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Original Article

On Pseudo S_b –Menger Space and Rational Type Contraction in S_b –Menger Space with Applications

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Abstract - *In the present paper, we define a pseudo S_b −Menger space with counterexamples and prove a decomposition theorem for a pseudo S_b −Menger space. Also, we prove fixed-point theorems for rational-type contraction in a newly defined S_b −Menger space. Our results extend the results of Gupta V et al [8] in S_b −Menger spaces. Applications are also given to prove the effectiveness of our results.*

Keywords - *S_b −Menger Space, Rational Expression, Integral Type Contraction, Control Function.*

Mathematics Subject Classification: *47H10, 54H25.*

1. Introduction and Preliminaries

The origin of generalized metric spaces dates back to 1928, when Karl Menger introduced the concept of a probabilistic metric space, now commonly referred to as a Menger space. This foundational work marked a significant milestone in the history of metric space theory and inspired numerous extensions and generalizations in the field of fixed-point theory. Over the decades, classical results like Banach's contraction principle have been adapted to more complex structures, including Menger spaces and mappings with special conditions. In recent years, novel structures such as S-metric spaces and their generalizations, including S_b −metric spaces have attracted increasing attention from researchers [2,4,5,6]. These advancements have enabled the formulation of new fixed-point theorems under broader and more flexible conditions. Very recently, Keer, P.K. et al defined the concept of S_b −Menger Spaces and proved the fixed point theorem in symmetric spaces S_b −Menger spaces with applications. Motivated by this rich history and ongoing research. In the present paper, we prove the fixed point theorem using rational type contraction; our result extends the results of Gupta V. et al [8] in the structure of S_b −Menger space. As an application, we prove an integral analogue of our results. In this work, after the introduction and preliminaries in Section 2, we define a pseudo S_b −Menger space, which is a generalization of a pseudo S_b −metric space with examples and counterexamples. In Section 3, we prove fixed-point theorems for rational-type contraction in S_b −Menger space. In Section 4, we also provide applications in support of our results.

We recall the following definitions, which will be needed in the sequel.

Definition 1.1. Let X be a nonempty set and $k \geq 1$ be a given real number. A Function $S_b: X^3 \to [0, \infty)$ is said to be S_b − metric iff for all $x, y, z, a \in X$ the following conditions are satisfied:
(S_{b1}) $S_b(x, y, z, t) = 0$ iff $x = y = z$,
(S_{b2}) $S_b(x, y, z, t) \leq k[(S_b(x, x, a, t) + S_b(y, y, a, t) + S_b(z, z, a, t)]$.
The pair (X, S_b) is called a S_b −metric space.

Definition 1.2.[1] The quadruple $(X, \boldsymbol{S_{bp}}, *, k)$ is said to be a pseudo S_b −metric space if X is an arbitrary (nonempty) set, $k \geq 1$ is a given real number satisfying the following conditions:
(S_{bp1}) $\forall\, x, y, z \in X$, $S_{bp}(x, y, z, t) \geq 0$,
(S_{bp2}) $\forall\, x, y, z \in X$, $S_{bp}(x, y, z, t) = 0$ *if* $x = y = z$,

(S_{bp3}) $\forall\, x, y, z, a \in X$ and $k \geq 1$,

$$S_{bp}(x, y, z, t) \leq k\big[\big(S_{bp}(x, x, a, t) + S_{bp}(y, y, a, t) + S_{bp}(z, z, a, t)\big)\big]$$

Definition 1.3.[3] A map $*: [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

(i) $*\,(a, 1, 1) = a$, $*(0,0,0) = 0$;
(ii) $*\,(a, b, c) = *\,(a, c, b,) = *\,(b, c, a)$;
(iii) $*\,(a_1, b_1, c_1) \geq *\,(a_2, b_2, c_2)$ for $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$.

Examples of t −norms are
(1): $x * y * z = x.y.z$ and
(2): $x * y * z = \min\{x.y.z\}$ minimum t −norm.

