International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V71I10P102
Volume 71 Issue 10, 9-14, October 2025

Original Article

Certified Domination Polynomials of Book Graph

K. Lal Gipson[1], Angelin Jenisha MJ[2]

[1,2]*Department of Mathematics & Research Centre, Scott Christian College (Autonomous), Nagercoil, Kanyakumari District, Tamil Nadu, India.*

Corresponding Author : lalgipson@yahoo.com

Received: 08 August 2025 Revised: 13 September 2025 Accepted: 05 October 2025 Published: 17 October 2025

Abstract - *Let $G = (V, E)$ be a simple graph of order n. The certified domination polynomial of G is the polynomial, $D_{cer}(G, x) = \sum_{i=\gamma_{cer}(G)}^{|V(G)|} d_{cer}(G, i)\, x^i$, where $\gamma_{cer}(G)$ is the minimum cardinality of a certified dominating set of G and $d_{cer}(G, i)$ is the number of certified dominating sets of G of size i. In this article, we study the certified dominating sets and the certified domination polynomial of the book graph B_n, and obtain some properties of this polynomial.*

Keywords - *Certified dominating set, Certified domination number, Certified domination polynomial.*

1. Introduction

Let $G = (V, E)$ be a simple, finite, undirected connected graph without loops and multiple edges. G's order and size are shown by numbers n and m, respectively. A vertex in a graph G has an open neighbourhood defined as the set $N_G(v) = \{u \in V(G) \ / \ uv \in E(G)\}$, and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. A dominating set is a subset of a graph's vertices such that every vertex not in the set is adjacent to at least one vertex within the set. The domination number denoted as $\gamma(G)$ is the minimum number of vertices required to form a dominating set in a graph G [10]. A dominating set is called a certified dominating set if every vertex in that set has either zero or at least two neighbours outside of the set. The certified domination number $\gamma_{cer}(G)$ is the minimum size of such a set for a given graph G [5]. This property ensures a certain level of "certification" for the dominating set. The concept of dominating sets and the domination number in Graph Theory was first introduced by Oystein Ore and Claude Berge in the 1960s. The concept of polynomial domination in graph theory was introduced by Saied Alikhani and Yee-hock Peng in 2009[10]. The concept of certified domination in graphs was introduced by Dettlaf et al.,2020[5]. They also further studied the concept in their subsequent work, including its applications in real-life situations. While extending the concept of domination polynomial and certified domination, we came across many interesting relations among the certified domination polynomials of different graphs. In the next section, we consider the book graph and compute its certified domination polynomials, and we can find a closed-form expression for the coefficients of the certified domination polynomial.

Definition 1.1: Let G be a simple connected graph. Let $D_{cer}(G, i)$ be a family of all certified dominating sets of G with cardinality i, and let $d_{cer}(G, i) = |D_{cer}(G, i)|$. Then the **certified domination polynomial** $D_{cer}(G, x)$ is defined as $D_{cer}(G, x) = \sum_{i=\gamma_{cer}(G)}^{|V(G)|} d_{cer}(G, i)\, x^i$, where $\gamma_{cer}(G)$ is the certified domination number of G.

Lemma 1.2: Let G be the graph with n vertices. Then

(i) The coefficient of x^n in the certified domination polynomial of G, one

(ii) The certified dominating sets of G with cardinality i are empty if and only if $i < \gamma_{cer}(G)$ or $i = n - 1$ or $i > n$

(iii) The certified domination polynomial has no constant term and $(n-1)^{th}$ term.

2. Certified Dominating Sets and Certified Domination Polynomial of B_m

A book graph B_n, is defined as follows $V(B_n) = \{u_1, u_2\} \cup \{v_i, w_i : 1 \le i \le n\}$ and $E(B_n) = \{u_1u_2\} \cup \{u_1v_i, u_2w_i, v_iw_i : 1 \le i \le n\}$. We cite S. Jahari and S. Alikhani for the definition of book graph [3]. Figure 1 shows the example of B_n. Let $D_{cer}(B_n, i)$ be the family of dominating sets of B_n with cardinality i and let $d_{cer}(B_n, i) = |D_{cer}(B_n, i)|$. In this section, we shall investigate the certified dominating sets and certified domination polynomials of the book graph B_n. The following theorem 2.2 is useful for finding the coefficients of certified domination polynomial of B_n

