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Original Article

Common Fixed Point Theorem for Two Self-Mappings in b-Rectangular Metric Spaces

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Abstract *- In this paper, the author has proved the common fixed point theorem on b-rectangular metric spaces. We have proved common fixed point results by using pair self-mappings in b-rectangular metric spaces. This result is the expansion of some existing results.*

Keywords *- Common Fixed Point, b-rectangular Metric Space, Existence and Uniqueness.*

Subject Classification: *54H25, 47H10.*

1. Introduction

Fixed point theory plays an important role in the development of nonlinear analysis. It is also used in different branches of engineering and science. The concept of b-metric space is introduced in 1989[1]. After that, in 1993, Czerwik [3] extended the results of b-metric spaces. Many researchers used this concept to generalize the renowned Banach fixed-point theorem in b-metric spaces. Boriceanu [2], Czerwik [3] extended the fixed point theorem in b-metric space. George et al. [4] is introduced the concept of b-rectangular metric spaces as a generalization of metric spaces. In this paper, the result is obtained by extending the results mentioned by author Qasim K. Kadim [6] for a b-rectangular metric space. The author has used these results for two self-mappings and proved new conditions for b-rectangular metric spaces.

2. Preliminaries

Definition 2.1 [1, 3]: a pair (X, d) is called a b-metric space where X be a non-empty set and $p \geq 1$ be a real number and d is a metric defined on X, that is, a function defined on $X \times X$ such that for all $x, y, z \in X$, the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq p[d(x, y) + d(y, z)]$ (b-triangular inequality)

Definition 2.2: A pair (X, d) is called a rectangular or generalized metric space, where X be a non-empty set, and let d be a mapping defined on $X \times X$ such that for all $x, y, z \in X$ and all different points $u, v \in X$, each distinct from x and y, if

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, u) + d(u, v) + d(v, z)$ (Rectangular inequality)

Definition 2.3[4]: A pair (X, d) is called a b-rectangular metric space or a b-generalized metric space, where X be a non-empty set, $p \geq 1$ be a given real number, and let d be a mapping defined on X such that for all $x, y, z \in X$ and distinct points $u, v \in X$ distinct from x and y, if

(i) $d(x,y)=0$ if and only if $x=y$;

(ii) $d(x,y)=d(y,x)$;

(iii) $d(x,z)\leq p[d(x,u)+d(u,v)+d(v,z)]$ (b-rectangular inequality)

Then

Definition 2.4[4]: Suppose (X,d) be a b-rectangular metric space and $\{u_n\}$ be a sequence defined on X, $u\in X$. Then:

(i) The sequence $\{u_n\}$ is said to be convergent in (X,d), if $\varepsilon>0$, there exist $n_0\in N$ such that $d(x_n,x)<\varepsilon$ for all $n>n_0$ and $\lim_{n\to\infty} u_n=u$ or $u_n\to u$ as $n\to\infty$.

(ii) The sequence $\{u_n\}$ is said to be b-rectangular Cauchy in (X,d) if $\varepsilon>0$, there exist $n_0\in N$ such that $d(u_n,u_{n+\rho})<\varepsilon$ for all $n>n_0$,$\rho>0$, or $\lim_{n\to\infty} d(u_n,u)=0$ for all $\rho>0$.

(iii) A pair (X,d) is said to be complete if every b-rectangular Cauchy sequence in (X,d) converges to an element of X.

Lemma 2.5[5]: Lat (P,d) be a b-metric space with $p\geq 1$, and T be a mapping on P. Suppose that $\{u_n\}$ is a sequence in P induced by $u_{n+1}=Ru_n$ such that $d(u_n,u)\leq \alpha d(u_{n-1},u)$ for all $n\in N$ where $\alpha\in[0,1)$ is a constant, then $\{u_n\}$ is a Cauchy sequence.

3. Main Result

Let (P,d) be a complete b-rectangular metric space with $p\geq 1$ and $R,T:P\to P$ be two mappings on P satisfying the following condition.

$$d(Ra,Tb)\leq \alpha d(a,Ra)+\beta d(b,Tb)+\gamma d(a,b) \tag{3.1}$$

where α,β *and* γ are non-negative, and a,b in P, then R and T have a unique common fixed point.