Definition 1.4. [9] The S_b −Menger space

The quadruple $(X, S_b, *, k)$ is said to be S_b a Menger space if X it is a nonempty set, and $k \geq 1$ is a given real number. S_b is a function defined on X^3 to the set of distribution functions and $*$ is a continuous third-order t-norm such that the following conditions are satisfied:

(i) $S_{b(x,y,z)}(0) = 0$ for all $x, y, z \in X$
(ii) $S_{b(x,x,y)}(t) < 1$ for $t > 0$ with $x \neq y$,
(iii) $S_{b(x,y,z)}(t) = 1$ for all $t > 0$, if and only if $x = y = z$,
(iv) $S_{(x,y,z)}(kt) \geq *\,[\big(S_{b(x,x,a)}(t_1), S_{b(y,y,a)}(t_2), S_{b(z,z,a)}(t_3)\big)]$
Where $t = t_1 + t_2 + t_3$ and $t, t_1, t_2, t_3 > 0$ for all $x, y, z, a \in X$

Definition 1.5. [9] The S_b- Menger space is called symmetric if $S_{b(x,x,y)}(t) = S_{b(y,y,x)}(t)$

Definition 1.6. [9] Let $(X, S_b, *, k)$ be a symmetric S_b- Menger space, then a sequence $\{x_n\} \in X$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} S_{b(x_n,x_n,x)}(t) = 1$ for all $t > 0$.

Definition 1.7. [9] Let $(X, S_b, *, k)$ be a symmetric S_b- Menger space, then a sequence $\{x_n\} \in X$ is called a Cauchy sequence $x \in X$ if $\lim_{n\to\infty} S_{b(x_n,x_n,x_{n+s})}(t) = 1$ for all $t, s > 0$.

Example 1.8.[9] Let $X = [0,1]$ and $(X, S_b, *, k)$ be a complete symmetric S_b- Menger space where S_b defined by

$$S_{b(x,y,z)}(t) = \begin{cases} 0 & , t = 0 \\ \dfrac{t}{t + [|x - z| + |y - z|]^2} & t > 0 \end{cases}$$

2. Pseudo S_b- Menger space

In this section, we will define pseudo S_b- Menger space with a counterexample and prove the decomposition theorem.

Definition 2.1. A quadruple $(X, S_{bp}, *, k)$ is said to be pseudo S_b- Menger space if X is an arbitrary (nonempty) set, $*$ is a continuous t-norm, $k \geq 1$ is a given real number, and S_{bp} is a distribution function on $X^3 \times (0, \infty)$ satisfying the following conditions:
$(S_{bp}1)$ $\forall\, x, y, z \in X$ *and* $\forall\, t > 0, S_{bp(x,y,z)}(t) > 0$,
$(S_{bp}2)$ $\forall\, x, y, z \in X$ *and* $\forall\, t > 0, S_{bp(x,y,z)}(t) = 1$ *if* $x = y = z$,
$(S_{bp}3)$ $\forall\, x, y, z, a \in X$ and $\forall\, r, s, t > 0$,

$$S_{bp(x,y,z)}k(r + s + t) \geq \Big[\Big(S_{bp(x,x,a)}(r) + S_{bp(y,y,a)}(s) + S_{bp(z,z,a)}(t)\Big)\Big].$$

Example 2.2. Consider R the usual metric. Let $X = \{x_n\}: \{x_n\}$ it be convergent in $R\}$.Define $a * b * c = abc$ for all $a, b, c \in [0,1]$ and

$S_{bp(x_n,y_n,z_n)}(t) = e^{[\frac{(|x_n - z_n| + |y_n - z_n|)^2}{t}]^{-1}}$.

Noting that $(X, S_{bp}, *, k)$ it is a pseudo S_b −Menger space but not S_b −a Menger space.
To see this, let $\{x_n\} = \left\{\frac{1}{n}\right\}, \{y_n\} = \left\{\frac{2}{n}\right\}, \{z_n\} = \{\frac{3}{n}\}$
Then $\{x_n\} \neq \{y_n\} \neq \{z_n\}$ for $\{x_n\}, \{y_n\}$ and $\{z_n\} \in X$ but $S_{bp(x_n, y_n, z_n)}(t) = 1$.

Remark 2.3. Every $S_b - A$ Menger space is a pseudo S_b −Menger space, denoted by $(X, S_{bp}, *, k)$, but the converse is not true, see the above example.

Theorem 2.4. Let $(X, S_{bp}, *, k)$ be a pseudo S_b −Menger space and
$S_{bp(x)(x,y,z)} = \inf\{t > 0: S_{bp(x,y,z)}(t) > x, x \in (0,1)\}$.
Then $S = \{S_{bp}(x)\}_{x\in(0,1)}$ there is an ascending family of pseudo-Menger on X.

