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Review Article

Weighted Inequalities and Estimates of Operators in the Weighted Setting

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Abstract - *A historical development and scope of the A_p The condition of the scalar weight function are reviewed in the paper. Moreover, the dependence of the weighted norm, $\|T\|_{L^2(w)}$ of some classical operators T on the Muckenhoupt characteristic constant of the weight, $[w]_{A_2}$, qualitatively and quantitatively, and its chronology, as well as linear and multilinear estimates of some fundamental operators, and some important tools used are also studied.*

Keywords - *A_2-Conjecture, A_p-Characteristic Constant, Jones Factorization Theorem, Weight, Weighted Inequality.*

1. Introduction

A linear operator, $T: X \to Y$ defined on two normed spaces X, Y, is said to be bounded if the ratio of the norm of $T(x)$ to that of x is bounded for all nonzero vectors in X, that is, if there exists $C > 0$ such that, $\|T(x)\|_Y \leq C\|x\|_X$, for all, $x \in X, x \neq 0$. The smallest of such C is the norm of the operator T denoted by $\|T\|$. Thus, $\|T\| = \sup_{x \in D(T), x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X}$.

The weighted inequality of the norm of an operator in the weighted Lebesgue space $L^p(w)$ deals with the boundedness and estimation of the operator. Basically, weighted inequality includes weak and strong type inequalities of the form,

$$U(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p V(x) dx, \text{ and } \left(\int_{\mathbb{R}^n} |Tf(x)|^q U(x) dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |Tf(x)|^p V(x) dx\right)^{1/p}$$

Determining the existence of such C for any two weight functions U and V in the two-weight setting. Generally, the constant C is found in terms of the Muckenhoupt characteristic constant, $[w]_{A_p}$.

Characterization of many operators, mainly the Hardy-Littlewood maximal function, in terms of A_p The class was started by Muckenhoupt [5] in the early 1970s, despite the exploration of the necessary and sufficient conditions for the Hilbert transform boundedness in $L^2(w)$ by Helson and Szegö [13] in 1960, in a different way. See also (on page 148)[15]. One can follow [2, 27] for more details, and the chronology of the linear estimates of many operators in $L^2(w)$.

The paper reviewed the weighted inequalities of many types of operators: linear and multilinear in the weighted Lebesgue spaces. $L^p(w)$ for the scalar weight function w along with some mathematical tools.

Partial differential equations, the theory of quasiconformal mapping, complex analysis, factorization theory, Fourier analysis, and operator theory are some fields in which the theory of weighted inequality is very crucial [27]. Even though the weight theory related to the matrix weight function exists in the literature, we restrict our study to the scalar weight function only.

The organization of the paper is as follows: Section 2 consists of the necessary notations and definitions that are used in the paper. Section 3 consists of some methods that are very essential in dyadic harmonic analysis. Section 4 tells about the estimation of linear and multilinear operators, which are the focus points in the theory of weighted inequalities.

2. Preliminaries

This section includes some basic notations and definitions that are used throughout the paper. Exclusion of an implicit constant in the inequality A ≤ cB is denoted by A ≲ B, and if the constant depends on p, q, r, it is denoted by A ≲p,q,r B. Moreover, A ≲ B and B ≲ A are denoted by the symbol A ≈ B. $L^p(w)$ denotes the weighted Lebesgue space of functions g with the weighted norm $\|g\|_{L^p(w)} := \left(\int_{\mathbb{R}} |g(x)|^p\, w(x)dx\right)^{1/p} < \infty$.

2.1. Weight, A_p Condition, Maximal Function

A weight w, locally integrable function, 0 < w(x) < ∞, is said to be in A_p Class, 1 < p < ∞, if there exists a constant k such that, $\langle w\rangle_Q \langle \left(\frac{1}{w}\right)^{\frac{1}{p-1}}\rangle_Q^{p-1} < k$. The smallest such a constant K is called the A_p characteristic constant of w, denoted by $[w]_{A_p}$ or $\|w\|_{A_p}$.

