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Research Article

On The Rainbow Neighborhood Number Of Shiju-Graphs

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Abstract *- A rainbow neighborhood of a graph Γ is defined as the closed neighborhood N [v] of a vertex v ∈ V (Γ) that contains at least one vertex of every color present in a given proper chromatic coloring C of Γ. The total number of vertices in Γ whose closed neighborhoods form rainbow neighborhoods is termed the rainbow neighborhood number of Γ, denoted by* $r_\chi(\Gamma)$*. In this study, the authors introduce a new class of graphs, called Shiju-graphs, which are subgraphs of the strong product of two paths. Also, explore and analyze various properties of these graphs, including their chromatic number, rainbow neighborhood number, and several other related graph parameters.*

Keywords *- Color Class, Rainbow Neighborhood, Rainbow Neighborhood Number, Shiju Graphs.*

1. Introduction

For general notations and concepts in graphs and digraphs, see [1], [2], [3]. For further concepts of graph coloring, refer to [4], [5]. Unless stated otherwise, all graphs considered in this paper are simple and finite graphs. A coloring of a graph Γ is an assignment of colors to its elements (vertices and/or edges) subject to certain conditions. Many practical and real-life situations paved paths to different graph coloring problems. Graph coloring refers to the assignment of colors to the vertices of a given graph in such a way that no two adjacent vertices share the same color. Note that the color classes in a graph Γ are independent sets in Γ. A k-coloring of a graph Γ uses k colors to color its vertices by partitioning its vertex set V into a color classes. The minimum value of k for which Γ admits a k-coloring is called the chromatic number of Γ, denoted by $\chi(\Gamma)$ [6].

In graph theory, the concept of a rainbow neighborhood provides an interesting connection between vertex colorings and local structural properties of a graph. For a given graph Γ with a proper chromatic coloring C, the closed neighborhood of a vertex v∈V(Γ), denoted by N[v], is said to be a *rainbow neighborhood* if it contains at least one vertex of every color used in the coloring C. In other words, within N[v], all the distinct colors that appear in the chromatic coloring of Γ are represented at least once. The total number of vertices in Γ whose closed neighborhoods are rainbow neighborhoods is known as the rainbow neighborhood number of the graph, and it is denoted by $r_\chi(\Gamma)$. This parameter serves as a measure of how widely the colors are distributed throughout the graph and indicates the extent to which color diversity appears locally around vertices [7].

The concept of rainbow neighborhoods was introduced by Kok and collaborators as a coloring-based graph parameter linking chromatic coloring and local vertex structure[7]. For a graph Γ with a chromatic coloring C, a vertex v yields a rainbow neighborhood if its closed neighborhood N[v] contains at least one vertex of every color used in C. The number of such vertices defines the rainbow neighborhood number $r_\chi(\Gamma)$, which has been investigated for various standard graph families such as paths, cycles, complete graphs, and set-graphs. Subsequent studies explored its minimum and maximum versions, generalized it to distance-based neighborhoods, and analyzed its dependence on the structure and distribution of color classes. Parallelly, the chromatic and structural properties of strong product graphs, especially those formed from paths (also known as king's graphs), have been studied for their complex adjacency patterns and applications in grid-like networks. However, the rainbow neighborhood behavior of strong-product subgraphs remains largely unexplored. Motivated by this gap, the present study introduces a new class of graphs called Shiju-graphs—subgraphs of the strong product of two paths—and investigates their chromatic number, rainbow neighborhood number, and related parameters.

An independent set in Γ is a subset of vertices $I \subseteq V$ such that no two vertices in I are adjacent. A maximum independent set is an independent set of the largest possible size in Γ. The cardinality of such a set is called the independence number of the graph, denoted by $\alpha(\Gamma)$. The independent-maximal chromatic number of a graph Γ, denoted $\chi^{i-max}(\Gamma)$, is defined as the maximum number of colors needed in a proper coloring of Γ. Such that each color class is a maximal independent set. The $i-max$ number of a graph Γ, denoted by $\alpha^{i-max}(\Gamma)$, is defined as $\alpha^{i-max}(\Gamma) = min\{k: \chi^{i-max}(\Gamma) \leq \chi(\Gamma)+k,\ k \in \mathbb{N}\}$. The rainbow neighborhood number of a graph that possesses a maximax independence, maximum proper coloring is denoted by $r_\chi^{i-max}(\Gamma)$[8].

The rainbow neighborhood number of certain fundamental graph classes has been determined in [8], [9], [10]. Some of the relevant results in these papers are as follows.

(1) For $n \geq 1$, $r_\chi(P_n) = n$.

(2) For $n \geq 3, r_\chi(C_n) = \begin{cases} 3 & \text{if n is odd} \\ 2 & \text{if n is even} \end{cases}$

(3) For $n \geq 1$, $r_\chi(K_n) = n$.

In this study, introduce a new class of graphs called Shiju-graphs, which are defined as subgraphs of the strong product of two paths. The strong product structure allows these graphs to exhibit rich adjacency patterns, combining features of both Cartesian and direct products. We investigate several fundamental properties of Shiju-graphs, including their chromatic number**,** rainbow neighborhood number**,** and other related graph parameters. These explorations provide new insights into how vertex colorings interact with the structural characteristics of this novel graph family, contributing to the broader understanding of coloring-based graph invariants.

2. Rainbow Neighborhood Number of Shiju-Graphs

In this section, the authors introduce the notion of shiju-graph as explained below.

Consider the points (m,n) in the XY-plane where $m,n \in \mathbb{Z}^+$ and take a rectangular block of this plane with sides parallel to the axes. If the order of such a block is $m \times n$, it is called an m,n-block

