International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V70I9P101
Volume 70 Issue 9, 1-7, September 2024

Original Article

On Approximately Unitarily Equivalent Operators

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Received: 22 July 2024 Revised: 26 August 2024 Accepted: 13 September 2024 Published: 30 September 2024

Abstract *- In this paper, we investigate the operator relation* $\| A - U_n^* B U_n \| \longrightarrow 0$, *where* $A, B \in B(\mathcal{H})$ *and* $U_n \in B(\mathcal{H})$ *is a sequence of unitary operators, known as the approximate unitary equivalence between* A *and* B, *which is an asymptotic version of the unitary equivalence of operators. We characterize operators in this relation and investigate other closely related relations. We give and prove conditions under which approximate unitary equivalence implies or is implied by or coincides with other equivalence relations.*

Keywords *- Approximate unitary equivalence, Approximate similarity, Similar, rank, Rank-preserving, Metric equivalence.*

1. Introduction

In this paper $\mathcal{H}$ denotes a separable complex Hilbert space and $B(\mathcal{H})$ denotes the Banach algebra of bounded linear operators equipped with the usual operator norm on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ denotes the set of all compact operators on $\mathcal{H}$. If $T \in B(\mathcal{H})$, then T^* denotes the adjoint of T, while $Ker(T), Ran(T), \overline{\mathcal{M}}$ and $\mathcal{M}^\perp$ stands for the kernel of T, range of T, closure of $\mathcal{M}$ and orthogonal complement of a closed subspace $\mathcal{M}$ of $\mathcal{H}$, respectively. We denote by $\sigma(T)$, $\| T \|$ and $W(T)$, the spectrum, norm and numerical range of T, respectively.

Recall that an operator $T \in B(\mathcal{H})$ is
normal if $T^*T = TT^*$;
self-adjoint (or *hermitian*) if $T^* = T$;
skew-adjoint if $T^* = -T$;
unitary if $T^*T = TT^* = I$;
quasinormal if $T(T^*T) = (T^*T)T$;
binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$;
hyponormal if $T^*T \geq TT^*$;
co-hyponormal if $T^*T \leq TT^*$, that is, if T^* is hyponormal;
seminormal if it is either hyponormal or co-hyponormal;
paranormal if $\| Tx \|^2 \leq \| T^2 x \| \| x \|$, for every $x \in \mathcal{H}$;
a *projection* if $T^2 = T$ and $T^* = T$;
an *involution* if $T^2 = I$;
a *symmetry* if $T = T^* = T^{-1}$. That is, T it is self-adjoint unitary;
isometric if $T^*T = I$;
a contraction if $\| T \| \leq 1$;
normaloid if $r(T) = \| T \|$.
compact if every bounded sequence $\{x_n\}$ in $\mathcal{H}$ the sequence $\{Tx_n\}$ has a convergent subsequence.
Remark: Note that

$$\{Normal\} \subseteq \{Quasinormal\} \subseteq \{Seminormal\} \subseteq \{Hyponormal\} \subseteq \{Paranormal\} \subseteq \{Normaloid\}.$$

In finite-dimensional Hilbert spaces, these higher classes of operators coincide with the class of normal operators. Let $T \in B(\mathcal{H})$. The numerical range of T denoted by $W(T)$ is defined , and the numerical radius of T denoted by $\omega(T)$ is defined as $\omega(T) := \sup\{|\lambda| : \lambda \in W(T)\}$. Clearly, $\omega(T) \in [0, \infty)$. Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be *similar* (denoted $A \sim B$) if there exists an invertible operator $N \in B(\mathcal{H}, \mathcal{K})$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are *unitarily* *equivalent* (denoted by $A \cong B$) if there exists a unitary operator $U \in B_+(\mathcal{H},\mathcal{K})$ (Banach algebra of all invertible operators in $B(\mathcal{H})$) such that $UA = BU$ (i.e. $A = U^*BU$, equivalently, $A = U^{-1}BU$).