Thus, $[w]_{A_p} := \sup_{Q\subset\mathbb{R}^n} \langle w\rangle_Q \langle w^{-\frac{1}{p-1}}\rangle_Q^{p-1}$, where Q denotes the cubes with the sides parallel to the coordinate axes in R^n, on which the supremum is taken, and $\langle f\rangle_Q := \frac{1}{|Q|}\int_Q f(x)dx$ Is the average of f over Q. $p = 1$ corresponds to the class A_1. Existence of the constant $k > 0$ satisfying $Mw \le kw$ a.e. for an uncentered Hardy-Littlewood maximal function, $Mf(x) = \sup_{x\in Q} \langle |f|\rangle_Q$ determines that $w \in A_1$. The A_p constant for a pair of weights (u, v) in the A_p Class is defined as: $[u, v]_{A_p} := \sup_{Q\subset\mathbb{R}^n} \langle u\rangle_Q \langle v^{-\frac{1}{p-1}}\rangle_Q^{p-1}$. This is the generalization of the one weight case, $u = v = w$ [15]. Similarly, $Mu(x) \le kv(x)$ is the condition required for the pair (u, v) of weights belonging to the class A_1.

The A_p Condition characterizes the weak-type inequality for the Hardy-Littlewood maximal function. The weak-type inequality $u(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \le \frac{1}{\lambda^p}\int_{\mathbb{R}^n} |f|^p v$ holds if and only if $(u, v) \in A_p$ for a given $p, 1 \le p < \infty$, Theorem 7.17 [15]. The $A_{\vec{p}}$ Condition in the multilinear setting was investigated in [3], the multilinear analogue of the A_p Condition. Let, $\vec{P} = (p_j)_{j=1,\dots,m}$ with, $p_j \in [1,\infty), \sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$ and $\vec{w} = (w_1, \dots, w_m), m \ge 2$ consists of nonnegative weight functions, $w_1, \dots, w_m$. $\vec{w}$ is said to satisfy $A_{\vec{p}}$ condition if $\sup_Q \langle v_{\vec{w}}\rangle_Q^{\frac{1}{p}} \prod_{j=1}^m \langle w_j^{1-p_j'}\rangle_Q^{\frac{1}{p_j'}} < \infty$, for $v_{\vec{w}} = \prod_{j=1}^m w_j^{\frac{p}{p_j}}$. When $p_j = 1$, $\langle w_j^{1-p_j'}\rangle_Q^{\frac{1}{p_j'}}$ is understood as $\|w_j^{-1}\|_{L^\infty(Q)}$ or $\left(\inf_Q w_j\right)^{-1}$. The multilinear $A_{\vec{p}}$ Condition is necessary and sufficient for the boundedness of the multilinear maximal function given by $M(\vec{f})(x) := \sup_{x\in Q} \prod_{j=1}^m \langle |f_j(y_j)|\rangle_Q$ for some $\vec{f} = (f_1, \dots, f_m)$.
For the theory in detail, see [15, 20, 26].

2.2. Haar Multipliers

Haar multiplier $T_s f(x) := \sum_{I\in D} s(x, I)\langle f, h_I\rangle h_I(x)$ is analogous to a pseudodifferential operator, replacing the trigonometric function by the Haar function, introduced by Alfred Haar in 1910, given by $h_I(x) = \frac{1}{|I|^{\frac{1}{2}}}\left(1_{I_+} - 1_{I_-}\right)$, where I_- is the left half, and I_+ Is the right half of the dyadic interval $I \in D$. $D := \{I = [m2^{-n}, (m+1)2^{-n}) : m, n \in Z\}$ is the standard dyadic grid, and 1_I Is the characteristic function, which is 1 in the set, and 0 otherwise.

Basically, there are two types of Haar multipliers depending on the variable symbol $s(x, I) \equiv \sigma_I \left(\frac{w(x)}{\langle w\rangle_I}\right)^t$, where $\{\sigma_I\}_{I\in D}$ is a sequence, $\sigma_I \in \{\pm 1\}$, t is any real number, and w is a weight function. They are

- $T_\sigma f(x) := \sum_{I\in D} \sigma_I \langle f, h_I\rangle h_I(x)$, called the constant Haar multiplier, when $w = 1$, and $S(x, I) = \sigma_I$; independent of t, x.
- $T_w^t f(x) = \sum_{I\in D} \frac{w(x)}{\langle w\rangle_I}\langle f, h_I\rangle h_I(x)$, called the variable Haar multiplier, when $S(x, I) = \sigma_I$; depends on x.

