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Original Article

The Stability of Three-dimensional Incompressible MHD with Mixed Dissipation

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Abstract - *In this paper, we investigate the global well-posedness of the three-dimensional (3D) incompressible magnetohydrodynamics (MHD) system with mixed dissipation. More precisely, when the first and third components of velocity have only one direction dissipation and the second component of velocity and the magnetic field have two direction dissipation, the MHD system is stable.*

Keywords - *Global well-posedness, Magnetohydrodynamics, Mixed dissipation.*

1. Introduction

The MHD system is a coupled system by the Navier-Stokes equations of fluid dynamics and the Maxwell's equations of electromagnetism, which is used to simulate the motion of electrically conducting fluids in the magnetic field[9,17]. Extensive physical experiments and numerical simulations have shown an important phenomenon: a background magnetic field can actually stabilize and damp electrically conducting fluids[1-3,6,15]. This paper considers the following MHD system:

$$\begin{cases} \partial_t u + u\cdot\nabla u - b\cdot\nabla b + \nabla p = \begin{bmatrix} \partial_{33} \\ \partial_{11}+\partial_{33} \\ \partial_{11} \end{bmatrix} u + \partial_2 b, \\ \partial_t b + u\cdot\nabla b - b\cdot\nabla u = (\partial_1^2+\partial_2^2) b + \partial_2 u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ (u,b)|_{t=0} = (u_0, b_0), \end{cases} \tag{1,1}$$

where $x \in \mathbb{R}^3, t \geq 0$. The unknown function $u=(u_1,u_2,u_3)(x,t), b=(b_1,b_2,b_3)(x,t)$ and $p(x,t)$ are velocity, magnetic field, and pressure, respectively. Many efforts are focused on the MHD equations with partial dissipation or fractional dissipation to determine whether they possess global well-posedness and stability. Duvet-Lions[11] first established the global existence of classical solutions to the 2D MHD equations for initial data $(u_0,b_0)\in H^s(\mathbb{R}^2)(s>2)$. For the anisotropic incompressible MHD system, Cao-Wu[8] first proved the global existence of the classical solutions. Recently, Lin-Xu-Zhang [21] used the Lagrangian approach to deal with the stability problem of the 2D MHD system only with partial dissipation. Boardman-Lin-Wu[5] established the stability of the 2D MHD system while the velocity field involves only one direction damping. Some of the significant progress for the perturbations of the MHD system near the equilibrium state can be found in[4,7,10,13,14,18]. However, the results of stability for the 3D anisotropic incompressible MHD system are very rare.

For the ideal 3D MHD, Cai-Lei[7] and He-Xu-Yu[16] used a different method to establish the stability result. Later, Wei-Zhang[25] found that the viscosity and resistivity can be slightly different. For the 3D viscous and zero-resistive MHD, the stability results can be seen[12,23,24]. It is worth noting that the first stability result of the 3D MHD system with mixed dissipation was established by Wu-Zhu[26]. They found that the velocity only needs dissipation in the horizontal direction and the magnetic field only needs dissipation in the vertical direction. However, the work is more difficult when the velocity field dissipates in only one direction. Lin-Wu-Zhu[19] first proved the stability of the 3D MHD system with only one direction viscosity in space $H^4(\mathbb{R}^3)$. Then, they[19] improved their result and obtained the large-time behavior. Recently, Lai-Wu-

Zhang-Zhao[20] established the optimal decay rate in $H^4(\mathbb{R}^3)$ for the 3D MHD system with only one direction viscosity. Moreover, they also proved the stability of this system in $H^3(\mathbb{R}^3)$ by means of the structure of the velocity equation and magnetic equation.

Inspired of the result above, our main results can then be stated as follows.

Theorem 1.1 Consider (1.1) with the initial data $(u_0, b_0) \in H^2(\mathbb{R}^2)$ satisfies $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then there exists a positive constant $\varepsilon > 0$, such that if

$$\|(u_0, b_0)\|_{H^2} \leq \varepsilon,$$

then the system (1.1) has a unique global solution for any $t > 0$, satisfying

$$\|(u,b)(t)\|_{H^2}^2 + \int_0^t (\|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^2}^2 + \|(\partial_1 b, \partial_2 b)\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2) d\tau \leq C\varepsilon^2, \tag{1.2}$$

where $C > 0$ is a generic positive constant independent of ε and t.

The rest of this paper is divided into two sections. Section 2 presents several tool lemmas to be used in the proof of Theorem 1.1, and the Theorem 1.1 will be completed in Section 3.

2. Preliminaries

In this section, we provide two lemmas that will be very important in subsequent proofs. By using Lemma 2.1 and Lemma 2.2 can help us to treat the difficulty caused by the absence of dissipation.

Lemma 2.1 ([19]) Assume $f, \partial_1 f, g, \partial_2 g, h$ and $\partial_3 h$ all in $L^2(\mathbb{R}^3)$, it holds that

$$\int |fgh|\, dx \leq C\| f\|_{L^2}^{1/2} \|\partial_1 f\|_{L^2}^{1/2} \|g\|_{L^2}^{1/2} \|\partial_2 g\|_{L^2}^{1/2} \|h\|_{L^2}^{1/2} \|\partial_3 h\|_{L^2}^{1/2}.$$

Lemma 2.2([26]) The following estimates hold when the right-hand sides are all bounded in $\mathbb{R}^3$, we have

$$\int |fgh|\, dx \leq C\| f\|_{L^2}^{1/4} \|\partial_i f\|_{L^2}^{1/4} \|\partial_j f\|_{L^2}^{1/4} \|\partial_i \partial_j f\|_{L^2}^{1/4} \|g\|_{L^2}^{1/2} \|\partial_k g\|_{L^2}^{1/2} \|h\|_{L^2},$$

where $i, j, k \in 1,2,3$ and $i \neq j \neq k$.

3. The global Well-Posedness

The main purpose of this section is to prove Theorem 1.1.

3.1. L^2 *estimate of* (u,b)

Take the L^2-inner product of (1.1) with (u,b) to obtain

$$\frac{1}{2}\frac{d}{dt}\|(u,b)\|_{L^2}^2 + \|\partial_3 u_1\|_{L^2}^2 + \|(\partial_1, \partial_3) u_2\|_{L^2}^2 + \|\partial_1 u_3\|_{L^2}^2 + \|(\partial_1, \partial_2) b\|_{L^2}^2 = 0. \tag{3.1}$$

3.2. $\dot{H}^2$ *estimate of* (u,b)

Applying $\partial_i^2 (i = 1,2,3)$ to (1.1) and dotting them with $(\partial_i^2 u, \partial_i^2 b)$ in L^2, one can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3} \|(\partial_i^2 u, \partial_i^2 b)\|_{L^2}^2 + \sum_{i=1}^{3} \|\partial_3 \partial_i^2 u_1\|_{L^2}^2 + \sum_{i=1}^{3} \|(\partial_1 \partial_i^2, \partial_3 \partial_i^2) u_2\|_{L^2}^2 \\
&\quad + \sum_{i=1}^{3} \|\partial_1 \partial_i^2 u_3\|_{L^2}^2 + \sum_{i=1}^{3} \|(\partial_1 \partial_i^2, \partial_2 \partial_i^2) b\|_{L^2}^2 \\
&= -\sum_{i=1}^{3} \int \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 u dx + \sum_{i=1}^{3} \int \partial_i^2 (b \cdot \nabla b) \cdot \partial_i^2 u dx \\
&\quad - \sum_{i=1}^{3} \int \partial_i^2 (u \cdot \nabla b) \cdot \partial_i^2 b dx + \sum_{i=1}^{3} \int \partial_i^2 (b \cdot \nabla u) \cdot \partial_i^2 b dx \\
&:= I_1 + I_2 + I_3 + I_4. \qquad (3.2)
\end{aligned}$$