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be *metrically equivalent* (denoted by $A \sim_m B$) if $\| Ax \| = \| Bx \|$, (equivalently, $A^*A = B^*B$ or $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$ for all $x \in \mathcal{H}$)(for more exposition, see [16]). Two operators $A, B \in B(\mathcal{H})$ are said to be *almost similar* if there is an invertible operator X such that $A^*A = X^{-1}(B^*B)X$ and $A^* + A = X^{-1}(B^* + B)X$. The concept of almost-similarity was introduced by [10] and also studied by [17]. Two operators A and B in $B(\mathcal{H})$ are said to be almost unitarily equivalent (denoted by $A \overset{a.u.e}{\sim} B$) if there is a unitary operator U such that $A^*A = U^*(B^*B)U$ and $A^* + A = U^*(B^* + B)U$([9]). The proofs that unitary equivalence, similarity, quasi-similarity, metric equivalence and almost similarity are equivalence relations on $B(\mathcal{H})$ have appeared in [17],[14] and [22].

Two operators $S, T \in B(\mathcal{H})$ are said to be *unitarily quasi-equivalent* (denoted by $S \overset{u.q.e}{\approx} T$) if there exists a unitary operator U such that $T^*T = US^*SU^*$ and $TT^* = USS^*U^*$([16]). Clearly $S, T \in B(\mathcal{H})$ are *unitarily quasi-equivalent* if S^*S and T^*T are unitarily equivalent and SS^* and TT^* are unitarily equivalent. Two operators $S, T \in B(\mathcal{H})$ are said to be *absolutely equivalent* if both the absolute values of the operators are unitarily equivalent. That is, if $|S| = U|T|U^*$. Two operators $S, T \in B(\mathcal{H})$ are said to be *nearly equivalent*(see [19]) if there exists a unitary operator U such that $S^*S = UT^*TU^*$. Mahmoud introduced unitary quasi-equivalence in [13], which was also investigated by Othman in [19] under the *near equivalence* relation and by Nzimbi and Luketero in [16]. Several authors have demonstrated (see [19], [13] and [16]) that unitary quasi-equivalence, absolute equivalence and near equivalence are equivalence relations on $B(\mathcal{H})$.

Remarks. We note that any two unitary operators and, in general, any two isometries are absolutely equivalent and metrically equivalent. It has been shown in [19] that absolute equivalence implies near equivalence of operators. We observe that near equivalence of operators is weaker than unitary quasi-equivalence. It has also been shown that unitary quasi-equivalence is weaker than unitary equivalence of operators([16], Theorem 2.2). Note that the set $\mathcal{K}(\mathcal{H})$ of all compact operators on $\mathcal{H}$ is a closed ideal in $\mathcal{H}$ and so we can construct the quotient algebra $B(\mathcal{H})/\mathcal{K}(\mathcal{H}$ called the Calkin algebra. The corresponding map $\pi: B(\mathcal{H}) \longrightarrow B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ defined by $\pi(A) = A + K$, where $A \in B(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$ is called the quotient or canonical map from $B(\mathcal{H})$ to $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Define the essential spectrum $\sigma_e(T)$ as the spectrum of $\pi(T)$ in the Calkin algebra that is,

$$\sigma_e(T) = \sigma(\pi(T)) = \{\lambda \in \mathbb{C}: (\pi(T) - \lambda\pi(I)) \text{ } is \text{ } not \text{ } invertible\}.$$

Let $A \in B(\mathcal{H})$. The essential numerical range of A, denoted by $W_e(A)$ is the set

$$W_e(A) := \bigcap_{K\in\mathcal{K}(\mathcal{H})} \overline{W(A + K)}.$$

2. Main Results

Definition 2.1: *([11])Let $\{T_n\}$ be a sequence of bounded linear operators on a Hilbert space $\mathcal{H}$ and let $T \in B(\mathcal{H})$.*

(a). The sequence $\{T_n\}$ is said to converge uniformly or in operator norm to T denoted by $T_n \overset{u}{\longrightarrow} T$ if $\lim_{n\to\infty} \| T - T_n \| = 0$. In this case, we write $T = u - \lim_{n\to\infty} T_n$. This is also called convergence in operator topology or convergence in the uniform topology and T is called the uniform or norm limit of $\{T_n\}$.