Notice that the necessary and sufficient condition for the boundedness of T_σ is $\sigma = \{\sigma_I\}_{I\in D}$ is bounded, Lemma 10, and the necessary and sufficient condition for the boundedness of T_w^t is $w \in RH_p^d$, Theorem 1 [31].

2.2.1. Multilinear Haar Multipliers

Kunwar [4] introduced the multilinear analogue of the Haar multiplier, $T_\epsilon^{\vec{\alpha}}(f_1, \dots, f_m) := \sum_{I\in D} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}$ and explored its boundedness whenever $\epsilon = \{\epsilon_I\}_{I\in D}$ is bounded.

2.3. Paraproduct, Dyadic Square Function

Bony was the first to come up with the paraproduct in the work on nonlinear partial differential equations, and now it is central to harmonic analysis. The dyadic paraproduct $\pi_b f$ is defined by $\pi_b f := \sum_{I \in D} \langle b, h_I \rangle \langle f \rangle_I h_I(x)$, where b, f are locally integrable functions. Heuristically, a paraproduct can be thought of as "half a product": $bf \sim \pi_b f + \pi_f b$ since the product $bf = \pi_b f + \pi_f b + \pi_b^* f$ [27]. The dyadic paraproduct is bounded in L^p whenever, $b \in BMO^d$.

Kunwar [14] introduced $P^{\vec{\alpha}}, \pi_b^{\vec{\alpha}}$, which are multilinear analogues of the paraproduct on more than one function given as:
$P^{\vec{\alpha}}(f_1, \dots, f_m) := \sum_{I \in D} \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}$ and
$T_\epsilon^{\vec{\alpha}}(f_1, \dots, f_m) := \sum_{I \in D} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}$, where $\vec{\alpha} \in \{0,1\}^m, m > 1, \sigma(\vec{\alpha}) := \#\{i: \alpha_i = 0\}$. Moreover, $f_j(I, 0) := \langle f, h_I \rangle = \int_{\mathbb{R}} f(x) h_I(x) dx$, and $f_j(I, 1) := \langle f \rangle_I$.

The dyadic square function for the function f is given by $Sf(x) := \left(\sum_{I \in D} \frac{|\langle f, h_I \rangle|^2}{|I|} 1_I(x) \right)^{\frac{1}{2}}$, where h_I is the Haar function and 1_I Is the characteristic function corresponding to the dyadic interval I in D.

2.4. Sparse Family, Sparse Operator

Lerner [4] introduced an alternative and relatively simple method of proving A_2 theorem by introducing the following concepts. S is a family of cubes in $\mathbb{R}^n$. S is said to be η-sparse $(0 < \eta < 1)$ If for any $Q \in S$, there exists pairwise disjoint measurable sets $E_Q \subset Q, E_Q \cap E_{Q'} = \emptyset$ for distinct Q, Q' in S satisfying $|E_Q| \geq \eta |Q|$. Given a sparse family S, the sparse operator is given by $A_S f(x) := \sum_{Q \in S} \langle f \rangle_Q 1_Q(x)$.

3. Methods

Out of many tools in harmonic analysis, some are discussed below.

3.1. Bellman Function Method

The Bellman function, in principle, is the source of control theory, a branch of mathematics. A. Beurling created the Bellman function technique for the first time [47]. Donald Burkholder developed the Bellman function approach in harmonic analysis to get the norm in the L^p Space of the Martingale transform. Linear estimates of the Martingale transform, Dyadic square function, Beurling transform, Hilbert transform, and Dyadic paraproduct in the weighted space, $L^2(w)$, were obtained using the method [27]. Beznosova in [32] used Bellman function techniques to determine

- The dependence of the paraproduct norm in $L^2(w)$ on $[w]_{A_2}$ sharply
- The dependence of the Haar multiplier norm in the unweighted Lebesgue space L^2 on $[w]_{A_2}$ of the weight in the multiplier sharply
- Estimation of the weighted square function in two weighted settings.

D. Chung in [6] used the Bellman function method and proved that the operator norm of commutators of some transforms: Hilbert, Riesz, and Beurling with b; the BMO function depends quadratically on $[w]_{A_2}$; the characteristic of the weight.

