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Original Article

Hybrid Expansion Law for Bulk Viscous $f(T)$ Gravity

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Abstract - *The present investigation examines the bulk viscous Kantowski-Sachs cosmological model within the framework of f(T) modified gravity. We have derived the field equations by employing a specific functional form f(T) of gravity and analyzed the dynamics of essential physical parameters, including energy density, isotropic pressure, and the equation of state parameter. To obtain precise solutions, a hybrid expansion law has been utilized, which provides insight into the accelerated expansion of the universe. The results indicate that energy density consistently decreases over time, while isotropic pressure exhibits a declining trend, both contributing to cosmic acceleration. The equation of state parameter remains within the phantom regime, suggesting the presence of a dynamic dark energy component. This study highlights the versatility of f(T) modified gravity in elucidating both the early and late stages of cosmic evolution. Our findings align with observational data, enhancing the validity of modified gravity theories in explicating the dynamics of the universe's expansion.*

Keywords - *Kantowski-Sachs universe, $f(T)$ Theory of gravity, Hybrid expansion law.*

1. Introduction

The field of cosmology and its theoretical models have witnessed significant advancements, particularly with the observation of the universe's accelerated expansion. This acceleration has been corroborated by comprehensive observational data, including those from Type Ia supernovae [1, 2], Cosmic Microwave Background (CMB) radiation [3, 4], and Baryon Acoustic Oscillations (BAO) [5]. Standard cosmological models, which are founded on General Relativity (GR), explain this acceleration by introducing a cosmological constant (Λ) that correlates with the vacuum energy density. However, challenges such as the cosmological constant problem and fine-tuning issues have prompted researchers to explore alternative theoretical frameworks, including modified gravity theories [6, 7]. One noteworthy alternative is teleparallel gravity, which extends the traditional teleparallel concepts. In teleparallel gravity, the Lagrangian is defined in terms of the torsion scalar, as opposed to the Ricci scalar [8, 9] utilized in GR. This framework differentiates itself by conceptualizing gravity through torsion rather than the curvature of spacetime. This distinction opens new avenues for novel dynamic scenarios concerning cosmic evolution. The versatile nature of teleparallel gravity permits various modifications that can effectively simulate the influence of dark energy, rendering it a compelling alternative for elucidating late-time cosmic acceleration without necessitating an additional exotic energy source [10, 11].

In recent years, the significance of bulk viscosity in cosmology has become increasingly apparent as a potential mechanism for elucidating cosmic acceleration. Bulk viscosity arises in fluid dynamics when a system deviates from thermal equilibrium, resulting in internal friction that can influence the dynamics of the universe's expansion [12, 13]. The effects of bulk viscosity have been analyzed across various contexts within cosmology, including conventional models [14], scalar field models [15], and modified gravity frameworks [16]. When the bulk viscosity is accounted for, the effective pressure of the cosmic fluid is modified, potentially leading to accelerated expansion without necessitating a cosmological constant. Numerous studies have demonstrated that models incorporating bulk viscosity may exhibit characteristics reminiscent of those found in scenarios dominated by dark energy, indicating that viscosity can play a pivotal role in the evolution of the universe [17, 18]. Bulk viscosity introduces additional degrees of freedom in the realm of f(T) gravity, thereby enabling a broader spectrum of cosmological solutions that may align more closely with observational data [19].

The Kantowski-Sachs spacetime model, recognized as a homogeneous yet anisotropic cosmological model, offers a significant framework for investigating the implications of bulk viscosity in the context of modified gravity. Unlike the traditional Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which assumes isotropy, the Kantowski-Sachs metric enables the examination of anisotropic expansion. This characteristic is particularly relevant for analyzing the dynamics of the early universe and the anisotropies observed later [20, 21]. Numerous studies have explored Kantowski-Sachs cosmologies within the realms of both general relativity and modified gravity, revealing that anisotropies can profoundly influence cosmic evolution [22]. When incorporated with bulk viscosity, the Kantowski-Sachs model introduces new dissipative effects that significantly impact structure formation, cosmic expansion, and the transition from anisotropic to isotropic behavior over time [23]. Understanding these effects within the f(T) gravity framework is essential for developing robust cosmological models that adhere to observational constraints.