Due to the Newton-Leibniz formula and the fact of $\nabla \cdot u = 0$, it follows

$$I_1 = -\sum_{i=1}^{3}\int \partial_i^2 u\cdot\nabla u\cdot\partial_i^2 u dx - 2\sum_{i=1}^{3}\int \partial_i u\cdot\partial_i\nabla u\cdot\partial_i^2 u dx := I_{11} + I_{12},$$

where

$$I_{11} = -\int \partial_1^2 u\cdot\nabla u\cdot\partial_1^2 u dx - \int \partial_2^2 u\cdot\nabla u\cdot\partial_2^2 u dx - \int \partial_3^2 u\cdot\nabla u\cdot\partial_3^2 u dx := I_{111} + I_{112} + I_{113}.$$

For the first item I_{111}, by $\nabla\cdot u = 0$ and Sobolev's inequality, we have

$$\begin{aligned}
I_{111} &= -\sum_{j=2}^{3}\sum_{k=2}^{2}\int \partial_1^2 u_j\partial_j u_k\partial_1^2 u_k dx - \sum_{j=2}^{3}\int \partial_1^2 u_j\partial_j u_1\partial_1^2 u_1 dx \\
&\quad -\sum_{k=2}^{3}\int \partial_1^2 u_1\partial_1 u_k\partial_1^2 u_k dx - \int \partial_1^2 u_1\partial_1 u_1\partial_1^2 u_1 dx \\
&\le C\sum_{j=2}^{3}\sum_{k=2}^{3}\|\partial_1^2 u_j\|_{L^4}\|\partial_j u_k\|_{L^2}\|\partial_1^2 u_k\|_{L^4} + C\sum_{j=2}^{3}\|\partial_1^2 u_j\|_{L^4}\|\partial_j u_1\|_{L^2}\|\partial_1^2 u_1\|_{L^4} \\
&\quad +C\sum_{k=2}^{3}\|\partial_1^2 u_1\|_{L^4}\|\partial_1 u_k\|_{L^2}\|\partial_1^2 u_k\|_{L^4} + C\|\partial_1^2 u_1\|_{L^4}\|\partial_1 u_1\|_{L^2}\|\partial_1^2 u_1\|_{L^4} \\
&\le C\|u\|_{H^2}\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2.
\end{aligned}$$

Then, to estimate I_{112}, by $\nabla\cdot u = 0$, Lemma 2.1 and Lemma 2.2, one has

$$\begin{aligned}
I_{112} &= -\sum_{k=2}^{3}\int \partial_2^2 u_1\partial_1 u_k\partial_2^2 u_k dx - \int \partial_2^2 u_1\partial_1 u_1\partial_2^2 u_1 dx - \sum_{k=2}^{3}\int \partial_2^2 u_2\partial_2 u_k\partial_2^2 u_k dx \\
&\quad -\int \partial_2^2 u_2\partial_2 u_1\partial_2^2 u_1 dx - \sum_{k=1}^{2}\int \partial_2^2 u_3\partial_3 u_k\partial_2^2 u_k dx - \int \partial_2^2 u_3\partial_3 u_3\partial_2^2 u_3 dx \\
&\le C\sum_{k=2}^{3}\|\partial_2^2 u_1\|_{L^2}^{1/2}\|\partial_3\partial_2^2 u_1\|_{L^2}^{1/2}\|\partial_1 u_k\|_{L^2}^{1/2}\|\partial_2\partial_1 u_k\|_{L^2}^{1/2}\|\partial_2^2 u_k\|_{L^2}^{1/2}\|\partial_1\partial_2^2 u_k\|_{L^2}^{1/2} \\
&\quad +C\|\partial_2^2 u_1\|_{L^2}^{3/2}\|\partial_3\partial_2^2 u_1\|_{L^2}^{1/2}\|\partial_1 u_1\|_{L^2}^{1/4}\|\partial_1^2 u_1\|_{L^2}^{1/4}\|\partial_2\partial_1 u_1\|_{L^2}^{1/4}\|\partial_1^2\partial_2 u_1\|_{L^2}^{1/4} \\
&\quad +C\sum_{k=2}^{3}\|\partial_2^2 u_2\|_{L^2}^{1/2}\|\partial_1\partial_2^2 u_2\|_{L^2}^{1/2}\|\partial_2 u_k\|_{L^2}^{1/2}\|\partial_2^2 u_k\|_{L^2}\|\partial_3\partial_2^2 u_k\|_{L^2}^{1/2} \\
&\quad +C\|\partial_2^2 u_2\|_{L^2}^{1/2}\|\partial_1\partial_2^2 u_2\|_{L^2}^{1/2}\|\partial_2 u_1\|_{L^2}^{1/2}\|\partial_2^2 u_1\|_{L^2}\|\partial_3\partial_2^2 u_1\|_{L^2}^{1/2} \\
&\quad +C\sum_{k=1}^{2}\|\partial_2^2 u_3\|_{L^2}^{1/2}\|\partial_1\partial_2^2 u_3\|_{L^2}^{1/2}\|\partial_3 u_k\|_{L^2}^{1/2}\|\partial_2\partial_3 u_k\|_{L^2}^{1/2}\|\partial_2^2 u_j\|_{L^2}^{1/2}\|\partial_3\partial_2^2 u_k\|_{L^2}^{1/2} \\
&\quad +C\|\partial_2^2 u_3\|_{L^2}^{3/2}\|\partial_1\partial_2^2 u_3\|_{L^2}^{1/2}\|\partial_3 u_3\|_{L^2}^{1/4}\|\partial_2\partial_3 u_3\|_{L^2}^{1/4}\|\partial_3^2 u_3\|_{L^2}^{1/4}\|\partial_2\partial_3^2 u_3\|_{L^2}^{1/4} \\
&\le C\|u\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2).
\end{aligned}$$

Similarly as I_{111},

$$I_{113} \le C\|u\|_{H^2}\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2.$$

Thus,

$$I_{11} \le C\|u\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2).$$

In a similar manner, it is easily seen that

$$I_{12} = -2\int \partial_1 u \cdot \partial_1 \nabla u \cdot \partial_1^2 u dx - 2\int \partial_2 u \cdot \partial_2 \nabla u \cdot \partial_2^2 u dx - 2\int \partial_3 u \cdot \partial_3 \nabla u \cdot \partial_3^2 u dx$$
$$:= I_{121} + I_{122} + I_{123}.$$