Using the Bellman function, M. C. Pereyra in [28] explored the optimal dependence of operator norms of the Haar multipliers T_w^t to the corresponding RH_2^d or A_2^d characteristic of the weight in $L^2(\mathbb{R})$ for $t = 1, \pm\frac{1}{2}$.

3.2. Jones Factorization Theorem

The Jones factorization theorem is an important result in the study of A_p Weights. Muckenhoupt's A_p The condition provides the necessary and sufficient condition for the boundedness of many operators, such as maximal functions, square functions, and Hilbert transforms in weighted Lebesgue spaces. It is central to the weight theory.

The main essence of the Jones factorization theorem is the characterization of A_p weights in terms of A_1 weights.
Jones in [35] proved the following theorem: A weight w satisfies the A_p Condition, $1 < p < \infty$, if and only if it can be expressed as the product of w_1 and $(w_2)^{1-p}$, for $w_1, w_2 \in A_1$ [7, 11]. It was conjectured by Muckenhoupt in 1979 at the Williamstown conference [8]. A weaker version of the theorem is presented in [36], a paper by Coifman and Rochberg [35].

3.3. Rubio de Francia Extrapolation Theorem

Rubio de Francia's Extrapolation theorem, explored by Rubio de Francia, is one of the most powerful findings in harmonic analysis used for the study of weighted norm inequalities. Rubio de Francia declared it in [17] and explored it in [18]. The first version [16] is as follows:

"If, for some, p_0 A sublinear operator is bounded from $L^{p_0}(w)$ to $L^{p_0}(w)$ for all $w \in A_{p_0/\lambda}$ with $1 \leq \lambda < \infty$, and $\lambda \leq p_0 < \infty$, then it is bounded from $L^p(w)$ to $L^p(w)$ for all $w \in A_{p/\lambda}$ and $\lambda < p < \infty$ ". A second proof was given in [19] and reproduced in [20]. In these proofs, there are two cases: p is smaller than or greater than p_0. The Jones factorization theorem is crucially used in the extrapolation theorem [16]. Another version of the evidence (Theorem 7.8) is available in [15]. The theorem is as follows.

"Fix r, $1 < r < \infty$. If T is a bounded operator on $L^r(w)$ for any $w \in A_r$, with operator norm depending only on the A_r Constant of w, then T is bounded on $L^p(w)$, $1 < p < \infty$, for any $w \in A_p$".

Cruz-Uribe et al. in [9] explored a unified approach that dealt with both cases in the simpler form (Theorem 1.4) as follows: $\int_{\mathbb{R}^n} |Tf|^p w dx \leq C \int_{\mathbb{R}^n} |f|^p\, w dx$ holds good for any weight function $w \in A_p, p \in [1, \infty)$ whenever an operator T satisfies $\int_{\mathbb{R}^n} |Tf|^2 w dx \leq C \int_{\mathbb{R}^n} |f|^2\, w dx$ for any weight w of A_2. Explicitly, an operator will be bounded on weighted L^p for all $1 < p < \infty$ if it is bounded on $L^2(w)$. Antonio Cardoba, Rubio de Francia's colleague, summarized the philosophy of the result as " There are no. L^p spaces, only weighted L^2 " [7–9]. The theory also proves norm inequalities in a large class of Banach function spaces [8]. Dragičević et al. in [34] proved the theorem with sharp bounds. Moreover, a different version of the proof is given by Grafakos in [26]. Rubio de Francia's extrapolation is crucial to the proof of the so-called A_2 conjecture, which is the sharp constant estimate $\|Tf\|_{L^p(w)} \lesssim_{n,p,T} [w]_{A_p}^{max\left\{1,\frac{1}{p-1}\right\}} \|f\|_{L^p(w)}$, where T is a Calderón-Zygmund singular integral operator [29].

3.4. Sparse Operators

The sparse domination of different operators in harmonic analysis has been the subject of considerable attention. A. K. Lerner explored a simple proof of the A_2 conjecture proved by T. Hytönen using the technique of sparse domination [2]. It was an alternative proof of the A_2 conjecture using the sparse operator. An alternative approach to the results on pointwise domination of $w-$ Calderón-Zygmund operators in [30, 44] by sparse operators was also explored by A. K. Lerner in [4].