In this study, we investigated a Kantowski-Sachs cosmological model incorporating bulk viscosity within the framework of f(T)-modified gravity. By adopting a specific functional form of the gravitational field and integrating a hybrid expansion law, we derive precise solutions for key cosmological parameters, including the energy density, isotropic pressure, and equation of state parameter. Our analysis examined the effects of bulk viscosity on cosmic acceleration and the stability of the universe's evolution. The structure of this paper is organized as follows: Section 2 outlines the fundamental equations governing f(T)-modified gravity and Kantowski-Sachs spacetime. In Section 3, we present field equations and solutions derived from the proposed functional form of f(T) gravity. Section 4 explores the physical implications of our results and compares them with the observational data. Section 5 concludes by summarizing our findings and suggesting potential future research directions in the domains of modified gravity and bulk viscous cosmology.

2. Fundamental Equations Governing $f(T)$ Gravity

The initiative of broadening the Teleparallel Theory. i.e. f(T) theory as

$$S = \int [f(T) + L\_matter] \; e \, d\hat{}4\, x. \tag{1}$$

Here, f(T) is a mathematical function based on the torsion scalar T. The line element representing the Riemannian manifold is expressed as

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2}$$

This line element can be expressed in the Minkowskian framework using a transformation known as a tetrad, as follows:

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ij} \theta^i \theta^j, \tag{3}$$

$$dx^\mu = e_i^\mu \theta^i, \;\; \theta^i = e_\mu^i dx^\mu, \tag{4}$$

Where $\eta_{ij} = diag[1, -1, -1, -1]$ and $e_i^\mu e_\mu^i = \delta_\nu^\mu$ or $e_i^\mu e_\mu^j = \delta_i^j$.

The root of the metric determinant was defined by $\sqrt{-g} = det[e_\mu^i] = e$. In a manifold where the Riemann tensor components, excluding torsion terms, are zero, indicating that only the contribution from the Levi-Civita connection is present, and where non-zero torsion terms exist, the components of the Weitzenböck connection are defined as follows:

$$\Gamma_{\mu\nu}^\alpha = e_i^\alpha \partial_\nu e_\mu^i = -e_\mu^i \partial_\nu e_i^\alpha. \tag{5}$$

It has zero curvature yet possesses non-zero torsion. By utilizing this connection, it is possible to define the components of the torsion tensor appropriately.

$$T_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha - \Gamma_{\nu\mu}^\alpha = e_i^\alpha \left(\partial_\mu e_\nu^i - \partial_\mu e_\mu^i\right), \tag{6}$$

The difference between the Levi and Civita connection and the Weitzenböck connection is captured by a spacetime tensor, known as the contorsion tensor:

$$K_\alpha^{\mu\nu} = \left(-\frac{1}{2}\right)\left(T^{\mu\nu}{}_\alpha + T^{\nu\mu}{}_\alpha - T_\alpha^{\mu\nu}\right). \tag{7}$$

An additional tensor can be defined to facilitate the characterization of the Lagrangian and its corresponding equations of motion. $S_\alpha^{\mu\nu}$. This tensor is derived from the components of the torsion and contorsion tensors as

$$S_\alpha^{\mu\nu} = \left(\frac{1}{2}\right)\left(K^{\mu\nu}{}_\alpha + \delta_\alpha^\mu T^{\beta\nu}{}_\beta - \delta_\alpha^\nu T_\beta^{\beta\mu}\right). \tag{8}$$

The torsion scalar $T$is

$$T = T^{\alpha}_{\mu\nu}\, S^{\mu\nu}_{\alpha}. \tag{9}$$

Upon conducting the functional variation of action (1) with respect to the tetrads, we derived the subsequent equations of motion.

$$S^{\nu\rho}_{\mu}\partial_{\rho}T f_{TT} + \left[e^{-1}e^{i}_{\mu}\partial_{\rho}\left(e e^{\alpha}_{i} S^{\nu\rho}_{\alpha}\right) + T^{\alpha}{}_{\lambda\mu}S^{\nu\lambda}_{\alpha}\right](f_T) + \frac{1}{4}\delta^{\nu}_{\mu}(f) = 4\pi T^{\nu}_{\mu}, \tag{10}$$

The field equation (10), expressed in terms of tetrad and partial derivatives, significantly differs from Einstein's equations. whereT^{ν}_{μ} is the energy-momentum tensor, $f_T = df(T)/dT$ and by setting $f(T) = a_0$ =constant, this is dynamically equivalent to GR.