Thanks to $\nabla \cdot u = 0$, we can conclude that

$$I_{121} = -\sum_{k=2}^{3} \int \partial_1 u \cdot \partial_1 \nabla u_k \partial_1^2 u_k dx - \int \partial_1 u_1 \partial_1^2 u_1 \partial_1^2 u_1 dx - \sum_{j=2}^{3} \int \partial_1 u_j \partial_1 \partial_j u_1 \partial_1^2 u_1 dx$$
$$\le C\sum_{k=2}^{3} \|\partial_1 u\|_{L^2} \|\partial_1 \nabla u_k\|_{L^4} \|\partial_1^2 u_k\|_{L^4} + C\|\partial_1 u_1\|_{L^2} \|\partial_1^2 u_1\|_{L^4}^2$$
$$+C\sum_{j=2}^{3} \|\partial_1 u_j\|_{L^2} \|\partial_j \partial_1 u_1\|_{L^4} \|\partial_1^2 u_1\|_{L^4}$$
$$\le C\|u\|_{H^2} \|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^2}^2.$$

For the term I_{122}, we rewrite it as

$$I_{122} = -2(\int \partial_2 u_1 \partial_2 \partial_1 u_1 \partial_2^2 u_1 dx + \sum_{k=2}^{3} \int \partial_2 u_1 \partial_2 \partial_1 u_k \partial_2^2 u_k dx)$$
$$-2\int \partial_2 u_2 (\partial_2^2 u_1 \partial_2^2 u_1 + \partial_2^2 u_2 \partial_2^2 u_2 + \partial_2^2 u_3 \partial_2^2 u_3) dx$$
$$-2(\sum_{k=1}^{2} \int \partial_2 u_3 \partial_2 \partial_3 u_k \partial_2^2 u_k dx + \int \partial_2 u_3 \partial_2 \partial_3 u_3 \partial_2^2 u_3 dx)$$
$$:= I_{1221} + I_{1222} + I_{1223}.$$

By Lemma 2.1 and $\nabla \cdot u = 0$,

$$I_{1221} = -2\int \partial_2 u_1 \partial_2 \partial_1 u_1 \partial_2^2 u_1 dx - 2\sum_{k=2}^{3} \int \partial_2 u_1 \partial_2 \partial_1 u_k \partial_2^2 u_k dx$$
$$\le C\| \partial_2 u_1\|_{L^2}^{1/2} \| \partial_2^2 u_1\|_{L^2}^{1/2} \| \partial_2 \partial_1 u_1\|_{L^2}^{1/2} \| \partial_1^2 \partial_2 u_1\|_{L^2}^{1/2} \| \partial_2^2 u_1\|_{L^2}^{1/2} \| \partial_3 \partial_2^2 u_1\|_{L^2}^{1/2}$$
$$+C\sum_{k=2}^{3} \| \partial_2 u_1\|_{L^4} \| \partial_1 \partial_2 u_k\|_{L^4} \| \partial_2^2 u_k\|_{L^2}$$
$$\le C\| u\|_{H^2} (\|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^2}^2 + \| \partial_2 u\|_{H^1}^2).$$

To estimate the term I_{1222}, by Lemma 2.1 and Lemma 2.2, one can obtain

$$I_{1222} = -2\int \partial_2 u_2 \partial_2^2 u_1 \partial_2^2 u_1 dx - 2\int \partial_2 u_2 \partial_2^2 u_2 \partial_2^2 u_2 dx - 2\int \partial_2 u_2 \partial_2^2 u_3 \partial_2^2 u_3 dx$$
$$\le C\| \partial_2^2 u_1\|_{L^2}^{3/2} \| \partial_3 \partial_2^2 u_1\|_{L^2}^{1/2} \| \partial_2 u_2\|_{L^2}^{1/4} \| \partial_1 \partial_2 u_2\|_{L^2}^{1/4} \| \partial_2^2 u_2\|_{L^2}^{1/4} \| \partial_1 \partial_2^2 u_2\|_{L^2}^{1/4}$$
$$+C\| \partial_2 u_2\|_{L^2}^{1/2} \| \partial_2^2 u_2\|_{L^2}^{3/2} \| \partial_1 \partial_2^2 u_2\|_{L^2}^{1/2} \| \partial_3 \partial_2^2 u_2\|_{L^2}^{1/2}$$
$$+C\| \partial_2^2 u_3\|_{L^2}^{3/2} \| \partial_1 \partial_2^2 u_3\|_{L^2}^{1/2} \| \partial_2 u_2\|_{L^2}^{1/4} \| \partial_2^2 u_2\|_{L^2}^{1/4} \| \partial_3 \partial_2 u_2\|_{L^2}^{1/4} \| \partial_3 \partial_2^2 u_2\|_{L^2}^{1/4}$$
$$\le C\| u\|_{H^2} (\|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^2}^2 + \| \partial_2 u\|_{H^1}^2).$$

For the term I_{1223}, similar as I_{1221}, we have

$$\begin{aligned}I_{1223} &= -2\sum_{k=1}^{2}\int \partial_2 u_3 \partial_2 \partial_3 u_k \partial_2^2 u_k dx - 2\int \partial_2 u_3 \partial_2 \partial_3 u_3 \partial_2^2 u_3 dx \\ &\le C\sum_{k=1}^{2} \| \partial_2 u_3\|_{L^4} \| \partial_3 \partial_2 u_k\|_{L^4} \| \partial_2^2 u_k\|_{L^2} \\ &\quad + C\| \partial_2 u_3\|_{L^2}^{1/2} \| \partial_2^2 u_3\|_{L^2}^{1/2} \| \partial_2 \partial_3 u_3\|_{L^2}^{1/2} \| \partial_3^2 \partial_2 u_3\|_{L^2}^{1/2} \| \partial_2^2 u_3\|_{L^2}^{1/2} \| \partial_1 \partial_2^2 u_3\|_{L^2}^{1/2} \\ &\le C\| u\|_{H^2} (\|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^2}^2 + \| \partial_2 u\|_{H^1}^2).\end{aligned}$$

Therefore,

$$I_{12} \le C\| u\|_{H^2} (\|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^2}^2 + \| \partial_2 u\|_{H^1}^2).$$

And

$$I_1 \le C\| u\|_{H^2} (\|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^2}^2 + \| \partial_2 u\|_{H^1}^2). \tag{3.3}$$

Due to the Newton-Leibniz formula and $\nabla \cdot b = 0$, we estimate I_2, I_4 together,

$$\begin{aligned}I_2 + I_4 &= \sum_{i=1}^{3}\int \partial_i^2 b \cdot \nabla b \cdot \partial_i^2 u dx + 2\sum_{i=1}^{3}\int \partial_i b \cdot \partial_i \nabla b \cdot \partial_i^2 u dx \\ &\quad + \sum_{i=1}^{3}\int \partial_i^2 b \cdot \nabla u \cdot \partial_i^2 b dx + 2\sum_{i=1}^{3}\int \partial_i b \cdot \partial_i \nabla u \cdot \partial_i^2 b dx \\ &:= I_{21} + I_{22} + I_{41} + I_{42}.\end{aligned}$$

To bind the term I_{21}, we do the decomposition

$$I_{21} = \sum_{i=1}^{2}\int \partial_i^2 b \cdot \nabla b \cdot \partial_i^2 u dx + \int \partial_3^2 b \cdot \nabla b \cdot \partial_3^2 u dx = I_{211} + I_{212}.$$