4. Linear and Multilinear Estimates of Some Operators

4.1. Boundedness Characterization of Many Operators: Muckenhoupt A_p Condition: A Stepping Stone

Although in 1960, Helson and Szegö [13] used complex variable methods to come up with the necessary and sufficient conditions for the weights to make the Hilbert transform in weighted spaces. $L^2(w)$ space-bounded, the study of estimating operators in the weighted Lebesgue space $L^p(w)$ Started in the early 1970s. The primary concern of many mathematicians was to determine the necessary and sufficient conditions for the boundedness of an operator T in a weighted Lebesgue space. $L^p(w)$, where $1 < p < \infty$. Muckenhoupt [5] was the first to show that the maximal function is bounded on $L^p(w)$ If and only if w satisfies the A_p Condition in 1972. In 1973, this result was extended to the Hilbert transform by Richard Hunt, Benjamin Muckenhoupt and Richard Wheeden in [37]. In 1974, Coifman and Fefferman explored that. A_p Condition is necessarily satisfied by w for the singular integrals or Hardy-Littlewood maximal functions to be bounded on $L^p(w)$ i.e., $w \in A_p$ implies $\int_{\mathbb{R}^n} |Tf(x)|^n\, w(x)dx \leq c \int_{\mathbb{R}^n} |f(x)|^n\, w(x)dx$, where "T " is either the singular integral operator or the maximal function of Hardy-Littlewood. The same result also holds for their commutators with the function in the BMO space. This was a simplified proof of the weighted norm inequalities of Richard Hunt, Benjamin Muckenhoupt, and Richard Wheeden [38].

4.2. Sharp Dependence of Operator Norm with $[w]_{A_p}$

Although it was established that the norm of an operator in a weighted space is dependent on $[w]_{A_p}$ Of the weight w, qualitatively, the exact nature of the relationship quantitatively remained unknown. Later, mathematicians sought to explain how the norm of some operators in a weighted space depends on the so-called. A_p Characteristic of the weight. The determination of the sharp estimate of the weighted norm of an operator with the A_p characteristic of the weight, $[\mathrm{w}]_{\mathrm{A_p}}$

It was a crucial issue. In other words, the main question is to find the least power $\alpha = \alpha(p)$ such that $\|Tf\|_{L^p(w)} \lesssim_{n,T} [w]_{A_p}^{\alpha(p)} \|f\|_{L^p(w)}$ for all $f \in L^p(w)$ And a given operator T. This type of estimate, for many operators, is used in partial differential equations. Buckley [39] in 1993 was responsible for the first outcome of this kind related to the maximal function. He proved that, $\|Mf\|_{L^p(w)} \lesssim_p [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}, p > 1$. The estimate is sharp because the exponent $\frac{1}{p-1}$ It cannot be replaced by a smaller one.

4.3. Linear Dependence of A_2 Characteristic of w, $[w]_{A_2}$ and $L^2(w)$ Norm

In 2000, Witter [21] (in Theorem 3.1) established a linear relationship between the A_2 characteristic of w, $[w]_{A_2}$ and the weighted norm of the Martingale transform, $\|T_\sigma(f)\|_{L^2(w)}$ in $L^2(w)$ space as: $\|T_\sigma f\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}$ for $w \in A_2$ and $f \in L^2(w)$. It was proved that the same linear bound applied to the dyadic square function [22, 40]. The same was shown for the Ahlfors-Beurling transform, which has important consequences in the theory of quasi-conformal mapping [41]. In 2007 and 2008, Petermichl [42, 43] explored the sharp boundedness of the Hilbert transform and the Riesz transform, respectively, on the weighted Lebesgue space, proving the dependence of their weighted norms on the A_2 characteristic of the weight. In 2008, Beznosova [33] proved that the weighted norm of the dyadic paraproduct depends on the weight's A_2 characteristic, $[w]_{A_2}$. See [2, 27].