3. Field Equations and Some Physical Quantities

The following expression describes the line element of Kantowski-Sachs spacetime:

$$ds^2 = dt^2 - A^2dr^2 - B^2(d\theta^2 + sin^2\theta\, d\varphi^2), \tag{11}$$

Metric potentials A and B are designated as functions that depend exclusively on time t.

The corresponding Torsion scalar is definitely expressed as follows:

$$T = -2\left(2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}^2}{B^2}\right). \tag{12}$$

For the perfect fluid distribution, the energy-momentum tensor T^{i}_{j} is taken as

$$T^{\nu}_{\mu} = (\bar{p} + \rho)u^{\nu}u_{\mu} - \bar{p}g^{\nu}_{\mu}, \tag{13}$$

Satisfying the equation of state with precision.

$$\bar{p} = p - 3H\xi, \tag{14}$$

together with commoving co-ordinates

$$u^{\nu} = (0,0,0,1) \text{ and } u^{\nu}u_{\nu} = 1, \tag{15}$$

where $u^{\nu}$is the *4*-velocity vector of the cosmic fluid, $\bar{p}, p$ and ρ are the effective pressure, anisotropic pressure, and energy density of the fluid, respectively.
From the equation of motion (10), Kantowski-Sachs spacetime (11) for the fluid of the stress energy tensor (13) can be written as

$$f + 4(f_T)\left\{\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\dot{A}}{A}\frac{\dot{B}}{B}\right\} + 4\frac{\dot{B}}{B}\,\dot{T}\, f_{TT} = 16\pi(-\bar{p}), \tag{16}$$

$$f + 2(f_T)\left\{\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + 3\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right\} + 2\left\{\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right\}\dot{T}\, f_{TT} = 16\pi(-\bar{p}), \tag{17}$$

$$f + 4(f_T)\left\{\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right\} = 16\pi(\rho). \tag{18}$$

Where the dot (·) denotes the derivative with respect to time *t*. For simplicity, we consider the $f(T)$ gravity model as $f(T) = T^{\beta}$, where β is any model parameter.

Finally, we have three differential equations with six unknowns, namely $A, B, f, p, \bar{p}, \rho$. To find the solution to the field equation, we need additional conditions. Hence, first we consider

1) The relation between the metric potentials as

$$A = B^n, \tag{19}$$

and

2) The hybrid scale factor (a) of the form

$$a = te^t. \tag{20}$$

For this model, the corresponding metric coefficients A and B, with the help of equations (19) and (20), and the metric potentials are obtained as

$$A = (e^t t)^{\frac{3n}{2+n}}, \tag{21}$$

$$B = (e^t t)^{\frac{3}{2+n}}. \tag{22}$$

4. Physical Implications of Our Model

With the help of the above metric potentials, the Torsion scalar T becomes

$$T = -\frac{6(1+t)^2}{t^2}. \tag{23}$$

The function of the torsion scalar is also observed as,

$$f(T) = 6^\beta (-\frac{(1+t)^2}{t^2})^\beta. \tag{24}$$

The Energy density of the universe becomes

$$\rho = \frac{1}{(2+n)^2}\left\{6^\beta((2+n)^2 - 6(1+2n)\beta)(-\frac{(1+t)^2}{t^2})^\beta\right\}. \tag{25}$$

The energy density of the universe evolves with time depending on the cosmological model. In an expanding universe, the energy density typically decreases because of the dilution of matter and radiation. When dark energy prevails, the energy density approaches a stable value, which unequivocally leads to accelerated expansion of the universe. In the early universe, the energy density was incontrovertibly high and predominantly governed by radiation. As time passed, it transitioned to matter domination followed by dark energy. Understanding this kind of evolution is crucial for studying cosmic history, structure formation, and future expansion scenarios, including whether the universe will expand forever or collapse in a “Big Bang.”