By Lemma 2.1, we can infer

$$\begin{aligned}I_{211} &= \sum_{i=1}^{2}\sum_{j=1}^{2}\int \partial_i^2 b_j \partial_j b \cdot \partial_i^2 u dx + \sum_{i=1}^{2}\sum_{k=2}^{3}\int \partial_i^2 b_3 \partial_3 b_k \partial_i^2 u_k dx + \sum_{i=1}^{2}\int \partial_i^2 b_3 \partial_3 b_1 \partial_i^2 u_1 dx \\ &\le C\sum_{i=1}^{2}\sum_{j=1}^{2} \|\partial_i^2 b_j\|_{L^4} \|\partial_j b\|_{L^4} \|\partial_i^2 u\|_{L^2} + C\sum_{i=1}^{2}\sum_{k=2}^{3} \|\partial_i^2 b_3\|_{L^4} \|\partial_3 b_k\|_{L^4} \|\partial_i^2 u_k\|_{L^2} \\ &\quad + C\sum_{i=1}^{2} \|\partial_i^2 b_3\|_{L^2}^{1/2} \|\partial_2 \partial_i^2 b_3\|_{L^2}^{1/2} \|\partial_3 b_1\|_{L^2}^{1/2} \|\partial_1 \partial_3 b_1\|_{L^2}^{1/2} \|\partial_i^2 u_1\|_{L^2}^{1/2} \|\partial_3 \partial_i^2 u_1\|_{L^2}^{1/2} \\ &\le C\|(u,b)\|_{H^2} (\|\partial_3 u_1\|_{H^2}^2 + \|(\partial_1, \partial_2) b\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2).\end{aligned}$$

For the term I_{212} ,

$$\begin{aligned}I_{212} &= \sum_{j=1}^{2}\sum_{k=1}^{2}\int \partial_3^2 b_j \partial_j b_k \partial_3^2 u_k dx + \int \partial_3^2 b_3 \partial_3 b_3 \partial_3^2 u_3 dx + \sum_{k=1}^{2}\int \partial_3^2 b_3 \partial_3 b_k \partial_3^2 u_k dx + \sum_{j=1}^{2}\int \partial_3^2 b_j \partial_j b_k \partial_3^2 u_3 dx \\ &\le C\sum_{j=1}^{2}\sum_{k=1}^{2} \|\partial_3^2 b_j\|_{L^2} \|\partial_j b_k\|_{L^4} \|\partial_3^2 u_k\|_{L^4} + C\|\partial_3^2 b_3\|_{L^4} \|\partial_3 b_3\|_{L^2} \|\partial_3^2 u_3\|_{L^4}\end{aligned}$$

$$+C\sum_{k=1}^{2}\|\partial_3^2 b_3\|_{L^2}^{1/2}\|\partial_1\partial_3^2 b_3\|_{L^2}^{1/2}\|\partial_3 b_k\|_{L^2}^{1/2}\|\partial_2\partial_3 b_k\|_{L^2}^{1/2}\|\partial_3^2 u_k\|_{L^2}^{1/2}\|\partial_3^3 u_k\|_{L^2}^{1/2}$$

$$+C\sum_{j=1}^{2}\|\partial_3^2 b_j\|_{L^2}^{1/2}\|\partial_1\partial_3^2 b_j\|_{L^2}^{1/2}\|\partial_j b_k\|_{L^2}^{1/2}\|\partial_2\partial_j b_k\|_{L^2}^{1/2}\|\partial_3^2 u_3\|_{L^2}^{1/2}\|\partial_3^3 u_3\|_{L^2}^{1/2}$$

$$\leq C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2).$$

Therefore,

$$I_{21}\leq C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2).$$

Similarly,

$$I_{22}\leq C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2).$$

For the term I_{41}, by Lemma 2.1 and $\nabla\cdot u=0$, we get

$$\begin{aligned}
I_{41}&=\sum_{i=1}^{2}\int\partial_i^2 b\cdot\nabla u\cdot\partial_i^2 b dx+\sum_{k=2}^{3}\int\partial_3^2 b_1\partial_1 u_k\partial_3^2 b_k dx+\int\partial_3^2 b_1\partial_1 u_1\partial_3^2 b_1 dx\\
&\quad+\int\partial_3^2 b_2\partial_2 u\cdot\partial_3^2 b dx+\sum_{k=1}^{2}\int\partial_3^2 b_3\partial_3 u_k\partial_3^2 b_k dx+\int\partial_3^2 b_3\partial_3 u_3\partial_3^2 b_3 dx\\
&\leq C\sum_{i=1}^{2}\|\nabla u\|_{L^2}\|\partial_i^2 b\|_{L^4}^2+\int\partial_3^2 b_1\partial_1 u_1\partial_3^2 b_1 dx\\
&\quad+C\sum_{k=2}^{3}\|\partial_3^2 b_1\|_{L^2}^{1/2}\|\partial_1\partial_3^2 b_1\|_{L^2}^{1/2}\|\partial_1 u_k\|_{L^2}^{1/2}\|\partial_1\partial_3 u_k\|_{L^2}^{1/2}\|\partial_3^2 b_k\|_{L^2}^{1/2}\|\partial_2\partial_3^2 b_k\|_{L^2}^{1/2}\\
&\quad+C\|\partial_3^2 b_2\|_{L^2}^{1/2}\|\partial_1\partial_3^2 b_2\|_{L^2}^{1/2}\|\partial_2 u\|_{L^2}^{1/2}\|\partial_2\partial_3 u\|_{L^2}^{1/2}\|\partial_3^2 b\|_{L^2}^{1/2}\|\partial_2\partial_3^2 b\|_{L^2}^{1/2}\\
&\quad+C\sum_{k=1}^{2}\|\partial_3^2 b_3\|_{L^4}\|\partial_3 u_k\|_{L^4}\|\partial_3^2 b_k\|_{L^2}+C\|\partial_3 u_3\|_{L^2}\|\partial_3^2 b_3\|_{L^4}^2\\
&\leq C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2)+I_{413},
\end{aligned}$$

by integration by parts and Lemma 2.2, where

$$\begin{aligned}
I_{413}&=\int\partial_3^2 b_1\partial_1 u_1\partial_3^2 b_1 dx=-2\int u_1\partial_1\partial_3^2 b_1\partial_3^2 b_1 dx\\
&\leq C\|\partial_1\partial_3^2 b_1\|_{L^2}^{3/2}\|\partial_3^2 b_1\|_{L^2}^{1/2}\|u_1\|_{L^2}^{1/4}\|\partial_2 u_1\|_{L^2}^{1/4}\|\partial_3 u_1\|_{L^2}^{1/4}\|\partial_2\partial_3 u_1\|_{L^2}^{1/4}\\
&\leq C\|(u,b)\|_{H^2}(\|\partial_1 b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2).
\end{aligned}$$