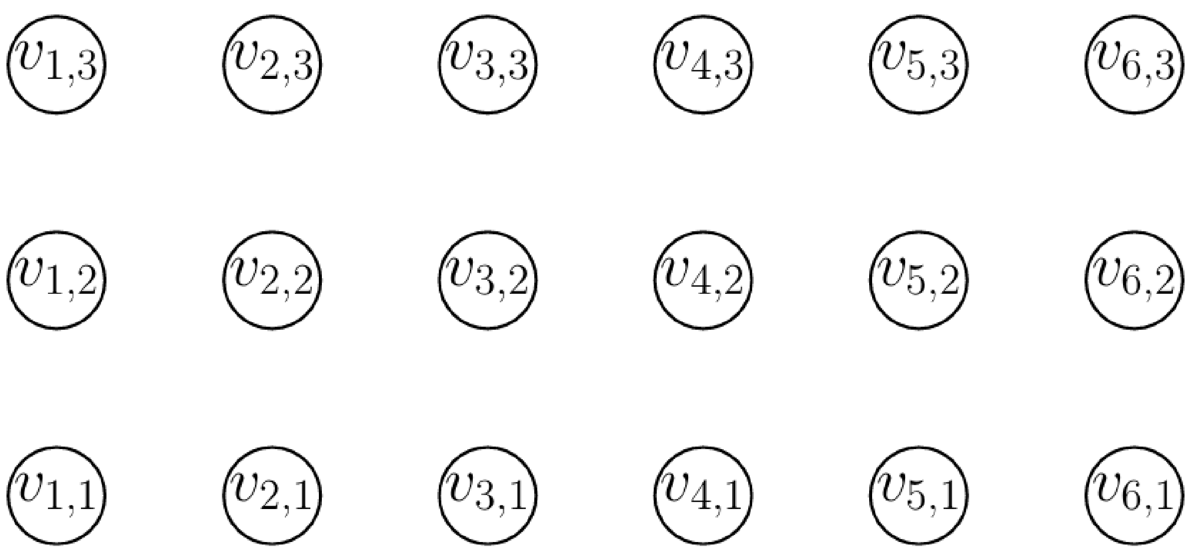

Fig. 1 6,3 Block

Now, define different types of shiju-graph as follows.

Definition 2.1. ${}^{m}_{n}$ ***shiju graph(y)*** *is a graph with mn vertices of an mn-block, where* $V=\{v_{1,1}\cdots,v_{1,n},v_{2,1}\cdots,v_{2,n},\cdots,v_{m,1}\cdots,v_{m,n}\}$ *and* $E=\{v_{i,j}v_{i,j+1}:1\le i\le m, 1\le j<n\}$ *and it is denoted by* ${}^{m}_{n}S(y)$.

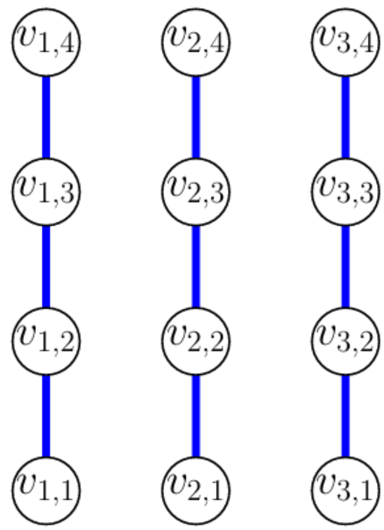

Fig. 2 ${}^{3}_{4}$ **Shiju Graph(*y*)**

${}^{m}_{n}S(y)$ contains m disjoint paths with n vertices. Note that ${}^{m}_{n}S(y)$ contains mn vertices and $m(n-1)$ edges. Clearly, ${}^{m}_{n}S(y)$ is a forest if $m,n\ge 2$.

Proposition 2.2. *Let* ${}^{m}_{n}S(y)$ *be the shiju-graph mentioned above. Then*

(a). *Chromatic number of* ${}^{m}_{n}S(y)$, $\chi({}^{m}_{n}S(y))=\begin{cases}1 & \text{if } n=1\\ 2 & \text{otherwise}\end{cases}$

(b). *Maximax independence, maximum proper coloring*

$$\chi^{\text{i-max}}({}^{m}_{n}S(y))=\begin{cases}1 & \text{if } n=1\\ 2 & \text{otherwise}\end{cases}$$

(c). *Rainbow neighborhood number,* $r_\chi({}^{m}_{n}S(y))=mn$.

(d). $r^{i-max}({}^{m}_{n}S(y))=mn$.

(e). $\alpha^{i-max}({}^{m}_{n}S(y))=0$.

Proof. (a) n=1it is a null graph and the case is trivial. Also $\chi(P_n)=2$ for all $n\in\mathbb{N}$ and $n>1$. Since ${}^{m}_{n}S(y)$ contains m disjoint paths, $\chi({}^{m}_{n}S(y))=2$.

(b) If n=1, the case is trivial. Suppose $n>1$and let $X_1=\{v_{i,2k+1}:1\le i\le m, 0\le k\le\left\lceil\frac{mn}{2}\right\rceil\}$. Then X_1 is the maximum independent set with $\left\lceil\frac{mn}{2}\right\rceil$ elements. Color all the vertices of X_1 with color c_1. Consider the graph ${}^{m}_{n}S(y)-X_1$. Let it be Γ'. Clearly, Γ' be a null graph with $\left\lceil \frac{mn}{2} \right\rceil$ elements. Color all the vertices of Γ' with color c_2. Hence the result.

(c) If $n = 1$, ${}^{m}_{n}S(y)$ is an empty graph with n vertices, and each vertex yields a rainbow neighborhood. Hence the result. Suppose $n \neq 1$, then ${}^{m}_{n}S(y)$ contains m disjoint paths with n vertices. Since $\chi(P_n) = 2$ for all $n \in \mathbb{N}$ and $n>1$, each vertex yields a rainbow neighborhood. Hence the result.

(d) The result follows through similar reasoning found in (c).

(e) Since $\chi({}^{m}_{n}S(y))=\chi^{i-max}({}^{m}_{n}S(y))$ in both cases, the result follows trivially.

Definition 2.3. ${}^{m}_{n}$ ***shiju graph(x)*** *is a graph with mn vertices of an mn-block, where* $V=\{v_{1,1}\cdots,v_{1,n},v_{2,1}\cdots,v_{2,n},\cdots,v_{m,1}\cdots,v_{m,n}\}$ and $E=\{v_{i,j}v_{i+1,j}: 1\leq i<m, 1\leq j\leq n\}$ *and it is denoted by* ${}^{m}_{n}S(x)$.