In our examined model, the energy density (ρ) exhibited a time-dependent behavior, where its value decreased as time increased. This phenomenon clearly indicates that the energy density of the model is intrinsically linked to cosmic evolution, which decreases as the universe expands. The decrease in energy density with increasing time can be attributed to the dilution of energy owing to the expanding universe. This behavior undeniably aligns with the fundamental principle of cosmology, which establishes that the universe is homogeneous and isotropic on a large scale. The implications of this behavior are far-reaching, particularly in the context of dark energy and cosmological models, as it delivers essential insights into the universe’s evolution and the powerful dynamics that drive its expansion.

Figure 1: The behavior of energy density of the model versus time with the appropriate choice of constants.

The pressure of the universe becomes

$$\bar{p} = \frac{1}{(2+n)t^2}\left\{6^\beta(-\frac{(1+t)^2}{t^2})^{-1+\beta}(2+n+4(-2+\beta)\beta + t(2+n-6\beta)(2+t))\right\} \tag{26}$$

Isotropic pressure represents the uniform pressure exerted in all spatial directions. In the early universe, radiation was dominated by high pressures. As the universe expanded, the pressure decreased, transitioning to matter dominance, where the pressure was negligible. If dark energy is characterized as a cosmological constant, it results in negative pressure, which is responsible for the accelerated expansion of the universe. Some models introduce dynamic dark energy, where the pressure evolves with time. Studying the pressure evolution helps to understand phase transitions, such as inflation and late-time acceleration. In alternative cosmological models, pressure variations may influence cosmic evolution, impacting large-scale structure formation and possible future scenarios, such as the Big Rip. In contrast, isotropic pressure, as depicted in Figure 1, displays a negative evolution, suggesting a potential driving force behind the accelerated expansion of the universe. This behavior aligns with the theoretical frameworks of cosmological evolution, offering intriguing insights into the universe’s intricate dynamics.

The Equation of state parameter of the universe becomes

$$\omega = -\frac{(2+n)(2+n+4(-2+\beta)\beta+t(2+n-6\beta)(2+t))}{((2+n)^2-6(1+2n)\beta)(1+t)^2} \tag{27}$$

The equation of state (EoS) parameter ω determines the cosmic evolution by relating pressure to energy density. In the radiation era, $\omega = \frac{1}{3}$, indicating that relativistic particles were dominant. In the matter era, $\omega = 0$, as dust-like matter has negligible pressure. For dark energy, $\omega = -1$ (cosmological constant), which leads to accelerated expansion. If $\omega < -1$ (phantom energy), expansion accelerates uncontrollably, possibly ending in a Big Rip. Dynamic dark-energy models allow for evolution over time. Studying ω helps constrain cosmological models and future expansion scenarios.

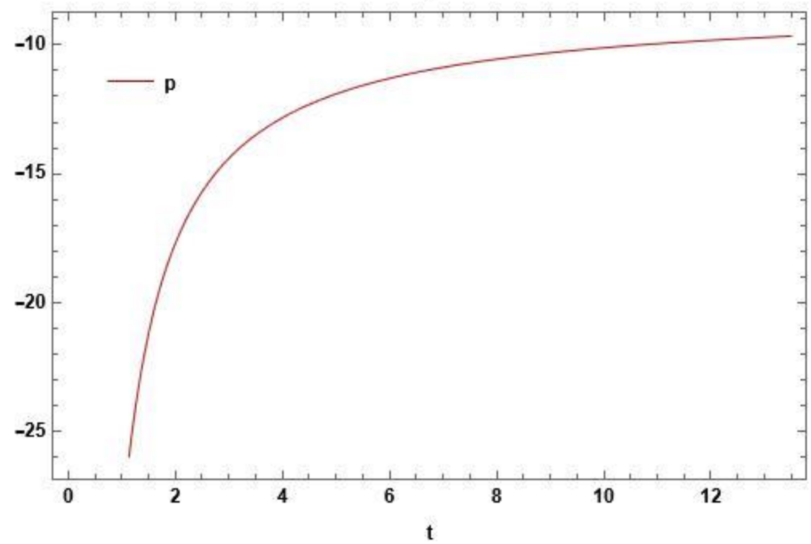

Fig. 1 The behavior of the isotropic pressure of the model versus redshift with the appropriate choice of constants.