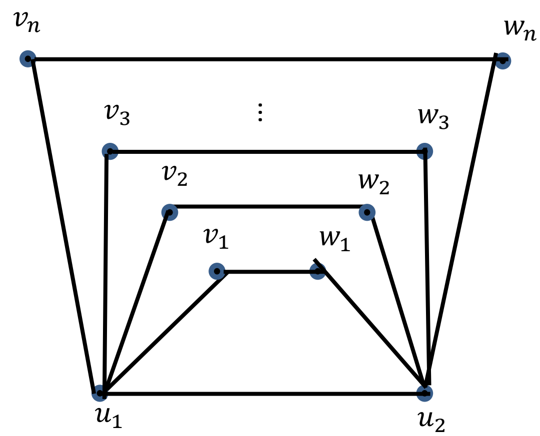

Fig. 1 Book Graph B_n

Lemma 2.1: Let B_n be the book graph with $2n+2$ vertices. Then $\gamma_{cer}(B_n) = 2$ for all $n \in N$

Proof: Let B_n be the book graph with $2n+2$ vertices. The $n-$ book graph consists of n pages sharing a common edge, which is the spine. For $n = 1$, $\{u_1, w_1\}$ and $\{u_2, w_1\}$ are the minimum certified dominating sets and for $n \geq 2$, $\{u_1, v_2\}$ is the minimum certified dominating set. Hence $\gamma_{cer}(B_n) = 2$ for all $n \in N$.

Theorem 2.2: Let B_n be the book graph with $2n+2$ vertices. Then for all $n \geq 2$,

If n is odd,

$$d_{cer}(B_n, i) = \begin{cases} \binom{n}{\frac{i}{2}-1} & if\ i = 2k, 1 \leq k \leq n+1\ but\ k \neq \frac{n+1}{2}, n \\ 2^n - 2 & if\ i = n \\ \binom{n}{\frac{i}{2}-1} + 2 & if\ i = n+1 \\ 0 & otherwise \end{cases}$$

If n is even,

$$d_{cer}(B_n, i) = \begin{cases} \binom{n}{\frac{i}{2}-1} & if\ i = 2k, 1 \leq k \leq n+1\ but\ k \neq \frac{n}{2}, n \\ \binom{n}{\frac{i}{2}-1} + 2^n - 2 & if\ i = n \\ 2 & if\ i = n+1 \\ 0 & otherwise \end{cases}$$

Proof: Let B_n be the book graph with $2n+2$ vertices.

Case 1: n is odd, $(n \geq 3)$

If $i = 2$, $\{u_1, v_1\}$ is the certified dominating set. Hence $d_{cer}(B_n, 2) = 1 = \binom{n}{\frac{2}{2}-1}$. If $i = 2k, 2 \leq k \leq n+1\ but\ k \neq \frac{n+1}{2}, n$, then the certified dominating sets of B_n is obtained by choosing $\frac{i}{2} - 1$ pages that share a common edge from the n pages that share a common edge. Hence there are $\binom{n}{\frac{i}{2}-1}$, possible ways. If $i = n$, then the certified dominating set of B_n is obtained by choosing one vertex from each page except the common vertices, but from this set of vertices $\{v_1, v_2, \dots, v_n\}$ and $\{w_1, w_2, \dots, w_n\}$ are not certified dominating sets. Thus we get $2^n - 2$ sets. If $i = n + 1$, then the certified dominating sets of B_n is obtained by choosing $\frac{i}{2} - 1$ pages that share a common edge from the n pages that share a common edge. In addition to that $\{u_1, w_1, w_2, \dots, w_n\}$ and $\{u_2, v_1, v_2, \dots, v_n\}$ are the certified dominating sets. Thus we get $\binom{n}{\frac{i}{2}-1} + 2$ sets. If $i = 2k + 1, 0 \le k \le n$ (but $i \ne n$), $i = 2n$ and $i > 2n + 2$, there are no certified dominating sets.