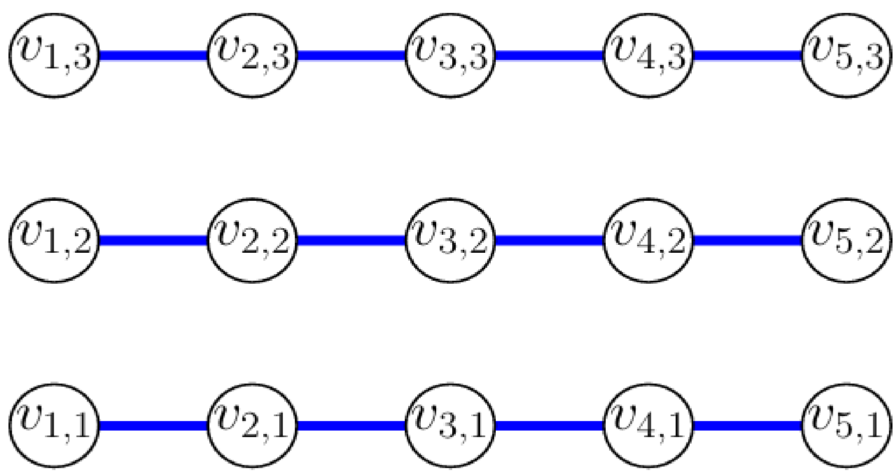

Fig. 3 ${}^{5}_{3}$ **Shiju Graph(*x*).**

${}^{m}_{n}S(x)$ contains n disjoint paths with m vertices. Note that ${}^{m}_{n}S(x)$ contains mn vertices and $n(m-1)$ edges. Clearly, ${}^{m}_{n}S(x)$ is a forest if $m,n\geq 2$. We can see that ${}^{m}_{n}S(y)\cong {}^{n}_{m}S(x)$. Hence, based on Proposition 2.2, the following results are obtained

Proposition 2.4. *Let* ${}^{m}_{n}S(x)$ *be the shiju-graph mentioned above. Then*

(a). Chromatic number of ${}^{m}_{n}S(x)$, $\chi({}^{m}_{n}S(x)) = \begin{cases} 1 & \text{if } m=1 \\ 2 & \text{otherwise} \end{cases}$

(b). Maximax independence, maximum proper coloring

$$\chi^{\text{i-max}}({}^{m}_{n}S(x)) = \begin{cases} 1 & \text{if } m=1 \\ 2 & \text{otherwise} \end{cases}$$

(c). Rainbow neighborhood number, $r_\chi({}^{m}_{n}S(x)) = mn$.

(d). $r^{i-max}({}^{m}_{n}S(x)) = mn$.

(e). $\alpha^{i-max}({}^{m}_{n}S(x)) = 0$.

Definition 2.5. ${}^{m}_{n}$ ***shiju graph(d)*** *is a graph with mn vertices of an m,n-block, where* $V=\{v_{1,1}\cdots,v_{1,n},v_{2,1}\cdots,v_{2,n},\cdots,v_{m,1}\cdots,v_{m,n}\}$ and

$E=\{v_{i,j}v_{i+1,j+1}:1\leq i<m, 1\leq j<n\}\cup\{v_{i,j}v_{i+1,j-1}:1\leq i<m, 2\leq j\leq n\}$ *and it is denoted by* ${}^{m}_{n}S(d)$.

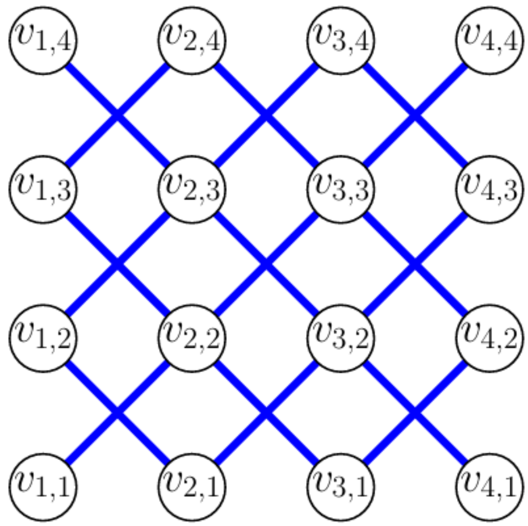

Fig. 4 ${}^{4}_{4}$ **Shiju graph(*d*).**

Note that ${}^{m}_{n}S(d)$ contains mn vertices and $2(m-1)(n-1)$ edges. We can see that ${}^{m}_{n}S(d)\cong{}^{n}_{m}S(d)$.

Proposition2.6. *Let* ${}^{m}_{n}S(d)$ *be the shiju-graph mentioned above. Then*

(a). *Chromatic number of* ${}^{m}_{n}S(d)$, $\chi({}^{m}_{n}S(d))=\begin{cases}1 & \text{if m}=1\text{ or n}=1\\ 2 & \text{otherwise}\end{cases}$

(b). *Maximax independence, maximum proper coloring*

$$\chi^{i-\max}({}^{m}_{n}S(d))=\begin{cases}1 & \text{if m}=1\text{ or n}=1\\ 2 & \text{otherwise}\end{cases}$$

(c). *Rainbow neighborhood number,* $r_\chi({}^{m}_{n}S(d))=mn$.

(d). $r^{i-max}({}^{m}_{n}S(d))=mn$.

(e). $\alpha^{i-max}({}^{m}_{n}S(d))=0$.

Proof. (a) If m=1 or n=1, the graph is an empty graph and the result is trivial. Suppose $m\neq 1$ and $n\neq 1$. Since ${}^{m}_{n}S(d)\cong{}^{n}_{m}S(d)$, it can be assumed, without loss of generality, we can assume that $n\geq m$. Further,

let m = 2. Then ${}^{m}_{n}S(d)$ contains 2 disjoint paths, hence the result. Next, assume that $m\geq 3$, then ${}^{m}_{n}S(d)$ contains two disjoint connected graphs which are formed by connected C_4's with or without thorns. Then $\chi({}^{m}_{n}S(d))$=2. Hence the result.

(b) If m=1 or n=1, the graph is an empty graph and the result is trivial. Suppose $m\neq 1$ and $n\neq 1$ and at least one of m *and* n is even. Since ${}^{m}_{n}S(d)\cong{}^{n}_{m}S(d)$, without loss of generality, it can be assumed that m is even. Then

$X_1=\{v_{1,1}\cdots,v_{1,n},v_{3,1}\cdots,v_{3,n},v_{m-1,1}{}^{2}\cdot,v_{m-1,n}\}$ is the maximum independent set with $\frac{mn}{2}$ elements. Color all the vertices of X_1 with color c_1. Consider the graph $\Gamma'={}^{m}_{n}S(d)-X_1$. Clearly, Γ' be a null graph with $\frac{mn}{2}$ elements. Color all the vertices of Γ' with color c_2. Hence the result. Suppose both m and n are odd. Since ${}^{m}_{n}S(d)\cong{}^{n}_{m}S(d)$, without loss of generality, it can be assumed that $m \geq n$. Then $X_1=\{v_{1,1}\cdots,v_{m,1},v_{1,3}\cdots,v_{m,3},v_{1,n}\cdots,v_{m,n}\}$ is the maximum independent with $\frac{m(n+1)}{2}$ elements. Color all the vertices of X_1 with color c_1. Consider the graph $\Gamma={}^{m}_{n}S(d)-X_1$. Clearly, Γ be a null graph with $\frac{m(n-1)}{2}$ elements. Color all the vertices of Γ' with color c_2. Hence the result.