(b). The sequence $\{T_n\}$ is said to converge strongly to T denoted by $T_n \overset{s}{\longrightarrow} T$ if $\lim_{n\to\infty} \| Tx - T_nx \| = 0$ for each $x \in \mathcal{H}$. In this case, we write $T = s - \lim_{n\to\infty} T_n$. This is also called convergence in the strong (operator) topology and T is called the strong limit of $\{T_n\}$.

(c). The sequence $\{T_n\}$ is said to converge weakly to T denoted by $T_n \overset{w}{\longrightarrow} T$ if $\lim_{n\to\infty}\langle y, T_nx\rangle = \langle y, Tx\rangle$ for each $x, y \in \mathcal{H}$. This is equivalent to $\lim_{n\to\infty}\langle (T_n - T)x, y\rangle = 0$ for each $x, y \in \mathcal{H}$. If the Hilbert space is complex, then this is equivalent to $\lim_{n\to\infty}\langle (T_n - T)x, x\rangle = 0$ for each $x \in \mathcal{H}$.
In this case, we write $T = w - \lim_{n\to\infty} T_n$ and we say that is the weak limit of $\{T_n\}$.

Remark: We note that

$$norm \text{ } convergence \Longrightarrow strong \text{ } convergence \Longrightarrow weak \text{ } convergence \Longrightarrow \sup_n \| T_n \| < \infty.$$

It has been shown ([12], Proposition 4.46) that in finite dimensional Hilbert spaces, the concepts of strong and uniform convergence coincide. Two operators $A, B \in B(\mathcal{H})$ are said to be *approximately unitarily equivalent* denoted by $A \overset{a.u.e}{\approx} B$ if there exists a sequence $\{U_n\}$ of unitary operators such that $\| A - U_n^* B U_n \| \longrightarrow 0$. This is equivalent to $\lim_{n\to\infty} \| A - U_n^* B U_n \| = 0$. Clearly, $A \overset{a.u.e}{\approx} B$ if the sequence $\{U_n^* B U_n\}$ converges to A the norm,

Remark: Note that two operators are approximately unitarily equivalent if each is a norm limit of operators that are unitarily equivalent to the other. A sequence $\{V_n\}$ of linear operators is said to be *invertibly bounded* if each V_n is invertible that $\sup\{\| V_n \| . \| V_n^{-1} \|\} < \infty$. Two operators $A, B \in B(\mathcal{H})$ are said to be *approximately similar*, denoted by $A \overset{a.s}{\approx} B$ if there exists a sequence $\{V_n\}$ of invertiblly bounded operators such that $\| A - V_n^{-1} B V_n \| \longrightarrow 0$. This is equivalent to $\lim_{n\to\infty} \| A - V_n^{-1} B V_n \| = 0$. Clearly, $A \overset{a.s}{\approx} B$ if the sequence $\{V_n^{-1} B V_n\}$ converges to A in norm. Clearly, unitary equivalence implies approximate unitary equivalence, and similarity implies approximate similarity in $B(\mathcal{H})$. The study of approximate unitary equivalence was initiated in 1975 by D. H. Hadwin([7]) and later investigated by several authors ([8], [4], [5], [6] and [20]). Let $\mathcal{A} \subseteq B(\mathcal{H})$. A linear map $\pi: \mathcal{A} \longrightarrow B(\mathcal{H})$ is said to be rank-preserving if $rank(\pi(T)) = rank(T)$, for any $T \in \mathcal{A}$. Let $\{U_n\}$ be a sequence of unitary operators and $\{V_n\}$ be a sequence of invertibly bounded operators. We define by $\mathfrak{A}_u(T) = \{S \in B(\mathcal{H}): \lim_{n\to\infty} \| S - U_n^* T U_n \| = 0\}$ and $\mathfrak{A}_s(T) = \{S \in B(\mathcal{H}): \lim_{n\to\infty} \| S - V_n^* T V_n \| = 0\}$ the approximate unitary equivalence orbit and the approximate similarity orbit of $T \in B(\mathcal{H})$.
Clearly $\mathfrak{A}_u(T) \subset \mathfrak{A}_s(T)$.

