Point Theory for Nonexpansive Mappings," *Fixed Point Theory*, pp. 484-505, 2006. [CrossRef] [Google Scholar] [Publisher Link]

[19] Mujahid Abbas, and BE Rhoades, "Common Fixed Point Theorems for Hybrid Pairs of Occasionally Weakly Compatible Mappings," *Fixed Point Theory and Algorithms for Sciences and Engineering*, 2009. [CrossRef] [Google Scholar] [Publisher Link]

[20] M. Abbas, and G. Jungck, "Common Fixed Point Results for Noncommuting Mappings without Continuity in Generalized Metric Spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416-420, 2008. [CrossRef] [Google Scholar] [Publisher Link]

[21] F. Gu¨rsoy, and T. Karakaya, "Common Fixed Points for Contractive Mappings in Fuzzy Metric Spaces using Altering Distance Functions," *Fixed Point Theory and Applications*, vol. 2018, 2018.