Thus,

$$I_{41}\leq C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2).$$

By Lemma 2.1, we find

$$
\begin{aligned}
I_{42} &= \sum_{i=1}^{2}\int \partial_i b\cdot\partial_i\nabla u\cdot\partial_i^2 b dx+\sum_{k=1}^{2}\int \partial_3 b\cdot\partial_3\nabla u_k\partial_3^2 b_k dx+\int \partial_3 b\cdot\partial_3\nabla u_3\partial_3^2 b_3 dx\\
&\le C\sum_{i=1}^{2}\|\partial_i b\|_{L^\infty}\|\partial_i\nabla u\|_{L^2}\|\partial_i^2 b\|_{L^2}\\
&\quad +C\sum_{k=1}^{2}\|\partial_3 b\|_{L^2}^{1/2}\|\partial_1\partial_3 b\|_{L^2}^{1/2}\|\partial_3\nabla u_k\|_{L^2}^{1/2}\|\partial_3^2\nabla u_k\|_{L^2}^{1/2}\|\partial_3^2 b_k\|_{L^2}^{1/2}\|\partial_2\partial_3^2 b_k\|_{L^2}^{1/2}\\
&\quad +C\|\partial_3 b\|_{L^2}^{1/2}\|\partial_2\partial_3 b\|_{L^2}^{1/2}\|\partial_3\nabla u_3\|_{L^2}^{1/2}\|\partial_1\partial_3\nabla u_3\|_{L^2}^{1/2}\|\partial_3^2 b_3\|_{L^2}^{1/2}\|\partial_3^3 b_3\|_{L^2}^{1/2}\\
&\le C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1 b,\partial_2 b)\|_{H^2}^2).
\end{aligned}
$$

Therefore,

$$
I_2+I_4\le C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2)\,. \tag{3.4}
$$

To establish the estimate of I_3, using the divergence-free condition of b, we find

$$
I_3=-\sum_{i=1}^{3}\int \partial_i^2 u\cdot\nabla b\cdot\partial_i^2 b dx-2\sum_{i=1}^{3}\int \partial_i u\cdot\partial_i\nabla b\cdot\partial_i^2 b dx:=I_{31}+I_{32}.
$$

By Lemma 2.1,

$$
\begin{aligned}
I_{31} &= -\sum_{j=1}^{3}\int \partial_1^2 u_j\partial_j b\cdot\partial_1^2 b dx-\int \partial_2^2 u\cdot\nabla b\cdot\partial_2^2 b dx-\sum_{j=1}^{2}\int \partial_3^2 u_j\partial_j b\cdot\partial_3^2 b dx-\int \partial_3^2 u_3\partial_3 b\cdot\partial_3^2 b dx\\
&\le C\sum_{j=1}^{3}\|\partial_1^2 u_j\|_{L^2}\|\partial_j b\|_{L^2}\|\partial_1^2 b\|_{L^4}+C\|\partial_2^2 u\|_{L^2}\|\nabla b\|_{L^4}\|\partial_2^2 b\|_{L^4}+C\sum_{j=1}^{2}\|\partial_3^2 u_j\|_{L^4}\|\partial_j b\|_{L^4}\|\partial_3^2 b\|_{L^2}\\
&\quad +C\|\partial_3^2 u_3\|_{L^2}^{1/2}\|\partial_3^3 u_3\|_{L^2}^{1/2}\|\partial_3 b\|_{L^2}^{1/2}\|\partial_1\partial_3 b\|_{L^2}^{1/2}\|\partial_3^2 b\|_{L^2}^{1/2}\|\partial_2\partial_3^2 b\|_{L^2}^{1/2}\\
&\le C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2).
\end{aligned}
$$

Next, we need to bound I_{32},

$$
\begin{aligned}
I_{32} &= -2\sum_{i=1}^{2}\int \partial_i u\cdot\partial_i\nabla b\cdot\partial_i^2 b dx-2\sum_{j=1}^{2}\int \partial_3 u_j\partial_3\partial_j b\cdot\partial_3^2 b dx-2\int \partial_3 u_3\partial_3^2 b\cdot\partial_3^2 b dx\\
&\le C\sum_{i=1}^{2}\|\partial_i u\|_{L^2}\|\partial_i\nabla b\|_{L^4}\|\partial_i^2 b\|_{L^4}+C\sum_{j=1}^{2}\|\partial_3 u_j\|_{L^4}\|\partial_j\partial_3 b\|_{L^4}\|\partial_3^2 b\|_{L^2}+I_{323}\\
&\le C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_3 u_2)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2)+I_{323},
\end{aligned}
$$

by integration by parts and Lemma 2.2, where

$$
\begin{aligned}
I_{323} &= -2\int \partial_3 u_3\partial_3^2 b\cdot\partial_3^2 b dx=-4\int u_1\partial_3^2 b\cdot\partial_1\partial_3^2 b dx-4\int u_2\partial_3^2 b\cdot\partial_2\partial_3^2 b dx\\
&\le C\sum_{j=1}^{2}\|\partial_j\partial_3^2 b\|_{L^2}\|\partial_3^2 b\|_{L^2}^{1/2}\|\partial_1\partial_3^2 b\|_{L^2}^{1/2}\|u_j\|_{L^2}^{1/4}\|\partial_2 u_j\|_{L^2}^{1/4}\|\partial_3 u_j\|_{L^2}^{1/4}\|\partial_2\partial_3 u_j\|_{L^2}^{1/4}\\
&\le C\|(u,b)\|_{H^2}(\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2).
\end{aligned}
$$

Then, we can conclude that

$$
I_3\le C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2). \tag{3.5}
$$

Adding the estimates (3.3), (3.4) and (3.5) into (3.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\|(\partial_i^2 u,\partial_i^2 b)\|_{L^2}^2+\sum_{i=1}^{3}\|\partial_3\partial_i^2 u_1\|_{L^2}^2+\sum_{i=1}^{3}\|(\partial_1\partial_i^2,\partial_3\partial_i^2)u_2\|_{L^2}^2$$
$$+\sum_{i=1}^{3}\|\partial_1\partial_i^2 u_3\|_{L^2}^2+\sum_{i=1}^{3}\|(\partial_1\partial_i^2,\partial_2\partial_i^2)b\|_{L^2}^2$$
$$\leq C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2+\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2). \tag{3.6}$$

3.3. L^2 estimate of $\partial_2 u$.

First, multiplying $(1.1)_2$ by $\partial_2 u$ and integrating over $\mathbb{R}^3$, it follows

$$\|\partial_2 u\|_{L^2}^2=\int\partial_2 u\cdot\partial_t b dx+\int u\cdot\nabla b\cdot\partial_2 u dx-\int b\cdot\nabla u\cdot\partial_2 u dx-\int(\partial_1^2+\partial_2^2)b\cdot\partial_2 u dx$$
$$:=J_1+J_2+J_3+J_4.$$

Applying the velocity equation in $(1.1)_1$ and the divergence-free condition of b ,

$$J_1=\frac{d}{dt}\int\partial_2 u\cdot b dx-\int\partial_2\left(\partial_2 b+\begin{bmatrix}\partial_{33}\\ \partial_{11}+\partial_{33}\\ \partial_{11}\end{bmatrix}u+b\cdot\nabla b-u\cdot\nabla u\right)\cdot b dx \tag{3.7}$$
$$:=J_{11}+J_{12}+J_{13}+J_{14}+J_{15}.$$

It is easily seen that

$$J_{12}+J_{13}\leq C\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^1}^2+C\|\partial_2 b\|_{L^2}^2.$$

By integration by parts and Lemma 2.1, we find

$$J_{14}\leq C\|b\|_{L^2}^{1/2}\|\partial_1 b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|\partial_2\nabla b\|_{L^2}^{1/2}\|\partial_2 b\|_{L^2}^{1/2}\|\partial_2\partial_3 b\|_{L^2}^{1/2}$$
$$\leq C\|b\|_{H^2}(\|\partial_1 b\|_{H^1}^2+\|\partial_2 b\|_{H^1}^2).$$