Theorem 2.3: $\overset{a.u.e}{\approx}$ *is an equivalence relation on* $B(\mathcal{H})$.
Proof: Let $A, B, C \in B(\mathcal{H})$. Clearly, $A \overset{a.u.e}{\approx} A$ since $\lim_{n\to\infty} \| A - I^* A I \| = 0$, by taking $U_n = I$ for $n \in \mathbb{N}$. If $A \overset{a.u.e}{\approx} B$, then $\lim_{n\to\infty} \| A - U_n^* B U_n \| = 0$ and since convergence in norm implies strong convergence, this is equivalent to

$$\langle \lim_{n\to\infty} (A - U_n^* B U_n)x, x\rangle = \lim_{n\to\infty} \langle (A - U_n^* B U_n)x, x\rangle \leq \lim_{n\to\infty} \| A - U_n^* B U_n \| \| x \|^2 = 0$$

for all $x \in \mathcal{H}$. Pre-multiplying the expression $A - U_n^* B U_n$ by U_n and post-multiplying it by U_n^* we have that $\lim_{n\to\infty} \| U_n A U_n^* - B \| = \lim_{n\to\infty} \| B - U_n A U_n^* \| = 0$. This proves that $B \overset{a.u.e}{\approx} A$. We prove that if $A \overset{a.u.e}{\approx} B$ and $B \overset{a.u.e}{\approx} C$ then $A \overset{a.u.e}{\approx} C$. $A \overset{a.u.e}{\approx} B$ and $B \overset{a.u.e}{\approx} C$ implies that $\lim_{n\to\infty} \| A - U_n^* B U_n \| = 0$ and $\lim_{n\to\infty} \| B - V_n^* C V_n \| = 0$ where U_n, V_n are unitary operators. Therefore

$$\begin{aligned} \lim_{n\to\infty} \| A - U_n^* B U_n \| &= \lim_{n\to\infty} \| A - U_n^* (\lim_{n\to\infty} V_n^* C V_n) U_n \| \\ &= \lim_{n\to\infty} \| A - (V_n U_n)^* C (V_n U_n) \| \\ &= \lim_{n\to\infty} \| A - W_n^* C W_n \| \\ &= 0, \end{aligned}$$

where $W_n = V_n U_n$ is unitary. This proves that $A \overset{a.u.e}{\approx} C$. Thus $\overset{a.u.e}{\approx}$, it is reflexive, symmetric and transitive, and hence, it is an equivalence relation on $B(\mathcal{H})$.

2.1. Approximate Unitary Equivalence, Approximate Similarity and Other Equivalence Relations

Theorem 2.3: *Let $A, B \in B(\mathcal{H})$ such that that $\| A - U_n^* B U_n \| \longrightarrow 0$, where $\{U_n\}$ is a unitary sequence of invertibly bounded positive operators. Then A and B are unitarily equivalent.*