Proof. $S_{bp(x)(x,x,x)} = \inf\{\, t > 0;\ S_{bp(x,x,x)}(t) > x\} = 0$.
Now,

$$k[(S_{bp(x)(x,x,a)} + S_{bp(x)(y,y,a)} + S_{bp(x)(z,z,a)})]$$
$$k[\inf\{\, r > 0: S_{bp(x,x,a)}(r) > x\} + \inf\{\, s > 0: S_{bp(y,y,a)}(s) > x\} + \inf\{\, r > 0: S_{bp(z,z,a)}(t) > x]$$
$$= k[inf\{r + s + t > 0: S_{bp(x,x,a)}(r) > x, S_{bp(y,y,a)}(s) > x, S_{bp(z,z,a)}(t) > x]$$
$$= inf[k\{r + s + t > 0: S_{bp(x,x,a)}(r) > x * S_{bp(y,y,a)}(s) > x * S_{bp(z,z,a)}(t) > x]$$
$$\geq inf\{k(r + s + t) > 0: S_{bp(x,y,z)}k(r + s + t) > x\}$$
$$= S_{bp(x)(x,y,z)}.$$

It remains to prove that $\delta = \{S_{bp(x)}\}_{x\in(0,1)}$ It is an ascending family.
Let $x_1 \leq x_2$, then
$\{t > 0: S_{bp(x,y,z)}(t) > x_2\} \subseteq \{t > 0: S_{bp(x,y,z)}(t) > x_1\}$.
Thus,

$$inf\{t > 0: S_{bp(x,y,z)}(t) > x_2\} \geq inf\{t > 0: S_{bp(x,y,z)}(t) > x_1\},$$

namely

$$S_{bp(x_2)(x,y,z)}(t) \geq S_{bp(x_1)(x,y,z)}\ ,\ \forall\,(x,y,z)\in X^3.$$

Example 2.5.[5] Let $X = [-1,1]$, define $S_{bp(x,y,z)}(t)$ by

$$S_{pb} = \begin{cases} \dfrac{t}{t + [|x^2 - z^2| + |y^2 - z^2|]^2} & t > 0 \\ 0, & t = 0 \end{cases}$$

If all $\forall\, x, y, z \in X, *$ are a continuous t −norm, then $(X, S_{bp}, *, k)$ it is a pseudo S_b −Menger space. Define
$S_{bp(x,y,z)}(t) = inf\{t > 0: S_{bp(x,y,z)}(t) > x\}$.
Take $x_1 = \frac{1}{8}$, and $x_2 = \frac{1}{4}$ in (0,1), then it is quite natural.

$$\{t > 0: S_{bp(x,y,z)}(t) > \frac{1}{4}\} \subseteq \{t > 0: S_{bp(x,y,z)}(t)\} > \frac{1}{8}\}.$$

Now, by the property of infimum

$$inf\left\{t > 0: S_{bp(x,y,z)}(t) > \frac{1}{4}\right\} \geq inf\left\{t > 0: S_{bp(x,y,z)}(t) > \frac{1}{8}\right\}.$$

Hence,

$$S_{bp(\frac{1}{4})(x,y,z)}(t) > S_{bp(\frac{1}{8})(x,y,z)}\quad \forall\,(x,y,z)\in X^3$$

Similarly, we can find other distinct members in (0,1). Then $S = \{S_{bp(x)}\}_{x\in(0,1)}$ is an ascending family of pseudo S_b −Menger metrics on X.

3. Fixed-point theorems for rational-type contraction in S_b −Menger space

In this section, we prove two lemmas and Fixed-point theorems for rational-type contraction.

Lemma 3.1. Let $(X, S_b, *, k)$ be a symmetric S_b − Menger space, then $S_{b(x,x,y)}(.)$ is non-decreasing for all x, y in X.
Proof. Suppose that $S_{b(x,x,y)}(t) > S_{b(x,x,y)}(s)$ for some $0 < t < s$.

Then

$$S_{b(x,x,x)}\left(\frac{s}{2k}\right) * S_{b(x,x,x)}\left(\frac{s}{2k}-t\right) * S_{b(y,y,x)}(t) \le S_{b(x,x,y)}(s).$$

By the Definition of $S_b -$ a Menger space, we have

$$S_{b(x,x,x)}\left(\frac{s}{2k}\right) = 1$$

and

$$S_{b(x,x,x)}\left(\frac{s}{2k}-\frac{t}{k}\right) = 1.$$

Thus $S_{b(y,y,x)}(t) \le S_{b(x,x,y)}(s)$
$S_{b(x,x,y)}(t) \le S_{b(x,x,y)}(s)$ [As $S_{b(x,x,y)}(t) = S_{b(y,y,x)}(t)$]
Which is a contradiction.