Figure 2 shows the behavior of ω, in the context of the current cosmological model. This parameter plays a pivotal role in cosmology, as it encapsulates the essential characteristics of the dominant component driving the universe's expansion. The behavior of the EoS parameter was thoroughly analyzed in relation to time (t), providing valuable cosmological implications. A notable feature of the figure is that the EoS parameter ω, consistently remains below -1 across all redshift values. This phenomenon indicates that the model is drawn toward a hypothetical region of dark energy, which is believed to be responsible for the accelerated expansion of the universe. The phantom region was distinguished by an EoS parameter of less than -1 ($\omega < -1$), signifying a dynamic and evolving dark energy component.

In this model, the EoS parameter was fixed at $\omega = -1.53$ as observed in Ref. [24], which corresponds to a static and unevolving dark energy component. However, the model's propensity towards the phantom region at a high cosmic time suggests that it exhibits a dynamic dark-energy component. This behavior is consistent with observational evidence and provides a more profound insight into the evolution of the universe's dark energy component.

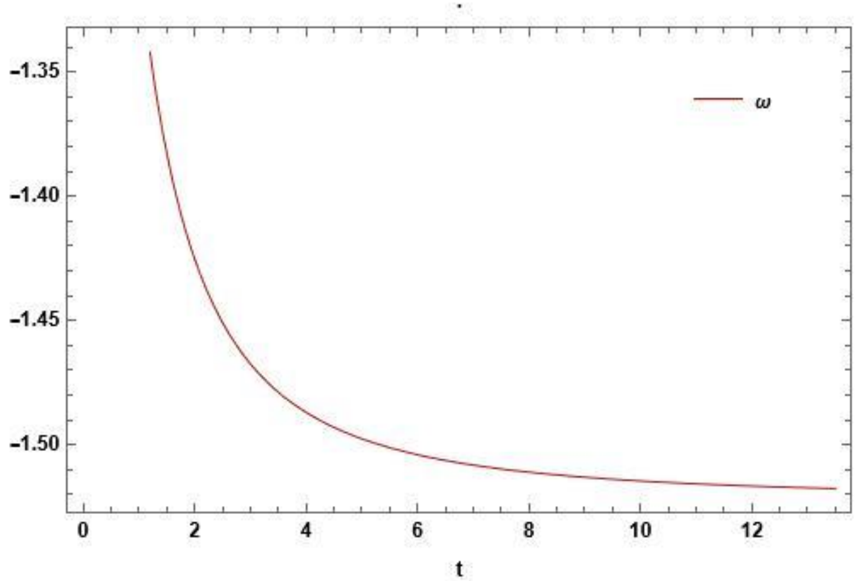

Fig. 2 The behavior of the equation of state parameter of the model versus redshift with the appropriate choice of constants.

The squared velocity of time of the universe becomes

$$\vartheta^2 = -\frac{(2+n)(2+n+t(t(2+n-6\beta)+2(1+n-4\beta))+4(-2+\beta)\beta)}{(1+t)^2((2+n)^2-6(1+2n)\beta)} \tag{28}$$

The speed of sound squared (ϑ^2) governs the stability of cosmological perturbations. If $\vartheta^2 > 0$, density fluctuations propagate as waves, ensuring stability. If $\vartheta^2 < 0$, the system is unstable, leading to exponential growth of perturbations, which could disrupt structure formation. In contrast, isotropic pressure, as depicted in Figure 3, displays a negative evolution. Hence, our derived model exhibited instability.