(c) If m=1or n=1, ${}^{m}_{n}S(d)$ is an empty graph with mn vertices and each vertex yields a rainbow neighborhood. Hence the result. Suppose $m \neq 1$ and $n \neq 1$. Since ${}^{m}_{n}S(d)\cong{}^{n}_{m}S(d)$, without loss of generality, it can be assumed that $n \geq m$, and suppose m=2. Then ${}^{m}_{n}S(d)$ contains 2 disjoint paths P_n, hence the result. Next, assume that $m \geq 3$, then ${}^{m}_{n}S(d)$ contains two disjoint connected graphs which formed by connected C_4's with or without thorns. Then $\chi^{i-max}({}^{m}_{n}S(d))=2$ and each vertex yields a rainbow neighborhood. Hence the result.

(d) The result follows through similar reasoning found in(c)

(e) Since $\chi({}^{m}_{n}S(d))=\chi^{i-max}({}^{m}_{n}S(d))$ in all cases, the result follows trivially.

Definition 2.7. ${}^{m}_{n}$ ***shiju graph(xy)*** *is a graph with mn vertices of an m,n-block, where*

$V=\{v_{1,1}\cdots,v_{1,n},v_{2,1}\cdots,v_{2,n},\cdots,v_{m,1}\cdots,v_{m,n}\}$ and

$E=\{v_{i,j}v_{i,j+1}:1\leq i \leq m, 1\leq j<n\} \cup \{v_{i,j}v_{i+1,j}:1\leq i <m, 1\leq j \leq n\}$ *and it is denoted by* ${}^{m}_{n}S(xy)$.

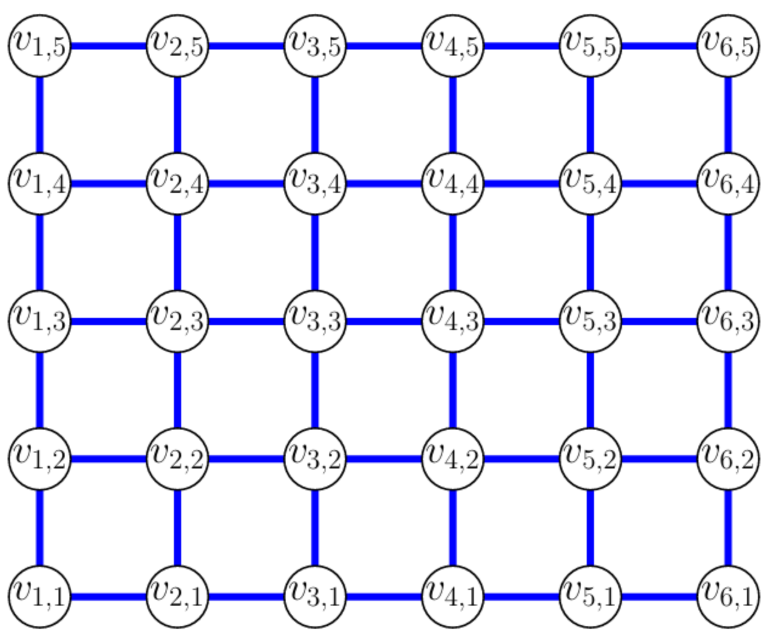

Fig. 5 ${}^{6}_{5}$shiju graph(*xy*).

Note that ${}^{m}_{n}S(xy)$ contains mn vertices and $2mn-(m+n)$ edges. We can see that ${}^{m}_{n}S(xy)\cong{}^{n}_{m}S(xy)$ and ${}^{m}_{n}S(xy)$ is connected.

Proposition 2.8. *Let ${}^{m}_{n}S(xy)$ be the shiju-graph mentioned above. Then*

(a). *Chromatic number of ${}^{m}_{n}S(xy)$,* $\chi({}^{m}_{n}S(xy)) = \begin{cases} 1 & \text{if m} = 1 \text{ and n} = 1 \\ 2 & \text{otherwise} \end{cases}$

(b). *Maximax independence, maximum proper coloring*

$$\chi^{i-\max}({}^{m}_{n}S(xy)) = \begin{cases} 1 & \text{if m} = 1 \text{ and n} = 1 \\ 2 & \text{otherwise} \end{cases}$$

(c). *Rainbow neighborhood number,* $r_\chi({}^{m}_{n}S(xy)) = mn$.

(d). $r^{i-max}({}^{m}_{n}S(xy)) = mn$.

(e). $\alpha^{i-max}({}^{m}_{n}S(xy)) = 0$.

Proof.(a).Consider the following cases.

Case 1: Suppose m=1and n=1, then ${}^{m}_{n}S(xy)$=$K_1$and the result is trivial.

Case 2: Suppose m=1and $n \neq 1$, then ${}^{m}_{n}S(xy)$=P_n and the result is trivial.

Case 3: Suppose $m \neq 1$ and n=1, then ${}^{m}_{n}S(xy)$=P_m and the result is trivial.

Case 4: Suppose $m \neq 1$ and $n \neq 1$, then ${}^{m}_{n}S(xy)$ is formed by connected C_4's.

Hence $\chi({}^{m}_{n}S(xy))$=2.

(b). Consider the following cases.

Case 1: Suppose m=1and n=1, then ${}^{m}_{n}S(xy)$=$K_1$and the result is trivial.

Case 2: Suppose m=1and $n \neq 1$, then ${}^{m}_{n}S(xy)$=P_n and the result is trivial.

Case 3: Suppose $m \neq 1$ and n=1, then ${}^{m}_{n}S(xy)$=P_m and the result is trivial.

Case 4: Suppose $m \neq 1$ and $n \neq 1$, then consider the following subcases

Subcase 4.1: Suppose m and n are even, then the set

X_1=$\{v_{1,1},v_{1,3},\cdots,v_{1,n-1},v_{2,2},v_{2,4},\cdots,v_{2,n},\cdots,v_{m,2},v_{m,4},\cdots,v_{m,n}\}$formsa maximum independent set with $\frac{mn}{2}$ elements.