Lemma 3.2. Let $(X, S_b, *, k)$ be a symmetric $S_b -$ Menger space. If there exists $q \in (0,1)$ such that $S_{b(x,x,y)}(t) \ge S_{b(x,x,y)}(\frac{t}{q})$ for all $x, y \in X$, $t > 0$ and

$$\lim_{t\to\infty} S_{b(x,y,z)}(t) = 1$$

Then $x = y$.

Proof. Suppose that there exists $q \in (0,1)$ such that $S_{b(x,x,y)}(t) \ge S_{b(x,x,y)}(\frac{t}{q})$ for all $x, y \in X$, $t > 0$. Then,

$$S_{b(x,x,y)}(t) \ge S_{b(x,x,y)}\left(\frac{t}{q}\right) \ge S_{b(x,x,y)}\left(\frac{t}{q^2}\right),$$

and so

$$S_{b(x,x,y)}(t) \ge S_{b(x,x,y)}\left(\frac{t}{q^n}\right).$$

For a positive integer n. Taking the limit as $n \to \infty$, $S_{b(x,x,y)}(t) \ge 1$ and hence $x = y$.

Theorem 3.3. Let $(X, S_b, *, k)$ be a complete symmetric $S_b -$Menger space and $f: X \to X$ be a mapping satisfying

$$\lim_{t\to\infty} S_{b(x,y,z)}(t) = 1, \tag{3.3.1}$$

$$\text{And } S_{b(fx,fx,fy)}(qt) \ge \mu_{(x,x,y)}(t) \tag{3.3.2}$$

$$\text{Where } \mu_{(x,x,y)}(t) = min\left\{\frac{S_{b(y,y,f(y))}(t)[1+S_{b(x,x,f(x))}(t)]}{[1+S_{b(x,x,y)}(t)]}, S_{b(x,x,y)}(t)\right\} \tag{3.3.3}$$

For all $x, y, z \in X$, $q \in \left(0, \frac{1}{3k}\right)$ where $k \ge 1$, then f has a fixed point.

Proof. Suppose $x \in X$ there is an arbitrary point in X. Now we construct a sequence $\{x_n\} \in X$ such that $f(x_n) = x_{n+1}$ for all $n \in N$. Claim $\{x_n\}$ is a Cauchy sequence.
Let us take $x = x_{n-1}$, and $y = x_n$ in (3.3.2), we get

$$S_{b(x_n,x_n,x_{n+1})}(qt) = S_{b(fx_{n-1},fx_{n-1}fx_n)}(qt) \ge \mu_{(x_{n-1},x_{n-1},x_n)}(t) \tag{3.3.4}$$

Now

$$\mu_{(x_{n-1},x_{n-1},x_n)}(t) = \min\left\{\frac{S_{b(x_n,x_n,fx_n)}(t)\left[1+S_{b(x_{n-1},x_{n-1},fx_{n-1})}(t)\right]}{\left[1+S_{b(x_{n-1},x_{n-1},x_n)}(t)\right]}, S_{b(x_{n-1},x_{n-1},x_n)}(t)\right\}$$

$$\mu_{(x_{n-1},x_{n-1},x_n)}(t) = \min\left\{\frac{S_{b(x_n,x_n,x_{n+1})}(t)\left[1+S_{b(x_{n-1},x_{n-1},x_n)}(t)\right]}{\left[1+S_{b(x_{n-1},x_{n-1},x_n)}(t)\right]}, S_{b(x_{n-1},x_{n-1},x_n)}(t)\right\}$$

$\Longrightarrow \mu_{(x_{n-1},x_{n-1},x_n)}(t) = \min\{S_{b(x_n,x_n,x_{n+1})}(t), S_{b(x_{n-1},x_{n-1},x_n)}(t)\}$,
Now if
$S_{b(x_n,x_n,x_{n+1})}(t) \le S_{b(x_{n-1},x_{n-1},x_n)}(t)$, then from equation (3.3.4)
$S_{b(x_n,x_n,x_{n+1})}(qt) \ge S_{b(x_n,x_n,x_{n+1})}(t)$

Hence, from Lemma (3.2), our claim follows immediately. Now suppose

$$S_{b(x_n,x_n,x_{n+1})}(t) \ge S_{b(x_{n-1},x_{n-1},x_n)}(t)$$

Then again from (3.3.4),
$S_{b(x_n,x_n,x_{n+1})}(qt) \ge S_{b(x_{n-1},x_{n-1},x_n)}(t)$