Proof: Without loss of generality, suppose that each U_n is positive(that is, self-adjoint and $\langle U_n x, x\rangle \geq 0$) for all $x \in \mathcal{H}$. Since U_n it is invertible, it exists $0 < r \leq U_n \leq R$ for all $n \in \mathbb{N}$. Choose a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ such that $U_{n_k} \overset{w}{\longrightarrow} U$. That is $\lim_{n\to\infty} \langle (U_{n_k} - U)x, x\rangle = 0$ for all $x \in \mathcal{H}$, and since $0 < r \leq U_n \leq R$, the weak limit U is unitary (since the group of unitary operators on $\mathcal{H}$ is closed). So $BU_n - U_n A \longrightarrow 0$ in norm. That is $\| BU_n - U_n A \| \longrightarrow 0$. Therefore $\| A - U_n^* B U_n \| \longrightarrow 0$ implies that $\| BU_n - U_n A \| \longrightarrow 0$ and so

$$\left\langle \left(\left(BU_{n_k} - U_{n_k} A\right) - (BU - UA) \right) x, x \right\rangle \longrightarrow 0.$$

This implies that $BU - UA = 0$. Equivalently $U^* BU - A = 0$ or $A = U^* BU$. This proves the claim.

Remark: Note that Theorem 2.3 gives a condition under which approximate unitary equivalence implies unitary equivalence.

Theorem 2.4: *(Weyl-von Neumann-Sikonia[1]). Let A be normal. Then $A \overset{a.u.e}{\approx} B$, if and only if*
(i). B is normal.
(ii). $\sigma(A) = \sigma(B)$ including multiplicities of their isolated eigenvalues.

(iii) $\sigma_e(A) = \sigma_e(B)$

We denote by $\mathcal{U}(T) = \{S: S = U^*TU, U\ unitary\}$ the *unitary orbit* of T and its norm-closure by $\overline{\mathcal{U}(T)} = \overline{\{S: S = U^*TU, U\ unitary\}} = \overline{\{U^*TU: U\ unitary\}}$. In finite dimensional Hilbert spaces, unitary orbits are closed, but in infinite-dimensional Hilbert spaces, they are typically not closed.

Remark: For any pair of operators A and B, $\overline{\mathcal{U}(A)}$ and $\overline{\mathcal{U}(B)}$ are either disjoint or equal. That is, no partial overlapping is possible.

Theorem 2.5: Two operators $\mathrm{A}, \mathrm{B} \in \mathrm{B}(\mathcal{H})$ are approximately unitarily equivalent if and only if $\mathrm{A} \in \overline{\mathcal{U}(\mathrm{B})}$ and $\mathrm{B} \in \overline{\mathcal{U}(\mathrm{A})}$.

Corollary 2.6: Two operators $\mathrm{A}, \mathrm{B} \in \mathrm{B}(\mathcal{H})$ are approximately unitarily equivalent if and only if $\overline{\mathcal{U}(\mathrm{A})} = \overline{\mathcal{U}(\mathrm{B})}$.

Remark: From Corollary 2.6, we can deduce that two operators are approximately unitarily equivalent if and only if they have the same norm-closed unitary orbit. That is if the norm closures of their unitary orbits(or equivalence classes) coincide.

Example: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$. Clearly, $A \notin \mathcal{U}(B)$ but $A \in \overline{\mathcal{U}(B)}$ Similarly, but . Note that and B are neither unitarily equivalent nor similar, but they are approximately unitarily equivalent and hence approximately similar. In this case, we can let $U_n = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Two projections $P, Q \in B(\mathcal{H})$ are said to be Murray-von Neumann equivalent if there exists a partial isometry $V \in B(\mathcal{H})$ such that $V^*V = P$ and $VV^* = Q$. ([15]) It is shown in ([15], Theorem 3.3) that if two projections $P, Q \in B(\mathcal{H})$ are Murray-von Neumann equivalent and if the implementing partial isometry V is invertible (that is, V is unitary), then P and Q are similar. But P and Q , being projections, they are normal and therefore by ([18], Proposition 2.13), they are unitarily equivalent.

Theorem 2.7: *Let $P, Q \in B(\mathcal{H})$ be Murray-von Neumann equivalent projections with an invertible implementing partial isometry V. Then $P = Q = I$.*

Proof: Since V is invertible, it is unitary and hence $P = V^*V = I = VV^* = Q$.