Obviously,

$$J_{15}=-\sum_{k=2}^{3}\int u_1\,\partial_1 u_k\partial_2 b_k dx-\int u_1\,\partial_1 u_1\partial_2 b_1 dx-\int u_2\,\partial_2 u\cdot\partial_2 b dx$$
$$-\sum_{k=1}^{2}\int u_3\,\partial_3 u_k\partial_2 b_k dx-\int u_3\,\partial_3 u_3\partial_2 b_3 dx$$
$$\leq C\sum_{k=2}^{3}\|u_1\|_{L^\infty}\|\partial_1 u_k\|_{L^2}\|\partial_2 b_k\|_{L^2}$$
$$+C\|u_1\|_{L^2}^{1/2}\|\partial_3 u_1\|_{L^2}^{1/2}\|\partial_1 u_1\|_{L^2}^{1/2}\|\partial_1^2 u_1\|_{L^2}^{1/2}\|\partial_2 b_1\|_{L^2}^{1/2}\|\partial_2^2 b_1\|_{L^2}^{1/2}$$
$$+C\|u_2\|_{L^\infty}\|\partial_2 u\|_{L^2}\|\partial_2 b\|_{L^2}+C\sum_{k=1}^{2}\|u_3\|_{L^\infty}\|\partial_3 u_k\|_{L^2}\|\partial_2 b_k\|_{L^2}$$
$$+C\|u_3\|_{L^2}^{1/2}\|\partial_2 u_3\|_{L^2}^{1/2}\|\partial_3 u_3\|_{L^2}^{1/2}\|\partial_1\partial_3 u_3\|_{L^2}^{1/2}\|\partial_2 b_3\|_{L^2}^{1/2}\|\partial_2\partial_3 b_3\|_{L^2}^{1/2}$$
$$\leq C\|u\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^1}^2+\|\partial_2 b\|_{H^1}^2+\|\partial_2 u\|_{L^2}^2).$$

Thus,

$$J_1 \leq \frac{d}{dt}\int \partial_2 u\cdot b dx + C\|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^1}^2 + C\|\partial_2 b\|_{L^2}^2$$
$$+C\|(u,b)\|_{H^2}(\|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^1}^2 + \|(\partial_1,\partial_2)b\|_{H^1}^2 + \|\partial_2 u\|_{L^2}^2). \tag{3.8}$$

Obviously,

$$J_2 = \int u\cdot\nabla b\cdot\partial_2 u dx = \sum_{j=1}^{2}\int u_j\,\partial_j b\cdot\partial_2 u dx + \int u_3\,\partial_3 b\cdot\partial_2 u dx$$
$$\leq C\sum_{j=1}^{2}\|u_j\|_{L^\infty}\|\partial_j b\|_{L^2}\|\partial_2 u\|_{L^2}$$
$$+C\|u_3\|_{L^2}^{1/2}\|\partial_1 u_3\|_{L^2}^{1/2}\|\partial_3 b\|_{L^2}^{1/2}\|\partial_2\partial_3 b\|_{L^2}^{1/2}\|\partial_2 u\|_{L^2}^{1/2}\|\partial_2\partial_3 u\|_{L^2}^{1/2}$$
$$\leq C\|(u,b)\|_{H^2}(\|\partial_1 u_3\|_{L^2}^2 + \|(\partial_1,\partial_2)b\|_{H^1}^2 + \|\partial_2 u\|_{H^1}^2). \tag{3.9}$$

By Lemma 2.1, we conclude that

$$J_3 = -\int b\cdot\nabla u\cdot\partial_2 u dx = -\sum_{k=1}^{2}\int b\cdot\nabla u_k\partial_2 u_k dx - \int b\cdot\nabla u_3\partial_2 u_3 dx$$
$$\leq C\sum_{k=1}^{2}\|b\|_{L^2}^{1/2}\|\partial_1 b\|_{L^2}^{1/2}\|\nabla u_k\|_{L^2}^{1/2}\|\partial_3\nabla u_k\|_{L^2}^{1/2}\|\partial_2 u_k\|_{L^2}^{1/2}\|\partial_2^2 u_k\|_{L^2}^{1/2} \tag{3.10}$$
$$+C\|b\|_{L^2}^{1/2}\|\partial_2 b\|_{L^2}^{1/2}\|\nabla u_3\|_{L^2}^{1/2}\|\partial_1\nabla u_3\|_{L^2}^{1/2}\|\partial_2 u_3\|_{L^2}^{1/2}\|\partial_2\partial_3 u_3\|_{L^2}^{1/2}$$
$$\leq C\|(u,b)\|_{H^2}(\|(\partial_3 u_1, \partial_3 u_2, \partial_1 u_3)\|_{H^1}^2 + \|(\partial_1,\partial_2)b\|_{L^2}^2 + \|\partial_2 u\|_{H^1}^2).$$

Based upon the Young's inequalities, it is easily deduced that

$$J_4 = -\int(\partial_1^2+\partial_2^2)\,b\cdot\partial_2 u dx \leq C\|(\partial_1^2+\partial_2^2)b\|_{L^2}\|\partial_2 u\|_{L^2}$$
$$\leq \frac{1}{2}\|\partial_2 u\|_{L^2}^2 + C\|(\partial_1,\partial_2)b\|_{H^1}^2. \tag{3.11}$$

Putting (3.8)--(3.11) into (3.7), one has

$$\|\partial_2 u\|_{L^2}^2 \leq 2\frac{d}{dt}\int b\cdot\partial_2 u dx + C\|(\partial_1,\partial_2)b\|_{H^1}^2$$
$$+C\|(u,b)\|_{H^2}(\|(\partial_3 u_1, \partial_1 u_2, \partial_3 u_2, \partial_1 u_3)\|_{H^2}^2 + \|(\partial_1,\partial_2)b\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2). \tag{3.12}$$

3.4. *H^1 estimate of $\partial_2 u$*.

Applying $\partial_i (i=1,2,3)$ to (1.1) and dotting it with $\partial_2\partial_i u$ in L^2, it follows

$$\sum_{i=1}^{3}\|\partial_2\partial_i u\|_{L^2}^2 = \sum_{i=1}^{3}\int\partial_2\,\partial_i u\cdot\partial_t\partial_i b dx + \sum_{i=1}^{3}\int\partial_i\,(u\cdot\nabla b)\cdot\partial_2\partial_i u dx$$
$$-\sum_{i=1}^{3}\int\partial_i\,(b\cdot\nabla u)\cdot\partial_2\partial_i u dx - \sum_{i=1}^{3}\int\partial_i\,(\partial_1^2+\partial_2^2)b\cdot\partial_2\partial_i u dx$$
$$:= W_1 + W_2 + W_3 + W_4. \tag{3.13}$$

By the special structure of $(1.1)_1$ and $\nabla\cdot b = 0$, we get

$$W_1 = \sum_{i=1}^{3} \frac{d}{dt}\int \partial_2\partial_i u\cdot\partial_i b dx - \sum_{i=1}^{3}\int \partial_2\partial_i\left(\partial_2 b + \begin{bmatrix}\partial_{33}\\ \partial_{11}+\partial_{33}\\ \partial_{11}\end{bmatrix} u + b\cdot\nabla b - u\cdot\nabla u\right)\cdot\partial_i b dx \tag{3.14}$$
$$:= W_{11}+W_{12}+W_{13}+W_{14}+W_{15}.$$