Proof: Let For a_0 be any arbitrary point in P, and define the sequence $\{a_n\}$ in P such that

$$a_{2n+1}=Ra_{2n}, a_{2n+2}=Ta_{2n+1}=TRa_{2n}$$

Let us assume that there is some $n\in N$ such that. $a_n=a_{n+1}$

If $n=2k$ then $a_{2k}=a_{2k+1}$ and from

$$d(a_{2k+1},a_{2k+2})=d(Ra_{2k,}Ta_{2k+1})$$

$$\leq \alpha d(a_{2k},Ra_{2k})+\beta d(a_{2k+1},Ta_{2k+1})+\gamma d(a_{2k},a_{2k+1})$$

$$\leq \alpha d(a_{2k},a_{2k+1})+\beta d(a_{2k+1},a_{2k+2})+\gamma d(a_{2k},a_{2k+1})$$

$$\leq 0+\beta d(a_{2k+1},a_{2k+2})+0$$

$d(a_{2k+1},a_{2k+2})\leq \beta d(a_{2k+1},a_{2k+2})$

Which is a contradiction.

Thus $d(a_{2k+1},a_{2k+2})=0$

Hence $a_{2k+1}=a_{2k+2}$. Thus we have $a_{2k}=a_{2k+1}=a_{2k+2}$

Which implies that $a_{2k}=Ra_{2k}=Ta_{2k}$, a_{2k} is a common fixed point of T.

If $n=2k+1$, then using the same process, one can be prove that a_{2k+1} is a common fixed point of R and T.

Now, suppose $a_n \neq a_{n+1}$ for all $n \in N$.

$$d(a_{2n+1}, a_{2n+2}) = d(Ra_{2n}, Ta_{2n+1})$$

$$\leq \alpha d(a_{2n}, Ra_{2n}) + \beta d(a_{2n+1}, Ta_{2n+1}) + \gamma d(a_{2n}, a_{2n+1})$$

$$\leq \alpha d(a_{2n}, a_{2n+1}) + \beta d(a_{2n+1}, a_{2n+2}) + \gamma d(a_{2n}, a_{2n+1})$$

$$\leq 0 + \beta d(a_{2n+1}, a_{2n+2}) + 0$$

$$\leq \beta d(a_{2n+1}, a_{2n+2})$$

which is a contradiction

By induction

$$d(a_{2n+1}, a_{2n}) \leq \beta^2 d(t_1, t_0)$$

Taking limit $n \to \infty$

$$\lim_{n\to\infty} d(a_{2n+1}, a_{2n}) = 0$$

Therefore $\{a_n\}$ is a b-rectangular Cauchy sequence in (P, d). By completeness of (P, d), there exists $u \in E$ such that. $a_{2n} = Ra_{2n+1} \to u$ as $n \to \infty$.

$$d(u, Ru) \leq p[d(u, a_n) + d(a_n, a_{n+1}) + d(a_{n+1}, Ru)]$$

$$\frac{1}{p} d(u, Ru) \leq d(u, a_n) + d(a_n, a_{n+1}) + d(Ta_{2n}, Ru)$$

$$\frac{1}{p} d(u, Ru) \leq d(u, a_n) + d(a_n, a_{n+1}) + \alpha d(u, Ru) + \beta d(a_{2n}, Ta_{2n}) + \gamma d(u, a_{2n})$$

$$\leq d(u, a_n) + \alpha d(u, Ru) + \beta d(a_{2n}, a_{2n+1}) + \gamma d(u, a_{2n})$$

$$\leq \alpha d(u, Ru)$$

$$d(u, Ru) \leq p\alpha d(u, Ru)$$

$$d(u, Ru) = 0 \Rightarrow Ru = u$$

Now, we prove that R and T have a unique common fixed point. Suppose u and v are common fixed point of R and T with $u \neq v$. By (3.1)

$$d(u, v) = d(Ru, Tv)$$

$$\leq \alpha d(u, Ru) + \beta d(v, Tv) + \gamma d(u, v)$$

$$\leq \gamma d(u, v)$$

which is a contradiction.

Therefore, R and T have a unique common fixed point.

Example 3.1: Let $(E, d) = [0,1]$ and $R, T: P \to P$ be two mappings defined by $R(x) = {}^{x}/_{2}$ and $T(x) = {}^{x}/_{4}$ for all x in R, and let a b-rectangular metric space d is defined as $d(x, y) = \min\{1, |x - y|\}$ then R and T have a unique common fixed point.

Corollary 3.2: Let (P, d) be a complete b-rectangular metric space with $p \geq 1$ and $R, T: P \rightarrow P$ be two mappings on P satisfying the condition

$$d(Ra, Tb) \leq \alpha[d(a, Ra) + d(b, Tb) + d(a, b)] + \beta[d(a, RTb) + d(RTa, b)]$$

For all a, b in E and $\alpha, \beta \in [0,1)$ with < 1, then R and T have a unique common fixed point.

Conclusion

In this paper, some basic definitions and fixed point results in the complete b-rectangular metric spaces are presented, and a fixed point theorem for two self-mappings in a b-rectangular metric space was deduced.

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