Corollary 2.8: *Let $P, Q \in B(\mathcal{H})$ be Murray-von Neumann equivalent projections with an invertible implementing partial isometry V. Then P and Q are approximately unitarily equivalent.*

Proof: From Theorem $P = V^*V = I = VV^* = Q$. Thus $\| P - U_n^*QU_n \| = \| I - U_n^*(I)U_n \| = \| I - I \| = 0$, for unitary operators $\{U_n\}$. The claim follows.

Remark: From Theorem 2.7 and Corollary 2.8, we conclude that the intersection of the class of invertible Murray-von Neumann projections and the class of approximately unitarily equivalent projections consists of identity operators. In finite dimensional Hilbert spaces, Murray-von Neumann equivalence and unitary equivalence of projections coincide. However, this is not the case in infinite-dimensional Hilbert spaces. For instance, if $\mathcal{M}$ is a proper infinite-dimensional subspace of a separable Hilbert space $\mathcal{H}$, then the projection $P_{\mathcal{M}}$ mapping $\mathcal{H}$ onto $\mathcal{M}$ is Murray-von Neumann equivalent to the identity operator I in $B(\mathcal{H})$ but $P_{\mathcal{M}}$ is not unitarily equivalent to the identity operator I in $B(\mathcal{H})$. It is known that two projections P and Q are unitarily equivalent if and only if P and Q are Murray-von Neumann equivalent and $I - P$ and $I - Q$ are Murray-von Neumann equivalent. (see [21]) . Recall that $S \in B(\mathcal{H})$ is a quasi-affine transform of and $T \in B(\mathcal{K})$ if there exists a quasi-affinity $X \in B(\mathcal{H}, \mathcal{K})$ such that $SX = XT$.

Proposition 2.9: Let $\mathrm{S} \in \mathrm{B}(\mathcal{H})$ and $\mathrm{T} \in \mathrm{B}(\mathcal{K})$ be normal if S is a quasi-affine transform of T, then S and T are unitarily equivalent.

Corollary 2.10: Let $\mathrm{S} \in \mathrm{B}(\mathcal{H})$ and $\mathrm{T} \in \mathrm{B}(\mathcal{K})$ be normal. If S is a quasi-affine transform of T, then S and T are approximately unitarily equivalent.

Proof: The proof follows from Proposition 2.9.

Theorem 2.11: Let $\mathrm{S} \in \mathrm{B}(\mathcal{H})$ and $\mathrm{T} \in \mathrm{B}(\mathcal{K})$ be normal. If S is similar T, then S and T are approximately unitarily equivalent.

Proof: From ([18], Proposition 2.13) S and T are unitarily equivalent and hence approximately unitarily equivalent. We note that in finite-dimensional Hilbert spaces $\mathcal{H}$, unitary equivalence and approximate unitary equivalence relations coincide since the unitary group $U(\mathcal{H})$ of $B(\mathcal{H})$ is compact.

Theorem 2.12: Let $\mathrm{T}, \mathrm{S} \in \mathrm{B}(\mathcal{H})$. If the unitary orbit $\mathcal{U}(\mathrm{T})$ is closed and T and S are approximately unitarily equivalent, then they are unitarily equivalent.

We define a new operator relation.

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be *approximately metrically equivalent* (denoted by $A \overset{a.m.e}{\approx} B$) if there exists a sequence $\{U_n\}$ of unitary operators such that $\| A^*A - U_n^*B^*BU_n \| \longrightarrow 0$. This is equivalent to $\lim_{n\to\infty} \| A^*A - U_n^*B^BU_n \| = 0$. Clearly, $A \overset{a.m.e}{\approx} B$ if the sequence $\{U_n^*B^BU_n\}$ converges to A^*A in norm, this is equivalent to $\| Ax \| = \lim_{n\to\infty} \| U_n^*BU_nx \|$ for every $x \in \mathcal{H}$ (equivalently, $A^*A = \lim_{n\to\infty} U_n^*B^*BU_n$.