Integrating by parts gives

$$W_{12}+W_{13} \le C\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2 + C\|\partial_2 b\|_{H^1}^2.$$

By integration by parts and Lemma 2.1, we have

$$\begin{aligned}
W_{14} &= \sum_{i=1}^{3}\int \partial_i b\cdot\nabla b\cdot\partial_2\partial_i b dx + \sum_{i=1}^{3}\int b\cdot\nabla\partial_i b\cdot\partial_2\partial_i b dx \\
&\le C\sum_{i=1}^{3}\|\partial_i b\|_{L^2}^{1/2}\|\partial_1\partial_i b\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/2}\|\partial_2\nabla b\|_{L^2}^{1/2}\|\partial_2\partial_i b\|_{L^2}^{1/2}\|\partial_2\partial_3\partial_i b\|_{L^2}^{1/2} \\
&\quad + C\sum_{i=1}^{3}\|b\|_{L^2}^{1/2}\|\partial_1 b\|_{L^2}^{1/2}\|\partial_i\nabla b\|_{L^2}^{1/2}\|\partial_2\partial_i\nabla b\|_{L^2}^{1/2}\|\partial_2\partial_i b\|_{L^2}^{1/2}\|\partial_2\partial_3\partial_i b\|_{L^2}^{1/2} \\
&\le C\|b\|_{H^2}(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 b\|_{H^2}^2).
\end{aligned}$$

To bound W_{15}, by the Newton-Leibniz formula, we yield

$$W_{15} = -\sum_{i=1}^{3}\int \partial_i u\cdot\nabla u\cdot\partial_2\partial_i b dx - \sum_{i=1}^{3}\int u\cdot\nabla\partial_i u\cdot\partial_2\partial_i b dx := W_{151}+W_{152}.$$

By Lemma 2.1,

$$\begin{aligned}
W_{151} &= -\sum_{j=2}^{3}\int \partial_1 u_j\partial_j u\cdot\partial_2\partial_1 b dx - \int \partial_1 u_1\partial_1 u\cdot\partial_2\partial_1 b dx - \int \partial_2 u\cdot\nabla u\cdot\partial_2^2 b dx \\
&\quad - \sum_{j=1}^{2}\int \partial_3 u_j\partial_j u\cdot\partial_2\partial_3 b dx - \int \partial_3 u_3\partial_3 u\partial_2\partial_3 b dx \\
&\le C\sum_{j=2}^{3}\|\partial_1 u_j\|_{L^4}\|\partial_j u\|_{L^2}\|\partial_2\partial_1 b\|_{L^4} \\
&\quad + C\|\partial_1 u_1\|_{L^2}^{1/2}\|\partial_3\partial_1 u_1\|_{L^2}^{1/2}\|\partial_1 u\|_{L^2}^{1/2}\|\partial_2\partial_1 u\|_{L^2}^{1/2}\|\partial_2\partial_1 b\|_{L^2}^{1/2}\|\partial_2\partial_1^2 b\|_{L^2}^{1/2} \\
&\quad + C\|\partial_2 u\|_{L^4}\|\nabla u\|_{L^2}\|\partial_2^2 b\|_{L^4} + C\sum_{j=1}^{2}\|\partial_3 u_j\|_{L^4}\|\partial_j u\|_{L^2}\|\partial_2\partial_3 b\|_{L^4} \\
&\quad + C\|\partial_3 u_3\|_{L^2}^{1/2}\|\partial_1\partial_3 u_3\|_{L^2}^{1/2}\|\partial_3 u\|_{L^2}^{1/2}\|\partial_2\partial_3 u\|_{L^2}^{1/2}\|\partial_2\partial_3 b\|_{L^2}^{1/2}\|\partial_2\partial_3^2 b\|_{L^2}^{1/2} \\
&\le C\|u\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2 + C\|\partial_2 b\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2).
\end{aligned}$$

For the term W_{152}, it is easily seen that

$$W_{152}=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\int u_j\cdot\partial_i\partial_j u_k\partial_2\partial_i b_k dx\le C\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\|u_j\|_{L^\infty}\|\partial_j\partial_i u_k\|_{L^2}\|\partial_2\partial_i b\|_{L^2}$$
$$\le C\|u\|_{H^2}(\|(\partial_3u_1,\partial_1u_2,\partial_3u_2,\partial_1u_3)\|_{H^2}^2+C\|\partial_2 b\|_{H^2}^2+\|\partial_2u\|_{H^1}^2).$$

Hence, we can be bound W_{15} by

$$W_{15}\le C\|u\|_{H^2}(\|(\partial_3u_1,\partial_1u_2,\partial_3u_2,\partial_1u_3)\|_{H^2}^2+C\|\partial_2 b\|_{H^2}^2+\|\partial_2u\|_{H^1}^2).$$

Therefore,

$$W_1\le\sum_{i=1}^{3}\frac{d}{dt}\int\partial_2\,\partial_i u\cdot\partial_i b dx+C\|(\partial_3u_1,\partial_1u_2,\partial_3u_2,\partial_1u_3)\|_{H^2}^2+C\|\partial_2 b\|_{H^2}^2$$
$$+C\|(u,b)\|_{H^2}(\|(\partial_3u_1,\partial_1u_2,\partial_3u_2,\partial_1u_3)\|_{H^2}^2+C\|(\partial_1,\partial_2)b\|_{H^2}^2+\|\partial_2u\|_{H^1}^2). \tag{3.15}$$

Next, to bound W_2. By Newton-Leibniz formula and Lemma 2.2,

$$\begin{aligned}W_2&=\sum_{i=1}^{3}\int\partial_i u\cdot\nabla b\cdot\partial_2\partial_i u dx+\sum_{i=1}^{3}\int u\cdot\partial_i\nabla b\cdot\partial_2\partial_i u dx\\&\le C\sum_{i=1}^{3}\|\partial_2\partial_i u\|_{L^2}^{3/2}\|\partial_i u\|_{L^2}^{1/2}\|\nabla b\|_{L^2}^{1/4}\|\partial_1\nabla b\|_{L^2}^{1/4}\|\partial_3\nabla b\|_{L^2}^{1/4}\|\partial_1\partial_3\nabla b\|_{L^2}^{1/4}\\&\quad+C\sum_{i=1}^{3}\|\partial_2\partial_i u\|_{L^2}\|\partial_i\nabla b\|_{L^2}^{1/2}\|\partial_1\partial_i\nabla b\|_{L^2}^{1/2}\|u\|_{L^2}^{1/4}\|\partial_3 u\|_{L^2}^{1/4}\|\partial_2 u\|_{L^2}^{1/4}\|\partial_2\partial_3 u\|_{L^2}^{1/4}\\&\le C\|(u,b)\|_{H^2}(\|\partial_1 b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2).\end{aligned} \tag{3.16}$$

Similarly,

$$W_3=-\sum_{i=1}^{3}\int\partial_i b\cdot\nabla u\cdot\partial_2\partial_i u dx-\sum_{i=1}^{3}\int b\cdot\partial_i\nabla u\cdot\partial_2\partial_i u dx:=W_{31}+W_{32}.$$