Subcase 4.2: Suppose m even and n odd, then the set

X_1=$\{v_{1,1},v_{1,3},\cdots,v_{1,n},v_{2,2},v_{2,4},\cdots,v_{2,n-1},\cdots,v_{m,2},v_{m,4},\cdots,v_{m,n-1}\}$forms a maximum independent set with $\frac{mn}{2}$ elements.

Subcase 4.3: Suppose m odd and n even, then the set

X_1=$\{v_{1,1},v_{1,3},\cdots,v_{1,n-1},v_{2,2},v_{2,4},\cdots,v_{2,n},\cdots,v_{m,1},v_{m,3},\cdots,v_{m,n-1}\}$forms a maximum independent set with $\frac{mn}{2}$ elements.

In the above three subcases, color all the vertices of X_1 with color c_1. Consider the graph Γ'=${}^{m}_{n}S(xy)$−X_1. Clearly, Γ' be a null graph with $\frac{mn}{2}$ elements. Color all the vertices of Γ' with color c_2. Hence the result.

Subcase 4.4: Suppose m and n are odd, then the set

X_1=$\{v_{1,1},v_{1,3},\cdots,v_{1,n},v_{2,2},v_{2,4},\cdots,v_{2,n-1},\cdots,v_{m,1},v_{m,3},\cdots,v_{m,n}\}$forms a maximum independent set with $\frac{mn+1}{2}$ elements. Color all the vertices of X_1 with color c_1. Consider the graph Γ'=${}^{m}_{n}S(xy)$−X_1. Clearly, Γ' be a null graph with $\frac{mn-1}{2}$ elements. Color all the vertices of Γ' with color c_2. Hence the result.

(c). Consider the following cases.

Case 1: Suppose m=1and n=1, then ${}^{m}_{n}S(xy)=K_1$and the result is trivial.

Case 2: Suppose m=1 and $n \neq 1$, then ${}^{m}_{n}S(xy)=P_n$ and the result is trivial.

Case 3: Suppose $m \neq 1$ and n=1, then ${}^{m}_{n}S(xy)=P_m$ and the result is trivial.

Case 4: Suppose $m \neq 1$ and $n \neq 1$. Since the chromatic number of ${}^{m}_{n}S(xy)$ is 2, every vertex yields a rainbow neighborhood. Hence the result.

(d) The result follows through similar reasoning found in (c)

(e) Since $\chi({}^{m}_{n}S(xy))=\chi^{i-max}({}^{m}_{n}S(xy))$ in all cases, the result follows trivially.

Definition2.9. ${}^{m}_{n}$ ***shiju graph*** *(yd)is a graph with mn vertices of an m,n-block, where* $V=\{v_{1,1}\cdots,v_{1,n},v_{2,1}\cdots,v_{2,n},\cdots,v_{m,1}\cdots,v_{m,n}\}$ *and* $E=\{v_{i,j}v_{i,j+1}:1\leq i \leq m, 1\leq j <n\} \cup \{v_{i,j}v_{i+1,j+1}:1\leq i <m, 1\leq j<n\} \cup \{v_{i,j}v_{i+1,j-1}:1\leq i<m, 2\leq j \leq n\}$ *and it is denoted by* ${}^{m}_{n}S(yd)$.

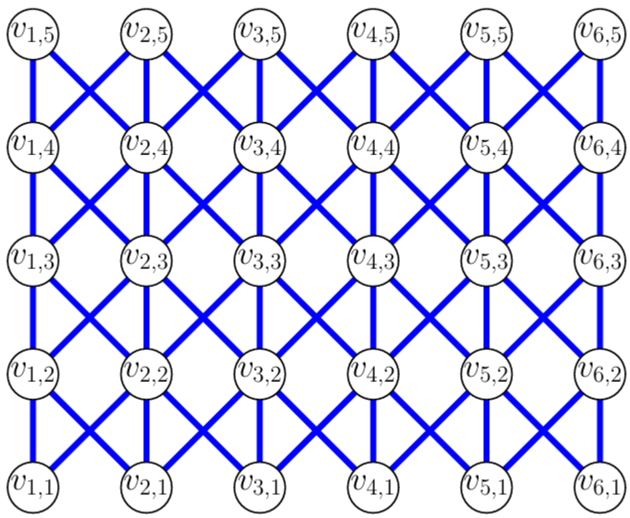

Fig. 6 ${}^{6}_{5}$ shiju graph(*yd*).

Note that ${}^{m}_{n}S(yd)$ contains mn vertices and $(3m-2)(n-1)$ edges. We can see that ${}^{m}_{n}S(yd)$ is connected.

Proposition2.10. *Let ${}^{m}_{n}S(yd)$ be the shiju-graph mentioned above. Then*

(a). Chromatic number of ${}^{m}_{n}S(yd)$, $\chi({}^{m}_{n}S(yd)) = \begin{cases} 1 & \text{if n} = 1 \\ 2 & \text{otherwise} \end{cases}$

(b). Maximax independence, maximum proper coloring

$$\chi^{i-\max}({}^{m}_{n}S(yd)) = \begin{cases} 1 & \text{if n} = 1 \\ 2 & \text{otherwise} \end{cases}$$

(c). Rainbow neighborhood number, $r_\chi({}^{m}_{n}S(yd)) = mn$.

(d). $r^{i-max}({}^{m}_{n}S(yd)) = mn$.

(e). $\alpha^{i-max}({}^{m}_{n}S(yd)) = 0$.

Proof.(a). Consider the following cases.

Case 1: Suppose m=1and n=1, then ${}^{m}_{n}S(yd)$=$K_1$and the result is trivial.

Case 2: Suppose m=1 and $n \neq 1$, then ${}^{m}_{n}S(yd)$=P_n and the result is trivial.

Case 3: Suppose $m \neq 1$ and n=1, then ${}^{m}_{n}S(yd)$ is an empty graph with m vertices, and the result is trivial.

Case 4: Suppose $m \neq 1$ and $n \neq 1$, then one can color odd columns by $c_1$and even columns by c_2. Hence $\chi({}^{m}_{n}S(yd)) = 2$.

(b). Consider the following cases.

Case 1: Suppose m=1and n=1, then ${}^{m}_{n}S(yd)$=$K_1$and the result is trivial.

