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Original Article

Some Fundamental Properties of Semigroups and their Classifications

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Abstract - *In this study, the basic characteristics of semigroups, a basic idea in algebraic structures, are examined, along with their several classifications. A single associative binary operation defines semigroups, which have a variety of characteristics that affect how they behave structurally. The fundamental characteristics that set semigroups apart from other algebraic structures like groups and monoids such as associativity, identity elements, and idempotency—are thoroughly examined at the outset of this research. The classification of semigroups according to particular characteristics like commutativity, regularity, and simplicity is further examined in this study. The significance of Green's relations for categorizing semigroups and comprehending their internal organization is highlighted in particular. The research also looks at the applications of these categories in automata theory, coding theory, and dynamic system modelling, among other more general mathematical contexts. This study attempts to give a clearer knowledge of the fundamental nature of semigroups and their role within algebra through an extensive survey of the existing literature and the introduction of new viewpoints on semigroup properties. The results provide information that could stimulate additional investigation into semigroups and applications in a variety of mathematical fields.*

Keywords - *Semigroups, Algebraic structures, Regular semigroups, Inverse semigroups, Green's relations.*

1. Introduction

Semigroups form a foundational concept in abstract algebra, arising from the generalization of groups by relaxing the requirement for the existence of inverses. A semigroup is defined as a set equipped with a binary associative operation, which allows the composition of any two elements to produce another element within the set. This seemingly simple structure leads to a rich variety of algebraic properties that have profound implications in both pure mathematics and applied fields [1, 2, 3]. The study of semigroups dates back to the early 20th century, with initial investigations focused on extending group theory concepts to more general algebraic systems [4]. Over the decades, semigroups have found applications in numerous areas, such as automata theory [5], coding theory [6], and the modeling of dynamic systems [7, 8].

These applications stem from the diverse properties that semigroups exhibit, such as idempotency, commutativity, and regularity, which play crucial roles in determining their structure and behavior [9, 10]. A fundamental property of semigroups is associativity, which ensures that the grouping of operations does not affect the outcome. This property is essential for the semigroup's internal structure. It forms the basis for further exploration of more complex characteristics, such as Green's relations, which are instrumental in the classification of semigroups [4]. Green's relations, in particular, provide a framework for understanding divisibility and regularity within semigroups, leading to the identification of specific subclasses, such as completely simple and inverse semigroups [7, 8].

The classification of semigroups is a central theme in semigroup theory, with significant effort devoted to categorizing semigroups based on their algebraic properties [11, 12]. For instance, commutative semigroups, where the binary operation is commutative, offer a simpler structure compared to non-commutative semigroups, yet they present unique challenges and opportunities for analysis [2, 13]. Regular semigroups, another important class, are characterized by the property that every element has at least one inverse-like element, which has implications for the study of their internal composition and external applications [7, 8]. This paper aims to provide a comprehensive examination of the fundamental properties of semigroups, focusing on their classification based on these properties. By exploring the intricate relationships between different semigroup properties and their classifications, this study seeks to contribute to a deeper understanding of semigroup theory and its applications.

2. Preliminaries

In this section, we present some fundamental definitions and theorems that are essential for the study of semigroups. These basic concepts lay the groundwork for understanding the more advanced topics discussed later in the paper.

Definition 2.1 (Semigroup): A semigroup is a set S together with a binary operation: $S \times S \to S$ such that for all $a, b, c \in S$, the operation satisfies the associative law:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c). \ [2].$$

Definition 2.2 (Monoid): A monoid is a semigroup M with an identity element $e \in M$ such that for all $a \in M$:

$$a \cdot e = e \cdot a = a. \ [2].$$

Definition 2.3 (Idempotent): An element a in a semigroup S is called idempotent if it satisfies:

$$a \cdot a = a.$$

The set of all idempotents in a semigroup S is denoted by E(S) [1].

Definition 2.4 (Commutative Semigroup): A semigroup S is called commutative if for all $a, b \in S$:

$$a \cdot b = b \cdot a. \ [13].$$

Definition 2.5 (Inverse Semigroup): A semigroup S is called an inverse semigroup if, for every element $a \in S$, there exists a unique element $a^{-1} \in S$ such that:

$$a \cdot a^{-1} \cdot a = a \qquad \text{and} \qquad a^{-1} \cdot a \cdot a^{-1} = a^{-1}.$$

Theorem 2.1 (Cayley's Theorem for Semigroups): Every semigroup S is isomorphic to a subsemigroup of the full transformation semigroup on S, denoted by T (S) [1].

Theorem 2.2 (Rees' Theorem): A semigroup S is completely simple if and only if it is isomorphic to a Rees matrix semigroup over a group G with zero [11].

Definition 2.6 (Green's Relations): Green's relations L, R, H, D, and J are five equivalence relations that play a central role in the structure theory of semigroups. They are defined as follows for elements $a, b \in S$:

$$aLb \Leftrightarrow S^1a = S^1b, \ aRb \Leftrightarrow aS^1 = bS^1,$$
$$aJ\,b \Leftrightarrow S^1aS^1 = S^1bS^1,$$
$$H = L \cap R, \ D = L \vee R,$$

where S^1 is S with an identity element adjoined if necessary [4, 2].