4.4. Hytönen's Celebrated A_2 Theorem and A_2 Conjecture

The A_2 The conjecture concerns the sharp dependency of the weighted. L^2 norm of the Calderón-Zygmund operator on the A_2 characteristic of w. More precisely, $\|Tf\|_{L^2(w)} \leq C_T[w]_{A_2}\|f\|_{L^2(w)}$, where T is a Calderón-Zygmund operator and $w \in A_2$. It was first proved by Hytönen [45, 46]. He proved in the scalar case, the weighted L^p Norm inequality's sharp constant is proposed to $[w]_{A_p}^{max\left\{1,\frac{1}{p-1}\right\}}$. Initially, it was proved for $1 < p < \infty$ by using the sharp version of the Rubio de Francia extrapolation theorem [34]. The proof of the celebrated A_2 The theorem for the Calderón-Zygmund operator by T Hytönen was quite difficult. In which he showed that the general Calderón-Zygmund operator could be represented as the average of "Haar shifts" [23]. Many authors studied the problem of the A_2 conjecture for different types of operators.

4.5. Sparse Operators Domination

Later, Lerner explored a relatively simple proof of the A_2 A conjecture based on the domination of Calderón-Zygmund operators by dyadic positive operators, called the sparse operator [1, 2]. Since then, in harmonic analysis, sparse operators and corresponding sparse bounds have become quite popular for many operators in various settings.

4.6. Estimates of Multilinear Operators

4.6.1. Estimate of Multilinear Maximal Operator: Extension of Muckenhoupt A_p Theorem

Multilinear singular integrals of the Calderón-Zygmund type may be precisely controlled with the help of multi(sub)linear maximal operator $\mathcal{M}$, and the linear weight theory was extended to a multilinear setting by A. K. Lerner et al. [3] in 2009. The authors introduced the multi(sub)linear maximal operator $\mathcal{M}$, and investigated that the operator is strictly smaller than the m-fold product of the Hardy-Littlewood maximal operator, M. They explored the characterization of the multilinear maximal function $\mathcal{M}$ in the weighted space, an extension of Muckenhoupt. A_p Theorem for the maximal function M as a linear to multilinear setting, in the Theorem 3.7, as:
Let $\vec{P} = (p_j)_{j=1,\dots,m}$ with $p_j \in [1,\infty)$, and $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}, j = 1, \dots, m, m \geq 2$. Then,

- (Theorem 3.7) $\mathcal{M}$ is bounded from $L^{p_1}(w_1) \times \dots \times L^{p_m}(w_m)$ to $L^p(v_{\vec{w}})$ if and only if $\vec{w} \in A_{\vec{P}}$ for $1 < p_j < \infty$ with the estimate

$$\|M(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

For the definitions of multilinear Calderón-Zygmund operators, follow [3, 48].

4.6.2. Estimation of Multilinear Calderón-Zygmund Operator

Calderón-Zygmund theory in the multilinear case was initially studied by Coifman and Mayer in 1970. The theory has been enlarged by many authors to date. Here is the chronological review of some steps of the theory. Maldonado and Naibo [10] in 2009, explored the weighted norm inequalities for the bilinear θ- type Calderón-Zygmund operators, which is the natural generalization of the linear case, on the product of weighted Lebesgue spaces with the Muckenhoupt weights. Yabuta [25] in 1985 introduced the θ- type Calderón-Zygmund operators to facilitate the study of pseudodifferential operators.

Lu and Zhang [12] in 2014 introduced the multilinear θ-type Calderón-Zygmund operators and their commutators as well, and explored the boundedness of m-linear operators, $m \geq 2$, and their commutators in the product of weighted Lebesgue spaces with the multiple Muckenhoupt weights. They proved the strong and weak type weighted estimates of multilinear θ-type Calderón-Zygmund operators on products of weighted Lebesgue spaces. Note that the classical multilinear Calderón-Zygmund operator is a special case of the θ−type multilinear Calderón-Zygmund operator. For more details, see [49, 50].

W. Damián et al. [48] (in Theorem 1.5) explored the sharp weighted norm inequalities for the multilinear Calderón-Zygmund operator, which is a multilinear version of the A_2 conjecture using the sparse operators domination. For a multi-linear Calderón-Zygmund operator $(\vec{f})$,

$\|T(\vec{f})\|_{L^p(v_{\vec{w}})} \lesssim_{T,m,n} [\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}$, whenever $p_1 = p_2 = \cdots = p_m = m+1$, see also in Theorem 3.3 [23].