Case 2: Suppose m=1 and $n \neq 1$, then ${}^{m}_{n}S(yd)$=P_n and the result is trivial.

Case 3: Suppose $m \neq 1$ and n=1, then ${}^{m}_{n}S(yd)$ is an empty graph with m vertices, and the result is trivial.

Case 4: Suppose $m \neq 1$ and $n \neq 1$, then consider the following subcases

Subcase 4.1: Suppose m and n are even, then the set

X_1=$\{v_{1,1},v_{1,3},\cdots,v_{1,n-1},v_{2,1},v_{2,3},\cdots,v_{2,n-1},\cdots,v_{m,1},v_{m,3},\cdots,v_{m,n-1}\}$forms a maximum independent set with $\frac{mn}{2}$ elements.

Subcase 4.2: Suppose m odd and n even, then the set

X_1=$\{v_{1,1},v_{1,3},\cdots,v_{1,n-1},v_{2,1},v_{2,3},\cdots,v_{2,n-1},\cdots,v_{m,1},v_{m,3},\cdots,v_{m,n-1}\}$forms a maximum independent set with $\frac{mn}{2}$ elements.

In the above two subcases, color all the vertices of X_1 with color c_1. Consider the graph Γ'=${}^{m}_{n}S(yd)-X_1$. Clearly, Γ' be a null graph with $\frac{mn}{2}$ elements. Color all the vertices of Γ' with color c_2. Hence the result.

Subcase 4.3: Suppose m even and n odd, then the set

X_1=$\{v_{1,1},v_{1,3},\cdots,v_{1,n},v_{2,1},v_{2,3},\cdots,v_{2,n},\cdots,v_{m,1},v_{m,3},\cdots,v_{m,n}\}$ forms a maximum independent set with $\frac{m(n+1)}{2}$ elements.

Subcase 4.4: Suppose m and n are odd, then the set

X_1=$\{v_{1,1},v_{1,3},\cdots,v_{1,n},v_{2,1},v_{2,3},\cdots,v_{2,n},\cdots,v_{m,1},v_{m,3},\cdots,v_{m,n}\}$forms a maximum independent set with $\frac{m(n+1)}{2}$ elements.

In the above two subcases, color all the vertices of X_1 with color c_1. Consider the graph Γ'=${}^{m}_{n}S(yd)-X_1$. Clearly, Γ' be a null graph with $\frac{m(n-1)}{2}$ elements. Color all the vertices of Γ' with color c_2. Hence the result.

(c). Consider the following cases.

Case 1: Suppose m=1and n=1, then ${}^{m}_{n}S(yd)$=$K_1$and the result is trivial.

Case 2: Suppose m=1 and $n \neq 1$, then ${}^{m}_{n}S(yd)$=P_n and the result is trivial.

Case 3: Suppose $m \neq 1$ and n=1, then ${}^{m}_{n}S(yd)$ is an empty graph with m vertices, and the result is trivial.

Case 4: Suppose $m \neq 1$ and $n \neq 1$. Since the chromatic number of ${}^{m}_{n}S(yd)$ is 2, every vertex yields a rainbow neighborhood. Hence the result.

(d) The result follows through similar reasoning found in (c)

(e) Since $\chi({}^{m}_{n}S(yd))$=$\chi^{i-max}({}^{m}_{n}S(yd))$ in all cases, the result follows trivially.

Definition2.11. ${}^{m}_{n}$ ***shiju graph***(*xd*) *is a graph with mn vertices of an m,n-block, where*

V=$\{v_{1,1}\cdots,v_{1,n},v_{2,1}\cdots,v_{2,n},\cdots,v_{m,1}\cdots,v_{m,n}\}$ *and*

E=$\{v_{i,j}v_{i+1,j}$:$1\leq i<m, 1\leq j\leq n\} \cup \{v_{i,j}v_{i+1,j+1}$:$1\leq i<m, 1\leq j<n\} \cup \{v_{i,j}v_{i+1,j-1}$:$1\leq i<m, 2\leq j\leq n\}$ *and it is denoted by* ${}^{m}_{n}S(xd)$.

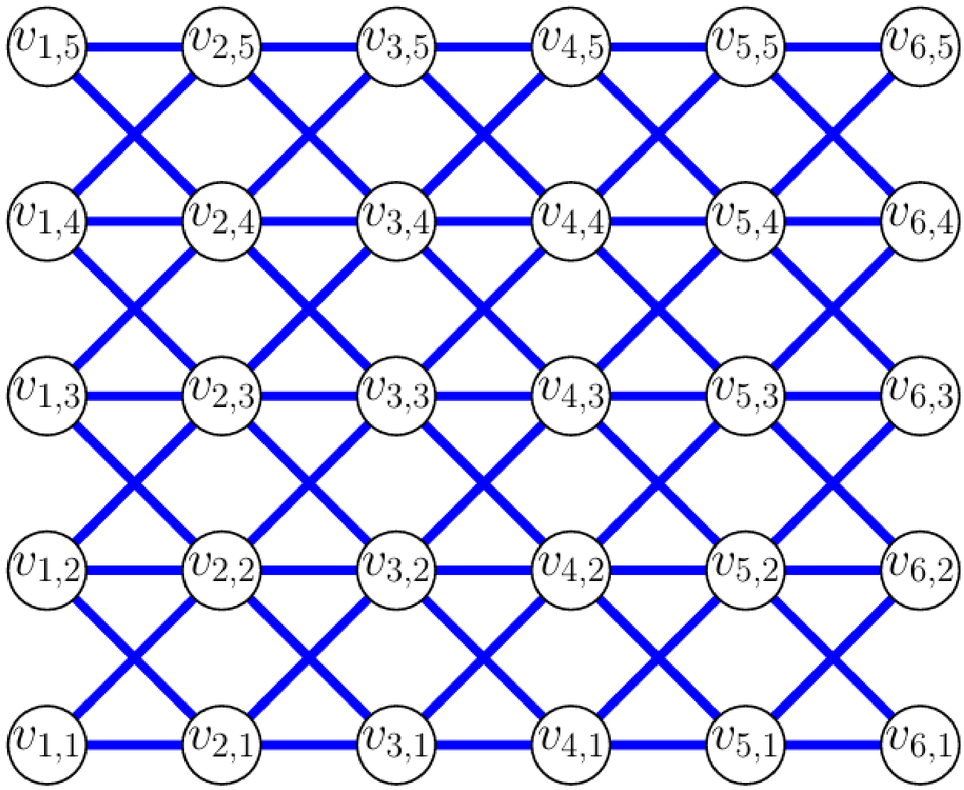

Fig. 7. $^{6}_{5}$ **Shiju graph(*xd*).**

Note that $^{m}_{n}S(xd)$ contains mn vertices and $(3n-2)(m-1)$ edges. We can see that $^{m}_{n}S(yd) \cong {}^{n}_{m}S(xd)$ and $^{m}_{n}S(xd)$ is connected. Hence, based on Proposition 2.10, we get the following results are obtained.