Case 1: n is even

For $n = 2$,

If $i = 2$, $\{u_1, u_2\}, \{v_1, w_2\}$ and $\{v_2, w_1\}$ are the certified dominating sets. That is $\binom{2}{\frac{2}{2}-1} + 2^2 - 2$ sets. If $i = 3$, $\{u_1, w_1, w_2\}$ and $\{u_2, v_1, v_2\}$ are the certified dominating sets. If $i = 6$, obviously $d_{cer}(B_2, 6) = 1 = \binom{2}{\frac{3}{2}-1}$. If $i = 1$ and 5, there are no certified dominating sets.

For all $n \ge 4$,

If $i = 2$, $\{u_1, v_1\}$ is the certified dominating set. Hence $d_{cer}(B_n, 2) = 1 = \binom{n}{\frac{2}{2}-1}$. If $i = 2k, 2 \le k \le n + 1$ $(but\ k \ne \frac{n}{2}, n)$, then the certified dominating sets of B_n is obtained by choosing $\frac{i}{2} - 1$ pages that share a common edge from the n pages that share a common edge. Hence there are $\binom{n}{\frac{i}{2}-1}$ possible ways. If $i = n$, then the certified dominating sets of B_n is obtained by choosing $\frac{i}{2} - 1$ pages that share a common edge from the n pages that share a common edge. Also select one vertex from each page except the common vertices, but from this set of vertices $\{v_1, v_2, \dots, v_n\}$ and $\{w_1, w_2, \dots, w_n\}$ are not certified dominating sets. Hence there are $\binom{n}{\frac{i}{2}-1} + 2^n - 2$ possible ways. If $i = n + 1$, then $\{u_1, w_1, w_2, \dots, w_n\}$ and $\{u_2, v_1, v_2, \dots, v_n\}$ are the certified dominating sets. Hence, there are two certified dominating sets. If $i = 2k + 1, 0 \le k \le n$ (but $i \ne n + 1$), $i = 2n$ and $i > 2n + 2$, there are no certified dominating sets.

Lemma 2.3:

The certified domination polynomial of the book graph B_n with $2n + 2$ vertices is $x^2\{(1 + x^2)^n - nx^{2n-2}\} + x^n\{2x + 2^n - 2\}$ for all $n \in N$.

Proof: For $n = 1$,

If $i = 2$, $\{u_1, w_1\}$ and $\{u_2, v_1\}$ are the certified dominating sets, if $i = 4$, obviously $d_{cer}(B_1, 4) = 1$ and if $i = 1$ and 3, there are no certified dominating sets. Therefore $D_{cer}(B_1, x) = 2x^2 + x^4 = x^2\{(1 + x^2) - x^0\} + x\{2x + 2 - 2\}$

For n is odd $(n \ge 3)$,

$D_{cer}(B_n, x) = \sum_{i=2}^{2n+2} d_{cer}(B_n, i)\, x^i$

$= \binom{n}{0}x^2 + \binom{n}{1}x^4 + \dots + \binom{n}{\frac{n-3}{2}}x^{n-1} + (2^n - 2)x^n + \left\{\binom{n}{\frac{n-1}{2}} + 2\right\}x^{n+1} + \dots + \binom{n}{n-2}x^{2n-2} + \binom{n}{n}x^{2n+2}$ (from Theorem 2.2)

$= x^2 \sum_{r=0}^{n-2}\binom{n}{r} x^{2r} + \binom{n}{n}x^{2n+2} + (2^n - 2)x^n + 2x^{n+1}$

$= x^2\{\sum_{r=0}^{n}\binom{n}{r} x^{2r} - \binom{n}{n-1}x^{2n-2}\} + x^n\{2x + 2^n - 2\}$