4.6.3. Estimate of Multilinear Sparse Operator

K. Li et al. [24](in Theorem 3.2) explored the estimate of the multilinear form of the sparse operator as

$\|A_s(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C[\vec{w}]_{A_{\vec{P}}}^{max\left(1,\frac{p_1'}{p},\ldots,\frac{p_m'}{p}\right)} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}$, where $\vec{P} = (p_1, \ldots, p_m), p_i \in (1, \infty), i = 1, \ldots, m, \frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$, $\vec{f} = (f_1, \ldots, f_m)$ And S is a sparse. Also, see Theorem 3.2 [23].

4.6.4. Estimates of m-linear Dyadic Paraproducts and Haar Multipliers

Kunwar in [14] explored the estimates for the m-linear dyadic paraproducts, $T \equiv P^{\vec{\alpha}}, \pi_b^{\vec{\alpha}}$ and Haar multiplier $T \equiv T_\epsilon^{\vec{\alpha}}$. Let T be a paraproduct for $\vec{\alpha} \in \{0,1\}^m$ or a Haar multiplier for $\vec{\alpha} \in U_m := \{0,1\}^m \backslash (1, \ldots, 1)$, then for a bounded BMO function $b \in BMO^d$, and bounded $\epsilon = \{\epsilon_I\}_{I \in D}$, and a weight $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}^d$ with $\sum_{j=1}^{m} \frac{1}{p_j} = \frac{1}{p}$,

$\|T(f_1, \ldots, f_m)\|_{L^p(v_{\vec{w}})} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$, whenever $p_j \in (1, \infty), j = 1, \ldots, m$, and

$\|T(f_1, \ldots, f_m)\|_{L^{p,\infty}(v_{\vec{w}})} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$, whenever $p_j \in [1, \infty), j = 1, \ldots, m$.

Kunwar [14] (Theorem 5.2.2) also established the domination of m-linear Paraproducts and Haar multipliers by a sparse operator A_s on every compactly supported $\vec{f} = (f_1, \ldots, f_m) \in L^1 \times \ldots \times L^1$ as: $|T(\vec{f})| \leq cA_s(|\vec{f}|)$ For the sparse collection S.

Thus, by using this theorem and the theorem proved by K. Li et al. for estimates of the m-linear sparse operator, Kunwar [14] (Theorem 5.2.3) established the following estimate: $\|T(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C_{m,\vec{P},T}[\vec{w}]_{A_{\vec{P}}}^{max\left(1,\frac{p_1'}{p},\ldots,\frac{p_m'}{p}\right)} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}$.

5. Conclusion

The theory of scalar weight and weighted norm inequalities, which is rich in literature, is reviewed. Muck-enhoupt's A_p condition [5] being necessary and sufficient for the boundedness of several classes of singular integrals is central to the A_pWeight theory and vital in developing the modern harmonic analysis. The study focuses on a revisit of the estimation of several classical operators, linear and multilinear, in the weighted Lebesgue spaces (s) corresponding to the scalar weight. Buckley [39] first quantitatively explored the sharp dependence of the weighted norm of the maximal function on the A_p Norm of the corresponding weight function. Several singular integral operators are studied by decomposing them into a sum of simpler dyadic operators. S. Petermichl [42] explored an idea in the proof of the sharp weighted estimate for the Hilbert transform by decomposing it into the sum of simpler operators, called dyadic shifts. Hytönen proved the A_2 conjecture and culminated the partial efforts of many mathematicians. Then it became the famous A_2theorem.

Moreover, he also proved that the general Calderón-Zygmund operators could be represented as an average of Haar shifts by using the technique developed by S. Petermichl in [42]. A. K. Lerner [2] proved the A_2 theorem, which was already proved by Hytönen [45], in a simplified form by introducing the concept of sparse operators, which is quite useful and popular nowadays. A. K. Lerner et al. [3] introduced and investigated the Muckenhoupt condition for the multilinear case and enhanced the estimation of multilinear operators in the weighted Lebesgue spaces with multiple Muckenhoupt weights.

Kunwar [14] explored the weighted estimates for multilinear paraproducts and Haar multipliers using the sparse operators. Although there are no novel findings in the paper, it will be helpful to the authors and researchers to pave the way for further findings in this era.

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