Proposition 2.12. *Let* $^{m}_{n}S(xd)$ *be the shiju-graph mentioned above. Then*

(a). *Chromatic number of* $^{m}_{n}S(xd)$, $\chi(^{m}_{n}S(xd)) = \begin{cases} 1 & \text{if m} = 1 \\ 2 & \text{otherwise} \end{cases}$

(b). *Maximax independence, maximum proper coloring*

$$\chi^{i-\max}(^{m}_{n}S(xd)) = \begin{cases} 1 & \text{if m} = 1 \\ 2 & \text{otherwise} \end{cases}$$

(c). *Rainbow neighborhood number,* $r_\chi(^{m}_{n}S(xd)) = mn$.

(d). $r^{i-max}(^{m}_{n}S(xd)) = mn$.

(e). $\alpha^{i-max}(^{m}_{n}S(xd)) = 0$.

Definition 2.13. $^{m}_{n}$ ***shiju graph(****xyd****)*** *is a graph with mn vertices of an m,n-block, where*

V$=\{v_{1,1}\cdots,v_{1,n},v_{2,1}\cdots,v_{2,n},\cdots,v_{m,1}\cdots,v_{m,n}\}$ *and*
$E=\{v_{i,j}v_{i,j+1}:1 \le i \le m,\ 1 \le j < n\} \cup \{v_{i,j}v_{i+1,j}:\ 1 \le i < m,\ 1 \le j \le n\}$
$\cup \{v_{i,j}v_{i+1,j+1}:1 \le i < m,\ 1 \le j < n\} \cup \{v_{i,j}v_{i+1,j-1}:1 \le i < m,\ 2 \le j \le n\}$
and it is denoted by $^{m}_{n}S(xyd)$.

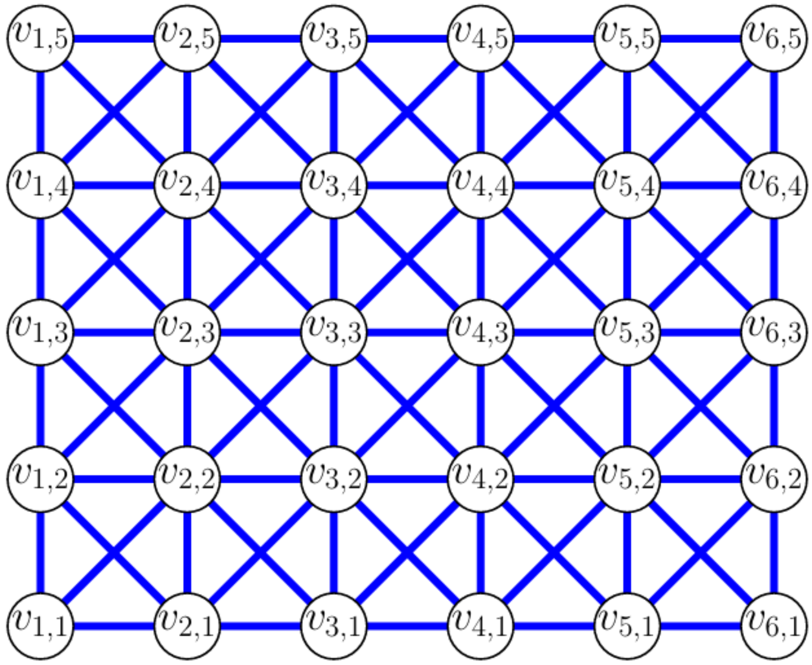

Fig. 8. $\,^{6}_{5}$ **Shiju graph(*xyd*).**

Note that $\,^{m}_{n}S(xyd)$ contains mn vertices and $4mn-(3m+n)+2$ edges. We can see that $\,^{m}_{n}S(xyd)$ is connected.

Proposition 2.14. *Let* $\,^{m}_{n}S(xyd)$ *be the shiju-graph mentioned above. Then*

(a). Chromatic number of $\,^{m}_{n}S(xyd)$, $\chi(\,^{m}_{n}S(xyd)) = \begin{cases} 1 & \text{if } m=1 \text{ and } n=1 \\ 2 & \text{if } m=1, n\neq 1 \text{ or } m\neq 1, n=1 \\ 4 & \text{otherwise} \end{cases}$

(b). Maximax independence, maximum proper coloring

$$\chi^{i-\max}(\,^{m}_{n}S(xyd)) = \begin{cases} 1 & \text{if } m=1 \text{ and } n=1 \\ 2 & \text{if } m=1, n\neq 1 \text{ or } m\neq 1, n=1 \\ 4 & \text{otherwise} \end{cases}$$

(c). Rainbow neighborhood number, $r_\chi(\,^{m}_{n}S(xd)) = mn$.

(d). $r^{i-max}(\,^{m}_{n}S(xyd)) = mn$.

(e). $\alpha^{i-max}(\,^{m}_{n}S(xyd)) = 0$.

Proof.(a).Consider the following cases.

Case 1: Suppose m=1and n=1, then $\,^{m}_{n}S(xyd)=K_1$and the result is trivial.

Case 2: Suppose m=1 and $n\neq 1$, then $\,^{m}_{n}S(xyd)=P_n$ and the result is trivial.

Case 3: Suppose $m\neq 1$ and n=1, then $\,^{m}_{n}S(xyd)=P_m$ and the result is trivial.

Case 4: Suppose $m\neq 1$ and $n\neq 1$, then $\,^{m}_{n}S(xyd)$ is formed by connected K_4's.

Hence $\chi(\,^{m}_{n}S(xyd))$=4.

(b). Consider the following cases.

Case 1: Suppose m=1 and n=1, then ${}_{n}^{m}S(xyd)=K_1$ and the result is trivial.

Case 2: Suppose m=1 and $n \neq 1$, then ${}_{n}^{m}S(xyd)=P_n$ and the result is trivial.