$= x^2\{(1 + x^2)^n - nx^{2n-2}\} + x^n\{2x + 2^n - 2\}$

Similarly, we can prove that for n is even

Lemma2.4: For all $n \geq 5$, $d_{cer}(B_n, i)$

$$= \begin{cases} d_{cer}(B_{n-1}, i) + d_{cer}(B_{n-1}, i-2) & if\ 2 \leq i \leq n-2,\ \ n+3 \leq i \leq 2n-3, i = 2n-1, 2n+1, 2n+2 \\ d_{cer}(B_{n-1}, i) + d_{cer}(B_{n-1}, i-2) - 2^{m-1} + 2 & if\ i = n-1 \\ d_{cer}(B_{n-1}, i) + d_{cer}(B_{n-1}, i-2) + 2^{m} - 4 & if\ i = n \\ d_{cer}(B_{n-1}, i) + d_{cer}(B_{n-1}, i-2) - 2^{m-1} + 4 & if\ i = n+1 \\ d_{cer}(B_{n-1}, i) + d_{cer}(B_{n-1}, i-2) - 2 & if\ i = n+2 \\ d_{cer}(B_{n-1}, i-2) + (m-1) & if\ i = 2n-2 \\ d_{cer}(B_{n-1}, i-2) & if\ i = 2n \end{cases}$$

Proof:

Case 1: $2 \leq i \leq n-2$

$d_{cer}(B_n, i) = \binom{n}{\frac{i}{2}-1}$ (by Theorem 2.2)

$= \binom{n-1}{\frac{i}{2}-1} + \binom{n-1}{\frac{i}{2}-2} = \binom{n-1}{\frac{i}{2}-1} + \binom{n-1}{\frac{i-2}{2}-1}$ (Since $\binom{m}{r} = \binom{m-1}{r} + \binom{m-1}{r-1}$)

$= d_{cer}(B_{n-1}, i) + d_{cer}(B_{n-1}, i-2)$

Similarly we can prove for $n+3 \leq i \leq 2n-3, i = 2n-1, 2n+1, 2n+2$

Case 2: $i = n-1$

If n is odd,

$d_{cer}(B_n, n-1) = \binom{n}{\frac{n-1}{2}-1}$(by Theorem 2.2)

$= \binom{n-1}{\frac{n-1}{2}-1} + \binom{n-1}{\frac{n-1}{2}-2} = \left\{\binom{n-1}{\frac{n-1}{2}-1} + 2^{n-1} - 2\right\} + \binom{n-1}{\frac{n-3}{2}-1} - 2^{n-1} + 2$

$= d_{cer}(B_{n-1}, n-1) + d_{cer}(B_{n-1}, n-3) - 2^{n-1} + 2$

Similarly, we can prove that for n is even.

Case 3: $i = n$

If n is odd,

$d_{cer}(B_n, n) = 2^n - 2$ (by Theorem 2.2)

$= 2 + 0 + 2^n - 2 - 2$

$= d_{cer}(B_{n-1}, n) + d_{cer}(B_{n-1}, n-2) + 2^n - 4$

Similarly, we can prove that for n is even

Case 4: $i = n+1$

If n is odd,

$d_{cer}(B_n, n+1) = \binom{n}{\frac{n+1}{2}-1} + 2$ (by Theorem 2.2)

$$= \binom{n-1}{\frac{n+1}{2}-1} + \binom{n-1}{\frac{n+1}{2}-2} + 2 = \binom{n-1}{\frac{n+1}{2}-1} + \left\{\binom{n-1}{\frac{n-1}{2}-1} + 2^{n-1} - 2\right\} + 2 - 2^{n-1} + 2$$

$$= d_{cer}(B_{n-1}, n+1) + d_{cer}(B_{n-1}, n-1) - 2^{n-1} + 4$$

Similarly, we can prove that for n is even.

Case 5: $i = n+2$

If n is odd,

$d_{cer}(B_n, n+2) = 0$ (by Theorem 2.2)

$$= 0 + 2 - 2$$

$$= d_{cer}(B_{n-1}, n+2) + d_{cer}(B_{n-1}, n) - 2$$

Similarly, we can prove that for n is even

Case 6: $i = 2n-2$

$d_{cer}(B_n, 2n-2) = \binom{n}{\frac{2n-2}{2}-1}$ (by Theorem 2.2)

$$= \binom{n-1}{\frac{2n-2}{2}-1} + \binom{n-1}{\frac{2n-2}{2}-2} = \binom{n-1}{n-2} + \binom{n-1}{\frac{2n-4}{2}-1}$$

$$= d_{cer}(B_{n-1}, 2n-4) + (n-1)$$

Case 7: $i = 2n$

$d_{cer}(B_n, 2n) = 0 = d_{cer}(B_{n-1}, 2n-2)$

Remark 2.5:

Sum of coefficients of a certified dominating polynomial of the book graph B_n is $2^{n+1} - n$ for all $n \in N$.

