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Review Article

Monomial Algebras and Electrical Network Monitoring Problem

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Abstract *- The PDS problem in graphs mathematically models the difficulty of monitoring electrical networks, which is inspired by the deployment of Phasor Measuring Units (PMUs) in power systems. This paper introduces a novel algebraic framework for studying power domination based on graph-derived monomial algebras. By encoding propagation rules as algebraic saturation operations, the power domination number is described in terms of minimal generating sets of monomial ideals. Examples of standard graph families are presented. The relationships with algebraic invariants, such as regularity and projective dimension, are investigated. Algebraic methods for computing and finding bounds for the power domination number are proposed. This paper connects commutative algebra and practical graph theory, with implications for electrical network research.*

Keywords *- Monomial Algebras, Electrical Network Monitoring Problem, Ideals.*

1. Introduction

Efficient monitoring of electrical power networks is critical for fault identification and stability. Placing Phasor Measurement Units (PMUs) at certain nodes in the network is a well-studied strategy for monitoring the entire system using direct measurement and propagation rules. A PMU situated at a bus vertex observes the bus and its gearbox lines, while Kirchhoff's principles allow for indirect observation of more buses without the need for additional PMUs. The aim is to identify the minimum number of phase measurement units needed to monitor the whole network.

The electrical network monitoring problem is mathematically modelled by the power domination problem in graphs [10]. Here, the electrical network is represented as a graph H= (N, L), where nodes correspond to buses and lines correspond to transmission lines. A set $P \subseteq N$ of chosen vertices represents PMU placements. Initially, the closed neighbourhood V[P] (the chosen vertices and their neighbours) is observed, reflecting direct monitoring by PMUs. Then a propagation rule is applied: if a monitored vertex has exactly one unmonitored neighbour, that neighbour is forced to become monitored. This model shows how electrical laws allow indirect monitoring of additional buses. If all vertices are monitored during this process, then S is a Power Dominating Set (PDS). The power domination number, γP(H), is the minimum cardinality of such a set. Thus, the electrical network monitoring problem is equivalent to finding the minimum PMUs. The PD problem involves identifying a set of nodes with minimum cardinality from which all nodes may be monitored using the particular propagation rules. The power domination number ($\gamma_P(H)$) refers to the smallest size of a given set.

Graph invariants frequently admit elegant algebraic interpretations. The classic example is the study of independence and vertex coverings using graph edge ideals. Similarly, the least rank problem and the zero forcing number connect combinatorial propagation processes with linear algebra [2], [6]. The PD problem, presented in the context of monitoring power grids, generalises the domination problem by incorporating a propagation rule similar to zero force. The power domination number, $\gamma_P(G)$, represents the smallest size of a vertex set from which all vertices may be observed using these rules.

Although power domination has been explored extensively from combinatorial and computational viewpoints, little attention has been paid to its algebraic structure. On the other hand, algebraic graph theory, particularly through edge ideals and monomial algebras, has yielded profound insights into graph parameters such as domination, independence, and vertex coverage [1], [5].

In this paper, an algebraic framework that encodes the electrical network monitoring problem in terms of monomial algebras is proposed. It is shown that power domination can be interpreted as a problem of generating sets in monomial ideals, and that algebraic invariants can serve as tools for bounding or computing $\gamma P(G)$.

2. Preliminaries

Power Dominating Set in graphs :

Suppose H = (N, L) is a finite simple graph with vertex set $N=\{n1, \ldots, nv\}$. A dominating set is a subset $D \subseteq N$ such that every vertex is either in D or adjacent to a vertex of D. A Power Dominating Set (PDS) is defined as follows:

Initially, all nodes in N[D] are monitored.
Propagation rule: If a monitored node has exactly one unmonitored neighbour, then that neighbour becomes observed.
If eventually all vertices are observed, then D is a PDS.

The PD number $\gamma P(H)$ is the minimum size of a PDS.

Monomial Algebras from Graphs:

Given H=(N,L) with $N=\{x1,\ldots,xv\}$, the edge ideal is $I(H)=\langle xixj:\{i,j\}\in E\rangle \subseteq k[x1,\ldots,xv]$. The monomial algebra associated with H is $A(H)=k[x1,\ldots, xv]/I(H)$. Vertex covers, independent sets, and domination sets have been studied via primary decompositions and algebraic invariants of I(H). This idea is extended to power domination sets.

Power Domination via Monomial Ideals:

For each vertex vi, define a monomial representing its closed neighborhood: $mi = \prod_{vj \in V[xv]} xj$.
Define the power domination ideal as $J(H)=\langle mi: vi \in V\rangle \subseteq k[x1, \ldots ,xv]$.

If a variable xj divides exactly one generator in the chosen set, propagation forces xj into the algebraic closure. Thus, the propagation rule corresponds to saturation in monomial ideals.

3. The Main Theorem

Theorem 1.

For a graph H, the PD number $\gamma P(H)$ equals the minimal cardinality of a generating set of monomials in J(H) whose saturation covers all variables.

Proof:

Each generator mi corresponds to choosing a vertex with its closed neighbourhood. Propagation corresponds to saturation: once a variable xj is uniquely divisible by a chosen generator, it is forced into the product. Iterating this recovers the propagation closure. Thus, the minimal generating set corresponds to a minimal PDS.

Example 1: Path Graph Pn: We know that $\gamma P(Pn)=1$. Also, $J(Pn)=\langle x1x2, x2x3, \ldots, xn-1xn\rangle$. A single generator suffices under saturation, confirming the algebraic characterization.