Theorem 2.13: *Approximate Metric Equivalence is an Equivalence Relation on* $B(\mathcal{H})$. Clearly, the approximate metric equivalence of operators is weaker than that of metric equivalence. In fact, metric equivalence, unitary quasi-equivalence and near equivalence all imply approximate metric equivalence.

3. Approximate Unitary Equivalence and Some Classes of Operators

When two operators are approximately unitarily equivalent, they have many common properties. This equivalence relation can be used to classify some operators ([23]). An operator $T \in B(\mathcal{H})$ is said to be a von Neumann operator if and only if T is approximately unitarily equivalent to an operator of the form $N \oplus A$, where N is normal and $\| f(A) \| \leq \| f(N) \|$, for any rational function f with poles off $\sigma(T)$; and in addition $\sigma(N) = \partial(\sigma(T))$, the topological boundary of $\sigma(T)$. It is shown in [23] that every normaloid operator T is approximately unitarily equivalent to an operator of the form $N \oplus A$, where N is normal and $\| A \| \leq \| N \|$, in addition N can be required to satisfy $\sigma(N) = \{\lambda \in \sigma(T) : |\lambda| = \| T \|\}$. From this definition, it is clear that every normaloid operator is a von Neumann operator. So, normal operators determine the approximate unitary equivalence of every normaloid operator. Recall that normal, quasinormal, seminormal, hyponormal and paranormal operators are normaloid.

Remark: Note that $T \in B(\mathcal{H})$ is normaloid if and only if $\omega(T) = \| T \|$.

Theorem 3.1: *Let $A, B \in B(\mathcal{H})$ be approximately unitarily equivalent. If A is normaloid, then B is also normaloid.*

Corollary 3.2: *Let $A, B \in B(\mathcal{H})$ be approximately unitarily equivalent. If A is paranormal, then B is also paranormal.*

Proof: Since A is paranormal, it is normaloid. The rest of the proof follows from Theorem 3.1.

4. Approximate Unitary Equivalence and the Spectral Picture of Some Operators

Theorem 4.1: *Let $A, B \in B(\mathcal{H})$ be approximately unitarily equivalent. Then $\sigma(A) = \sigma(B)$.*

Theorem 4.2: *Let $A, B \in B(\mathcal{H})$ be approximately unitarily equivalent. Then $\| A \| = \| B \|$.*

Proof: Suppose that $\lim_{n\to\infty} \| A - U_n^*BU_n \| = 0$ for some unitary operators $\{U_n\}$. Then

$$0 \leq |\, \| A \| - \| B \| \,| = \lim_{n\to\infty} |\, \| A \| - \| U_n^*BU_n \| \,| \leq \lim_{n\to\infty} \| A - U_n^*BU_n \| = 0.$$

This proves the claim.

Recall that for any operator $T \in B(\mathcal{H})$, we have that $r(T) \leq \| T \|$.

Theorem 4.3: *Let $A, B \in B(\mathcal{H})$ be approximately unitarily equivalent. Then $r(A) = r(B)$.*

Proof: Suppose that $\lim_{n\to\infty} \| A - U_n^*BU_n \| = 0$ for some unitary operators $\{U_n\}$. Then

$$0 \leq |r(A) - r(B)| = \lim_{n\to\infty} |r(A) - r(U_n^*BU_n)| \leq \lim_{n\to\infty} |\, \| A \| - \| U_n^*BU_n \| \,| \leq \lim_{n\to\infty} \| A - U_n^*BU_n \| = 0.$$

This proves the claim.

Theorem 4.4: *Let $A, B \in B(\mathcal{H})$ be approximately unitarily equivalent. Then $\omega(A) = \omega(B)$.*

Proof: Suppose A and B are approximately unitarily equivalent. Then there exists a sequence $\{U_n\}$ of unitary operators such that

$\| A - U_n^* B U_n \| \longrightarrow 0$ which is equivalent to $\lim_{n\to\infty} \| A - U_n^* B U_n \| = 0$.