Table 1. $d_{cer}(B_n, i)$, the number of coefficients of B_n with cardinality i

B_n \ i	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
B_1	2	0	1																		
B_2	3	2	0	0	1																
B_3	1	6	5	0	0	0	1														
B_4	1	0	18	2	6	0	0	0	1												
B_5	1	0	5	30	12	0	10	0	0	0	1										
B_6	1	0	6	0	77	2	20	0	15	0	0	0	1								
B_7	1	0	7	0	21	126	37	0	35	0	21	0	0	0	1						
B_8	1	0	8	0	28	0	310	2	70	0	56	0	28	0	0	0	1				
B_9	1	0	9	0	36	0	84	510	128	0	126	0	84	0	36	0	0	0	1		
B_{10}	1	0	10	0	45	0	120	0	1232	2	252	0	210	0	120	0	45	0	0	0	1

3. Conclusion

In this paper, we have derived an important relation for the coefficient of the certified domination polynomial of the book graph of B_n. Using this relation, we have to find out the certified domination polynomial of the book graph B_n .

References

[1] T. Anitha Baby, and Y.A. Shiny, "Connected 2-Domination Polynomials of Some Graph Operations," *Malaya Journal of Matematik*. vol. 8, no. 4, pp. 2329- 2332, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[2] A.M. Anto, and P. Paul Hawkins, "Perfect Dominating Sets and Perfect Domination Polynomial of Some Standard Graphs," *Turkish Journal of Computer and Mathematics Education*, vol. 11, no. 3, pp. 1178-1182, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[3] Saeid Alikhani, and Somayeh Jahari, "Domination Polynomial of Generalized Book Graphs," *Journal of Mathematical Extension*, vol. 10, no.3, pp. 35-45, 2016. [Google Scholar] [Publisher Link]

[4] Jyoti Rani, and Seema Mehra, "Total Domination Polynomial of Some Graphs," *Advances and Applications in Discrete Mathamatics*, vol. 25, no. 2, pp.161-171, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[5] Magda Dettlaf et al., "Certified Domination," *AKCE International Journal of Graphs and Combinatorics*, vol. 17, no. 1, pp 86-97, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[6] Rafia Yoosuf, and Preethi Kuttipulackal, "The Connected Domination Polynomial of Some Graph Constructions," *Journal of Physics: Conference Series*, vol. 1850, pp. 1-9, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[7] Nechrivan B. Ibrahim, and Assad A. Jund, "Edge Connected Domination Polynomial of a Graph," *Palestine Journal of Mathamatics*, vol. 7, no. 2, pp. 458-467, 2018. [Google Scholar] [Publisher Link]

[8] P.C. Priyanka Nair, and T. Anitha Baby, "2-Dominating Sets and 2-Domination Polynomials of Cycles," *Turkish Online Journal of Qualitative Inquiry*, vol.12, no. 10, pp. 4325-4332, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[9] V.S. Radhika, and A. Vijayan, "Connected Vertex-Edge Domination Polynomials of Some Graphs," *Advances and Applications in Discrete Mathematics*, vol. 27, no. 2, pp. 183-192, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[10] Saeid Alikhani, and Yee-Hock Peng, "Introduction to Domination Polynomial of a Graph," *arXiv:0905.2251*, pp. 1-10, 2009. [CrossRef] [Google Scholar] [Publisher Link]

[11] Y.A. Shiny, and T. Anitha Baby, "Connected 2-Dominating Sets and Connected 2-Domination Polynomials of the Complete Bipartite Graph $K_{2,m}$," *Ratio Mathematica*, vol. 44, pp. 51-55, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[12] A. Vijayan, and M. Felix Nes Mabel, "Connected Domination Polynomial of Some Graphs," *IOSR Journal of Mathematics,* vol. 12, no. 4, pp. 13-16, 2016. [Crossref] [Google Scholar] [Publisher Link]