Now, by simple Induction, for all n and $t > 0$, we get

$$S_{b(x_n,x_n,x_{n+1})}(qt) \geq S_{b(x_0,x_0,x_1)}\left(\frac{t}{q^{n-1}}\right) \tag{3.3.5}$$

By using (3.3.5), for any positive integer 's' we get

$$S_{b(x_n,x_n,x_{n+s})}(t) \geq S_{b(x_n,x_n,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(x_n,x_n,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(x_{n+s},x_{n+s},x_{n+1})}\left(\frac{t}{3k}\right)$$

$S_{b(x_n,x_n,x_{n+s})}(t) = S_{b(x_n,x_n,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(x_n,x_n,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(x_{n+1},x_{n+1},x_{n+s})}\left(\frac{t}{3k}\right)$ [by symmetry property]

$$S_{b(x_n,x_n,x_{n+s})}(t) \geq S_{b(x_n,x_n,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(x_n,x_n,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(x_{n+1},x_{n+1},x_{n+2})}\left(\frac{t}{(3k)^2}\right) * S_{b(x_{n+1},x_{n+1},x_{n+2})}\left(\frac{t}{(3k)^2}\right) * S_{b(x_{n+s},x_{n+s},x_{n+2})}\left(\frac{t}{(3k)^2}\right)$$

$S_{b(x_n,x_n,x_{n+s})}(t) = S_{b(x_n,x_n,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(x_n,x_n,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(x_{n+1},x_{n+1},x_{n+2})}\left(\frac{t}{(3k)^2}\right) * S_{b(x_{n+1},x_{n+1},x_{n+2})}\left(\frac{t}{(3k)^2}\right) * S_{b(x_{n+2},x_{n+2},x_{n+s})}\left(\frac{t}{(3k)^2}\right)$ [by symmetry property]

By using equation (3.3.5), we get

$S_{b(x_n,x_n,x_{n+s})}(t) \geq S_{(x_0,x_0,x_1)}\left(\frac{t}{q^n(3k)}\right) * S_{(x_0,x_0,x_1)}\left(\frac{t}{q^n(3k)}\right) * S_{(x_0,x_0,x_1)}\left(\frac{t}{q^{n+1}(3k)^2}\right) * S_{(x_0,x_0,x_1)}\left(\frac{t}{q^{n+1}(3k)^2}\right).$

By the definition of q −contraction (i.e., $q < 1$) together with condition (3.3.1) and letting limit $n \to \infty$

$\lim_{n\to\infty} S_{b(x_n,x_n,x_{n+s})}(t) = 1 * 1 * 1 * 1 \dots \dots 1 * 1 = 1.$

$\Longrightarrow \{x_n\}$ is a Cauchy sequence. Since $(X, S_b, *, k)$ it is a complete symmetric S_b −Menger space, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u \tag{3.3.6}$$

Claim u is a fixed point of f.

On consider

$$S_{b(u,u,fu)}(t) \geq S_{b(u,u,fx_n)}\left(\frac{t}{3k}\right) * S_{b(u,u,fx_n)}\left(\frac{t}{3k}\right) * S_{b(fu,fu,fx_n)}\left(\frac{t}{3k}\right) \tag{3.3.7}$$

$$= S_{b(u,u,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(u,u,x_{n+1})}\left(\frac{t}{3k}\right) * S_{b(fx_n,fx_n,fu)}\left(\frac{t}{3k}\right)$$

$$= S_{b(u,u,x_{n+1})}\left(\frac{t}{3}\right) * S_{b(u,u,x_{n+1})}\left(\frac{t}{3}\right) * \mu_{(x_n,x_n,u)}\left(\frac{t}{3kq}\right)$$

Now

$$\mu_{(x_n,x_n,u)}\left(\frac{t}{3kq}\right) = min\left\{\frac{S_{b(u,u,fu)}\left(\frac{t}{3kq}\right)[1 + S_{b(x_n,x_n,fx_n)}\left(\frac{t}{3kq}\right)]}{[1 + S_{b(x_n,x_n,u)}\left(\frac{t}{3kq}\right)]}, S_{b(x_n,x_n,u)}\left(\frac{t}{3kq}\right)\right\}$$

Taking $n \to \infty$. The above inequality and using (3.3.1), we get

$\mu_{(u,u,u)}\left(\frac{t}{3kq}\right) = \min\left\{S_{b(u,u,fu)}\left(\frac{t}{3kq}\right), 1\right\}$

Now if $S_{b(u,u,fu)}\left(\frac{t}{3kq}\right) \geq 1$ then $\mu_{(u,u,u)}\left(\frac{t}{3kq}\right) = 1$.