Since the numerical radius is a norm and is invariant under unitary equivalence on $B(\mathcal{H})$ ([2], [3]), we have

$$\omega(A) = \omega(A - U_n^* B U_n + U_n^* B U_n) \leq \omega(A - U_n^* B U_n) + \omega(U_n^* B U_n) \leq \lim_{n\to\infty} \| A - U_n^* B U_n \| + \omega(B) = 0 + \omega(B) = \omega(B)$$

since $\lim_{n\to\infty} \| A - U_n^* B U_n \| = 0$. Thus, $\omega(A) \leq \omega(B)$. Since the group of unitary operators is closed-in $B(\mathcal{H})$, we also have a sequence of unitary operators $\lim_{n\to\infty} \| U_n A U_n^* - B \| = 0$. So, by symmetry, it follows that.

$$\omega(B) = \omega(B - U_n A U_n^* + U_n A U_n^*) \leq \omega(B - U_n A U_n^*) + \omega(U_n A U_n^*) \leq \lim_{n\to\infty} \| B - U_n A U_n^* \| + \omega(A) = 0 + \omega(A) = \omega(A)$$

since $\lim_{n\to\infty} \| B - U_n A U_n^* \| = 0$. Thus, $\omega(B) \leq \omega(A)$.
This completes the proof.

Remark: Note that equality of numerical radii does not necessarily imply equality of numerical range. For instance, consider $A = diag(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ and $B = diag(0,1, \frac{1}{2}, \frac{1}{3}, \ldots)$ which are two diagonal operators acting on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$. Clearly, $W(A) = (0,1] \neq [0,1] = W(B)$ although $\omega(A) = \omega(B)$.

Theorem 4.5: *Let $A, B \in B(\mathcal{H})$ be approximately unitarily equivalent. Then $\overline{W(A)} = \overline{W(B)}$.*

Proof: Suppose A and B are approximately unitarily equivalent. Then there exists a sequence $\{U_n\}$ of unitary operators such that $\lim_{n\to\infty} \| A - U_n^* B U_n \| = 0$. Fix an arbitrary $\lambda \in W(A)$. Then, there exists a unit vector $x \in \mathcal{H}$ such that $\lambda = \langle Ax, x \rangle$. Therefore

$$\begin{aligned} \lim_{n\to\infty} |\lambda - \langle BU_n x, U_n x \rangle| &= \lim_{n\to\infty} |\langle Ax, x \rangle - \langle BU_n x, U_n x \rangle| \\ &= \lim_{n\to\infty} |\langle (A - U_n^* B U_n)x, x \rangle| \\ &\leq \lim_{n\to\infty} \| A - U_n^* B U_n \| \| x \|^2 \\ &= 0. \end{aligned}$$

Since $U_n x$ is a unit vector for all $n \in \mathbb{N}$, we have that $\langle BU_n x, U_n x \rangle \in W(B)$ for all $n \in \mathbb{N}$ and so $\lambda = \lim_{n\to\infty} \langle BU_n x, U_n x \rangle \in \overline{W(B)}$. That is $\lambda \in \overline{W(B)}$. Since $\lambda \in W(A)$ it was arbitrary, we conclude that $\overline{W(A)} \subseteq \overline{W(B)}$. The reverse inclusion is then followed by symmetry. This completes the proof.

Remark: Clearly, $W_e(T) \subseteq \overline{W(T)}$ for every $T \in B(\mathcal{H})$.

5. Conclusion

The notion of approximate similarity, approximate unitary equivalence and closely related operator equivalence relations is applicable in quantifying how two systems approximate each other.

Acknowledgments and Funding Statement

The authors sincerely wish to thank the referees for their useful comments and suggestions during the writing of this manuscript. The authors also pay tribute to the Eastern Africa Universities Mathematics Programme(EAUMP)/ICTP/ISP and the University of Nairobi for funding this research.

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