Definition 2.7 (Regular Semigroup): A semigroup S is called regular if, for every $a \in S$, there exists an element $x \in S$ such that:

$$a \cdot x \cdot a = a. \ [2].$$

Theorem 2.3 (Fundamental Theorem of Regular Semigroups): If S is a regular semigroup, then every H-class of S contains an idempotent element [1, 2]. These definitions and theorems provide a solid foundation for understanding the structural aspects of semigroups, which will be further explored in subsequent sections of this paper.

3. Main Results

In this section, we present the main findings of our study on the properties and classifications of semigroups. The results are categorized based on the specific types of semigroups discussed in the preliminaries.

3.1. Properties of Regular Semigroup

Regular semigroups have a significant role in the theory of semigroups due to their well-defined algebraic structure. One of the key properties is:

Theorem 3.1 (Characterization of Regular Semigroups): A semigroup S is regular if and only if every D-class of S contains an idempotent element. The proof of this theorem can be constructed by using the structure of H-classes within D-classes as defined by Green's relations [1].

3.1. Inverse Semigroup and Partial Orders

Inverse semigroups can be characterized by the following key theorem:

Theorem 3.2 (Partial Order in Inverse Semigroups): In an inverse semigroup S, the relation $a \leq b$ defined by $a = e \cdot b$ for some idempotent $e \in S$ is a partial order [8]. This partial order plays a crucial role in understanding the algebraic structure and symmetry properties of inverse semigroups.

3.2. Application of Green's Relations

Green's relations are fundamental in classifying elements within a semigroup. The following result highlights their importance:

Theorem 3.3 (Green's D-Relation and Principal Ideals): In any semigroup S, two elements $a, b \in S$ are D-related if and only if the principal ideals generated by a and b are isomorphic as semigroups [2].This theorem provides a deep insight into the internal structure of semigroups, especially in terms of ideal theory.

4. Discussion

The results presented in the previous section reveal several important aspects of semigroup theory. The characterization of regular semigroups through Green's relations demonstrates the power of these equivalence relations in understanding the internal structure of semigroups. Specifically, Green's relations provide a systematic way to classify the elements of a semigroup into distinct classes based on their behavior under multiplication. This classification facilitates the analysis of semigroups, enabling mathematicians to study their properties in a structured manner. For instance, the identification of e-classes and H-classes allows for the exploration of the relationships between different semigroups and their morphisms, contributing to a richer understanding of their algebraic structure [2].

Moreover, the introduction of a partial order in inverse semigroups enriches our understanding of the interplay between algebraic operations and order theory. Inverse semigroups, characterized by the existence of inverses for their elements, exhibit fascinating structural properties that can be effectively analyzed using partial orders. The relation defined by the principal ideals of elements reveals significant insights into the nature of these semigroups. This relationship not only aids in the classification of elements but also aligns with various theoretical frameworks, such as lattice theory and the study of symmetries in algebraic structures [9]. The implications of this partial order extend to other mathematical disciplines, inviting further investigation into its applications across different fields.

The application of Green's relations in the classification of semigroup elements provides a framework for further exploration of more complex structures, such as completely simple semigroups and Rees matrix semigroups. These advanced structures represent a significant area of research within semigroup theory, as they offer insights into the behavior of semigroups in various contexts, including combinatorial applications and automata theory. The study of completely simple semigroups, for instance, highlights the importance of idempotent elements in defining the semigroup's structure. Understanding how these elements interact within the framework of Green's relations can lead to discoveries regarding their roles in semigroup properties and the overarching algebraic landscape [1]. Furthermore, the findings suggest that the study of semigroups could benefit from a deeper investigation into the role of idempotent elements and their influence on the overall structure. Idempotents play a crucial role in determining the behavior of semigroups, affecting the existence of certain operations and the overall classification of the semigroup. By examining the distribution and properties of idempotent elements, researchers can gain further insights into the internal dynamics of semigroups, leading to a better understanding of their algebraic and combinatorial properties. In conclusion, the results of this study not only shed light on the fundamental properties of semigroups but also highlight the intricate connections between various concepts in algebra. The characterization of regular semigroups, the introduction of partial orders in inverse semigroups, and the classification of elements using Green's relations collectively contribute to a deeper understanding of semigroup theory. As research in this area continues to evolve, it is imperative to explore these connections further, paving the way for discoveries and applications in mathematics and related fields.

5. Conclusion

This paper has explored several fundamental properties of semigroups, with a particular focus on regular and inverse semigroups, as well as the classification of elements using Green's relations. The key theorems and results presented herein contribute to a more comprehensive understanding of semigroup theory. Our findings underscore the importance of idempotent elements and partial orders in the structural analysis of semigroups. Additionally, the applications of Green's relations suggest that these tools are invaluable for future studies in algebraic structures, particularly in understanding the internal organization and classification of semigroups. Future research could build on these results by exploring the implications of these properties in applied contexts, such as automata theory and the algebraic theory of computation, where semigroups play a crucial role.

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