Example 2: Cycle Cn: We know that $\gamma P(Cn)=1$. Also ,$J(Cn)=\langle x1x2, x2x3, \ldots, xn-1xn, xnx1\rangle$. One generator saturates all variables for all n.

Let M be a finitely generated graded module over the polynomial ring $R=k[x1,\ldots,xn]$. The Castelnuovo–Mumford regularity (denoted reg(M)) is an invariant that measures the "complexity" of the minimal free resolution of M.

If $0\to\bigoplus R(-dp)\beta p\to\cdots\to\bigoplus R(-d1)\beta 1\to R\beta 0\to M\to 0$ is the minimal graded free resolution of M, then $reg(M)=\max\{di-i: \beta i\neq 0\}$. Intuitively, the shifts di, tell us where generators appear in degree. Subtracting the homological step i corrects for resolution length. The regularity is the largest such corrected degree. It measures how complicated the syzygies (relations among generators) are.

When M=R/I for a monomial ideal I (like the edge ideal of a graph H), the regularity becomes a combinatorial invariant that reflects structural properties of the graph. For example, if I(H) is the edge ideal of H: $I(H)=\langle xixj:\{i,j\}\in E(H)\rangle$, then reg(R/I(H)) is tied to matching numbers, induced subgraphs, and projective dimension.

For forests H: reg(R/I(H))=induced matching number(H)+1. For general graphs: regularity provides bounds on domination and independence parameters.

If a power domination ideal is defined as J(H), then reg(R/J(H)) gives an upper bound on the number of propagation steps needed to observe the whole graph. Since γP(H) is the minimum number of PMUs (generators), and regularity reflects the "depth" of relations, studying reg(R/J(H)) may give inequalities of the form: γP(H) ≤ f(reg(R/J(H))), or vice versa.

Example: Take the path graph Pn. Then, the edge ideal, I(Pn)=⟨x1x2, x2x3, …, xn−1xn⟩ , reg(R/I(Pn)) =⌈n/3⌉ and PD number is γP(Pn)=1. This shows regularity captures propagation complexity (roughly, how many steps are needed to color all vertices), while power domination number captures initial seed size.

In short, Castelnuovo–Mumford regularity is an algebraic invariant from free resolutions. For monomial ideals from graphs, it encodes the combinatorial structure. It can serve as a bridge parameter between algebra (complexity of generators/relations) and power domination (complexity of propagation in networks).

4. Castelnuovo–Mumford Regularity and Power Domination

The Castelnuovo–Mumford regularity of a graded module provides a powerful tool for measuring the algebraic complexity of ideals associated with graphs. Let H = (V, E) be a finite simple graph, and let R = k[x_v: v \in V] be the polynomial ring over a field (k) with one variable corresponding to each vertex of G . If I(G) ⊆ R denotes a monomial ideal encoding a graph parameter (for example, the edge ideal or a power domination ideal), then the Castelnuovo–Mumford regularity of R/I(H), denoted reg(R/I(H)), captures the structural and combinatorial complexity of H [3,4,7].

In the context of network monitoring, the power domination number (γP(H)) measures the minimum number of vertices that must be chosen to observe the entire graph under the propagation rules of power domination. While γP(H) counts initial resources, the propagation process itself reflects the complexity of dependencies among vertices, which can be naturally encoded by the syzygies of the corresponding monomial ideal. Thus, reg(R/I(H))) can be interpreted as an algebraic measure of the "propagation depth" of H [8,9].

5. Applications

The algebraic perspective on power domination has numerous applications in computational, practical, and theoretical realms. Gröbner basis techniques in software like Macaulay2 can compute the ideal's minimal generating sets J(H), providing a new algebraic strategy for obtaining the power domination number γP(H). In practice, electrical networks modelled as graphs can be transformed into monomial algebras, where algebraic methods supplement combinatorial algorithms for optimal phasor measurement unit (PMU) positioning. From a theoretical approach, this framework bridges the gap between commutative algebra and graph theory, allowing structural results and invariants to be transferred between the two areas while also furthering the study of monitoring problems in complex networks.

6. Open Problems

Future research can expand the monomial algebra framework in several promising directions. One natural extension is to consider directed and weighted networks, where orientations and edge weights play a critical role in monitoring dynamics; this would require adapting the definition of J(H) to encode directional influence and weighted propagation costs. Another avenue is the investigation of binomial edge ideals and toric ideals, which may capture richer propagation behaviors compared to monomial ideals by reflecting pairwise dependencies and algebraic relations inherent in the network. Additionally, the study of Betti numbers of (J(H)) offers a potential algebraic analogue to the stages of propagation in power domination, with higher syzygies possibly corresponding to deeper layers of monitoring complexity. Finally, for large-scale electrical networks where exact computation is infeasible, the development of algebraic approximation techniques could provide scalable methods for estimating power domination parameters, thereby integrating algebraic insights with applied network optimization.

7. Conclusion

This paper presents an algebraic framework for addressing the power domination problem. It involves associating graphs with monomial ideals and analyzing their Castelnuovo-Mumford regularity and invariants. This viewpoint emphasizes how algebraic structures can encode both initial monitoring requirements and propagation dynamics in network observability. By combining Gröbner basis computations, Betti numbers, and regularity to power domination parameters, we create new avenues for analyzing network monitoring problems using commutative algebra techniques.

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