Case 3: Suppose $m \neq 1$ and n=1, then ${}_{n}^{m}S(xyd)=P_m$ and the result is trivial.

Case 4: Suppose $m \neq 1$ and $n \neq 1$, then consider the following subcases

Subcase 4.1: Suppose m and n are even, then the set $X_1=\{v_{1,1},v_{1,3},\cdots,v_{1,n-1},v_{3,1},v_{3,3},\cdots,v_{3,n-1},\cdots,v_{m-1,1},v_{m-1,3},\cdots,v_{m-1,n-1}\}$ forms a maximum independent set with $\frac{mn}{4}$ elements. Color all the vertices of X_1 with color c_1. Consider the graph $\Gamma'={}_{n}^{m}S(xyd)-X_1$. Let $X_2=\{v_{1,2},v_{1,4},\cdots,v_{1,n},v_{3,2},v_{3,4},\cdots,v_{3,n},\cdots,v_{m-1,2},v_{m-1,4},\cdots,v_{m-1,n}\}$ be a maximum independent set of Γ' with $\frac{mn}{4}$ elements. Color all the vertices of X_2 with color c_2. Consider the graph $\Gamma''=\Gamma'-X_2$. Then $X_3=\{v_{2,1},v_{2,3},\cdots,v_{2,n-1},v_{4,1},v_{4,3},\cdots,v_{4,n-1},\cdots,v_{m,1},v_{m,3},\cdots,v_{m,n-1}\}$ be a maximum independent set of Γ'' with $\frac{mn}{4}$ elements. Color all the vertices of X_3 with color c_3. Consider the graph $\Gamma'''=\Gamma''-X_3$. Clearly, Γ''' be a null graph with $\frac{mn}{4}$ elements, color all the vertices of Γ''' with color c_4. Hence the result.

Subcase 4.2: Suppose either m even and n odd or m odd and n *is* even. We can see that ${}_{n}^{m}S(xyd)\cong{}_{m}^{n}S(xyd)$, without loss of generality, assume that m odd and n *is* even. Then the set $X_1=\{v_{1,1},v_{1,3},\cdots,v_{1,n-1},v_{3,1},v_{3,3},\cdots,v_{3,n-1},\cdots,v_{m,1},v_{m,3},\cdots,v_{m,n-1}\}$ forms a maximum independent set with $\frac{n(m+1)}{4}$ elements. Color all the vertices of X_1 with color c_1. Consider the graph $\Gamma'={}_{n}^{m}S(xyd)-X_1$. Let $X_2=\{v_{1,2},v_{1,4},\cdots,v_{1,n},v_{3,2},v_{3,4},\cdots,v_{3,n},\cdots,v_{m,2},v_{m,4},\cdots,v_{m,n}\}$ be a maximum independent set of Γ' with $\frac{n(m+1)}{4}$ elements. Color all the vertices of X_2 with color c_2. Consider the graph $\Gamma''=\Gamma'-X_2$. Then $X_3=\{v_{2,1},v_{2,3},\cdots,v_{2,n-1},v_{4,1},v_{4,3},\cdots,v_{4,n-1},\cdots,v_{m-1,1},v_{m-1,3},\cdots,v_{m-1,n-1}\}$ be a maximum independent set of Γ'' with $\frac{n(m-1)}{4}$ elements. Color all the vertices of X_3 with color c_3. Consider the graph $\Gamma'''=\Gamma''-X_3$. Clearly, Γ''' be a null graph with $\frac{n(m-1)}{4}$ elements, color all the vertices of Γ''' with color c_4. Hence the result.

Subcase 4.3: Suppose m and n are odd, without loss of generality, also assume that $n \geq m$. Then the set $X_1=\{v_{1,1},v_{1,3},\cdots,v_{1,n},v_{3,1},v_{3,3},\cdots,v_{3,n},\cdots,v_{m,1},v_{m,3},\cdots,v_{m,n}\}$ forms a maximum independent set with $\frac{(n+1)(m+1)}{4}$ elements. Color all the vertices of X_1 with color c_1. Consider the graph $\Gamma'={}_{n}^{m}S(xyd)-X_1$. Let $X_2=\{v_{1,2},v_{1,4},\cdots,v_{1,n-1},v_{3,2},v_{3,4},\cdots,v_{3,n-1},\cdots,v_{m,2},v_{m,4},\cdots,v_{m,n-1}\}$ be a maximum independent set of Γ' with $\frac{(n+1)(m+1)}{4}$ elements. Color all the vertices of X_2 with color c_2. Consider the graph $\Gamma''=\Gamma'-X_2$. Then $X_3=\{v_{2,1},v_{2,3},\cdots,v_{2,n},v_{4,1},v_{4,3},\cdots,v_{4,n},\cdots,v_{m-1,1},v_{m-1,3},\cdots,v_{m-1,n}\}$ be a maximum independent set of Γ'' with $\frac{(n-1)(m-1)}{4}$ elements. Color all the vertices of X_3 with color c_3. Consider the graph $\Gamma'''=\Gamma''-X_3$. Clearly, Γ''' be a null graph with $\frac{(n-1)(m-1)}{4}$ elements, color all the vertices of Γ''' with color c_4. Hence the result.

Consider the following cases.

Case 1: Suppose m=1 and n=1, then ${}^{m}_{n}S(xyd)=K1,$ and the result is trivial.

Case 2: Suppose m=1 and $n \neq 1$, then ${}^{m}_{n}S(xyd)=P_n$ and the result is trivial.

Case 3: Suppose $m \neq 1$ and n=1, then ${}^{m}_{n}S(xyd)=P_m$ and the result is trivial.

Case 4: Suppose $m \neq 1$ and $n \neq 1$. Since each vertex of ${}^{m}_{n}S(xyd)$ is a vertex of K_4, it yields a rainbow neighborhood. Hence the result.

(d) The result follows through similar reasoning found in (c)

(e) Since $\chi({}^{m}_{n}S(xyd))=\chi^{i-max}({}^{m}_{n}S(xyd))$ in all cases, the result follows trivially.

3. Conclusion

In this paper, the authors introduce a new class of graphs, referred to as Shiju-graphs. We explore and analyze various graph-theoretic parameters associated with these graphs, including the chromatic number, rainbow neighborhood number, and the maximum independence number. The investigation of these parameters provides deeper insights into the structural and coloring properties of Shiju-graphs, thereby contributing to a more comprehensive understanding of this newly proposed class.

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