By Lemma 2.2, it is easily deduced that

$$W_{31}\le C\sum_{i=1}^{3}\|\partial_2\partial_i u\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\partial_2\nabla u\|_{L^2}^{1/2}\|\partial_i b\|_{L^2}^{1/4}\|\partial_1\partial_i b\|_{L^2}^{1/4}\|\partial_3\partial_i b\|_{L^2}^{1/4}\|\partial_1\partial_3\partial_i b\|_{L^2}^{1/4}$$
$$\le C\|(u,b)\|_{H^2}(\|\partial_1 b\|_{H^2}^2+\|\partial_2 u\|_{H^1}^2).$$

Similarly,

$$W_{32}=-\int b\cdot\partial_1\nabla u_1\partial_2\partial_1u_1dx-\sum_{k=2}^{3}\int b\cdot\partial_1\nabla u_k\partial_2\partial_1u_kdx-\int b\cdot\partial_2\nabla u\cdot\partial_2^2udx$$
$$-\sum_{k=1}^{2}\int b\cdot\partial_3\nabla u_k\partial_3\partial_2u_kdx-\int b\cdot\partial_3\nabla u_3\partial_3\partial_2u_3dx$$
$$\le C\|\partial_2\partial_1u_1\|_{L^2}\|\partial_1\nabla u_1\|_{L^2}^{1/2}\|\partial_3\partial_1\nabla u_1\|_{L^2}^{1/2}\|b\|_{L^2}^{1/4}\|\partial_1 b\|_{L^2}^{1/4}\|\partial_2 b\|_{L^2}^{1/4}\|\partial_1\partial_2 b\|_{L^2}^{1/4}$$
$$+C\sum_{k=2}^{3}\|b\|_{L^\infty}\|\partial_1\nabla u_k\|_{L^2}\|\partial_1\partial_2u_k\|_{L^2}+C\|b\|_{L^\infty}\|\partial_2\nabla u\|_{L^2}\|\partial_2^2u\|_{L^2}$$
$$+C\textstyle\sum_{k=1}^{2}\|b\|_{L^\infty}\|\partial_3\nabla u_k\|_{L^2}\|\partial_3\partial_2u_k\|_{L^2}$$

$$+C\|\partial_2\partial_3 u_3\|_{L^2}\|\partial_3\nabla u_3\|_{L^2}^{1/2}\|\partial_3^2\nabla u_3\|_{L^2}^{1/2}\|b\|_{L^2}^{1/4}\|\partial_1 b\|_{L^2}^{1/4}\|\partial_2 b\|_{L^2}^{1/4}\|\partial_1\partial_2 b\|_{L^2}^{1/4}$$

$$\leq C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2 + C\|(\partial_1,\partial_2)b\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2).$$

It is easily seen that,

$$W_3 \leq C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2 + C\|(\partial_1,\partial_2)b\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2). \tag{3.17}$$

Now, to estimate the last term of W_4, Young's inequalities, we have

$$W_4 = -\sum_{i=1}^3\int(\partial_1^2+\partial_2^2)\,\partial_i b\cdot\partial_2\partial_i u dx \leq C\sum_{i=1}^3\|(\partial_1^2+\partial_2^2)\partial_i b\|_{L^2}\|\partial_2\partial_i u\|_{L^2}$$

$$\leq \frac{1}{2}\sum_{i=1}^3\|\partial_2\partial_i u\|_{L^2}^2 + C\|(\partial_1,\partial_2)b\|_{H^2}^2. \tag{3.18}$$

Hence, adding (3.15)--(3.18) together, one can bound (3.13) by

$$\sum_{i=1}^3\|\partial_2\partial_i u\|_{L^2}^2 \leq 2\sum_{i=1}^3\frac{d}{dt}\int\partial_2\partial_i u\cdot\partial_i b dx + C\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2 + C\|(\partial_1,\partial_2)b\|_{H^2}^2$$
$$+C\|(u,b)\|_{H^2}(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2) + C\|(\partial_1,\partial_2)b\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2). \tag{3.19}$$

3.5. ***Proof of Theorem 1.1.***

This subsection completes the proof of Theorem 1.1, which can be achieved by the bootstrapping argument [22]. As we known, the local well-posedness of Theorem 1.1 can be established via a standard procedure, we only need to establish the global bounds and then apply the bootstrapping argument to obtain the desired stability result. To use the bootstrapping argument, we introduce an energy functional specifically to achieve our desired estimates. Let

$$\mathrm{E}(t) = \mathrm{E}_1(t) + \mathrm{E}_2(t),$$

where

$$\mathscr{E}_1(t) = \sup_{0\leq\tau\leq t}\|(u,b)(\tau)\|_{H^2}^2 + 2\int_0^t(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2 + \|(\partial_1 b,\partial_2 b)\|_{H^2}^2 d)\tau,$$

$$\mathscr{E}_2(t) = \int_0^t\|\partial_2 u(\tau)\|_{H^1}^2 d\tau.$$

Combining (3.1) with (3.6) and integrating it over $[0,t]$ yields $\|(u,b)(t)\|_{H^2}^2 + 2\int_0^t(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2 +$ $\|(\partial_1,\partial_2)b\|_{H^2}^2)d\tau$

$$\leq C\|(u_0,b_0)\|_{H^2}^2 + C\sup_{0\leq\tau\leq t}\|(u,b)\|_{H^2}\int_0^t(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2$$
$$+\|(\partial_1,\partial_2)b\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2)d\tau$$
$$\leq C\mathscr{E}_1(0) + C\mathscr{E}_1^{3/2}(t) + C\mathscr{E}_2^{3/2}(t). \tag{3.20}$$

Next, adding up (3.12) and (3.19), one derives

$$\int_0^t\|\partial_2 u\|_{H^2}^2 d\tau \leq C\|(u_0,b_0)\|_{H^2}^2 + C\|(u,b)\|_{H^2}^2 + C\int_0^t\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2$$
$$+C\|(\partial_1,\partial_2)b\|_{H^2}^2 d\tau + C\sup_{0\leq\tau\leq t}\|(u,b)\|_{H^2}\int_0^t(\|(\partial_3 u_1,\partial_1 u_2,\partial_3 u_2,\partial_1 u_3)\|_{H^2}^2$$
$$+C\|(\partial_1,\partial_2)b\|_{H^2}^2 + \|\partial_2 u\|_{H^1}^2)d\tau$$
$$\leq C\mathscr{E}_1(0) + C\mathscr{E}_1(t) + C\mathscr{E}_1(t)^{3/2} + C\mathscr{E}_2(t)^{3/2}. \tag{3.21}$$

For any$t > 0$, and $C_0 > 0$ is a pure constant. Adding (3.21) to (3.20) by the appropriate constant yields,

$$E(t) \le C_0 E(0) + C_0 E^{3/2}(t). \tag{3.22}$$

We take

$$\|(u_0, b_0)\|_{H^2}^2 \le \frac{1}{16C_0^3}.$$

The bootstrapping argument starts with the ansatz that

$$E(t) \le \frac{1}{4C_0^2}.$$

It follows from (3.22) that

$$\mathscr{E}(t) \le C_0\mathscr{E}(0) + C_0\mathscr{E}^{1/2}(t)\mathscr{E}(t) \le C_0\mathscr{E}(0) + \frac{1}{2}\mathscr{E}(t) \le 2C_0\mathscr{E}(0).$$

The bootstrapping argument then implies that, for any $t \ge 0$.

$$E(t) \le \frac{1}{8C_0^2}.$$

This completes the proof of Theorem1.1.

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