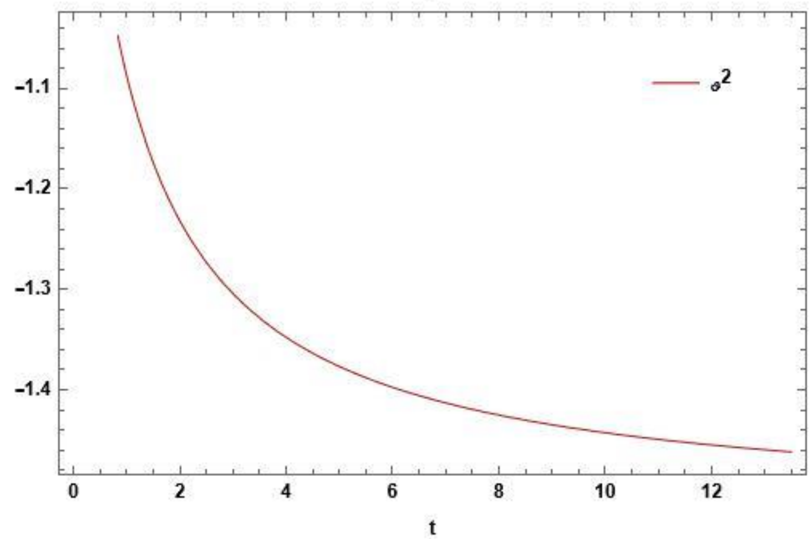

Fig. 3 The behavior of the squared speed of sound parameter of the model versus redshift with the appropriate choice of constants.

5. Conclusion

In the present study, we explored the Kantowski-Sachs cosmological model within the framework of $f(T)$ gravity, specifically focusing on a bulk viscous fluid. By adopting a functional form of the scale factor, we derived and solved field equations using a hybrid expansion law. This approach provides a comprehensive understanding of the evolution of various cosmological parameters and their implications for the accelerated expansion of the universe.

Our analysis shows that the energy density of the universe decreases over time, which aligns with the expected dilution caused by the cosmic expansion. This aligns perfectly with the standard cosmological model, in which the energy density evolves in accordance with the dominant cosmic component. The isotropic pressure exhibited a declining trend, indicating its crucial role in facilitating the observed accelerated expansion of the cosmos. This behavior aligns with theories involving dark energy or potential modifications to gravitational dynamics that impact the universe's evolution during its late phase.

The equation of state (EoS) parameter remained in the phantom regime ($\omega < -1$), clearly indicating the presence of a dynamic dark energy component. This behavior implies that the expansion of the universe not only accelerates, but is dominated by a form of energy with properties different from the conventional cosmological constant ($\omega = -1$). The presence of a phantom-like EoS parameter suggests that the universe could evolve towards a scenario such as Big Rip.

Furthermore, we analyzed the squared velocity of the sound parameter, which plays a critical role in determining the stability of the cosmic perturbations. Our findings indicate that the model is unstable under certain conditions. However, future research should investigate more comprehensive perturbation scenarios to ascertain the robustness of the framework.

From a theoretical perspective, our study reinforces the viability of $f(T)$ gravity as an alternative to general relativity. The ability to select various functional forms of $f(T)$ enables a diverse array of cosmological behaviors. This approach effectively explained the accelerated expansion of the universe. Torsion-based modification presents a compelling alternative to dark energy, significantly reducing the need for an exotic energy component.

This model exhibits a significant correlation with contemporary data derived from Type Ia supernovae, Cosmic Microwave Background (CMB) radiation, and Baryon Acoustic Oscillations (BAO). These results reinforce the perspective that modified gravity theories may serve as credible explanations for the observed acceleration of the universe. In conclusion, our work demonstrates that the bulk viscous Kantowski-Sachs cosmological model in $f(T)$ gravity provides a compelling framework for understanding the evolution of the universe. The results clearly demonstrate the model's capability to effectively explain accelerated expansion while perfectly aligning with observational data. Future research should extend this analysis by considering more complex forms of $f(T)$, additional cosmological tests, and perturbative analyses to explore the potential instabilities. This will help refine our understanding of modified gravity theories and their implications for the fundamental nature of the cosmos.

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