Therefore, from equation (3.3.7) and using the definition (1.2), we get u that is a fixed point of f.

Now if $S_{b(u,u,fu)}\left(\frac{t}{3kq}\right) \leq 1$ then $\mu_{(u,u,u)}\left(\frac{t}{3kq}\right) = S_{b(u,u,fu)}\left(\frac{t}{3kq}\right)$.

Hence, from equation (3.3.7) we get

$$S_{b(u,u,fu)}(t) \geq S_{b(u,u,fu)}\left(\frac{t}{3kq}\right) \tag{3.3.8}$$

Since $q \in (0, \frac{1}{3})$ now taking $n \to \infty$ in the above inequality and using (3.3.1), and Lemma (3.3.2), we get $fu = u$.

This completes the proof of Theorem (3.3.1)
Let us define $\theta = \{\frac{\phi}{\phi}: [0,1] \to [0,1]\}$ a continuous function such that $\phi(1) = 1, \phi(0) = 0, \phi(a) > a$ for each $0 < a < 1$.

Theorem 3.4. Let $(X, S_b, *, k)$ be a complete symmetric S_b −Menger space with

$$\lim_{n\to\infty} S_{b(x,y,z)}(t) = 1 \quad (3.4.1)$$

and $f: X \to X$ be a mapping satisfying

$$S_{b(f(x),f(x),f(y))}(qt) \geq \phi\{\mu_{(x,x,y)}(t)\} \quad (3.4.2)$$

Where

$$\mu_{(x,x,y)}(t) = \min\left\{\frac{S_{b(y,y,f(y))}(t)[1+S_{b(x,x,f(x)}(t)]}{[1+S_{b(x,x,y)}(t)}, S_{b(x,x,y)}(t)\right\} \quad (3.4.3)$$

For all $x, y \in X$, and $q \in \left(0, \frac{1}{3k}\right)$, $\phi \in \Phi$. Then f has a fixed point.

Proof: Since $\phi \in \Phi$. This implies that $\phi(a) > a$ for each $0 < a < 1$. , thus from ((3.4.2)
$S_{b(f(x),f(x),f(y))}(qt) \geq \phi\{\mu_{(x,x,y)}(t)\} \geq \mu_{(x,x,y)}(t)$
Now, applying Theorem (3.3.1), we obtain the desired result.

4. Applications

In this section, we give some applications related to our theoretical findings. Let us define $\psi: [0,\infty) \to [0,\infty)$, it is $\psi(t) = \int_0^1 \varphi(t)dt$, $\forall\, t > 0$,
be a non-decreasing and continuous function. Furthermore, for every ε >0, we assume that $\varphi(\varepsilon) > 0$. also implies that $\varphi(t) = 0$ iff $t = 0$.

Theorem 4.1 Let $(X,\ S_{b,}\ *,\ k)$ be a complete symmetric S_b −Menger space with $\lim_{n\to\infty} S_{b(x,y,z)}(t) = 1$ and $f: X \to X$ be a mapping satisfying

$$\int_0^{S_{b(f(x),f(x),f(y))}(qt)} \varphi(t)dt \geq \int_0^{\mu_{(x,x,y)}(t)} \varphi(t)dt$$

Where

$$\mu_{(x,x,y)}(t) = \min\left\{\frac{S_{b(y,y,f(y))}(t)[1+S_{b(x,x,f(x)}(t)]}{[1+S_{b(x,x,y)}(t)]}, S_{b(x,x,y)}(t)\right\}$$

For all $x, y \in X$, $\varphi \in \psi$, and $q \in \left(0, \frac{1}{3k}\right)$ Then f has a fixed point.

Proof: By taking $\varphi(t)$=1 and applying Theorem (3.3.1), we obtain the result.

Theorem 4.2 Let $(X, S_b, *, k)$ be a complete symmetric S_b −Menger space with $\lim_{n\to\infty} S_{b(x,y,z)}(t) = 1$ and $f: X \to X$ be a mapping satisfying

$$\int_0^{S_{b(f(x),f(x),f(y))}(qt)} \varphi(t)dt \geq \phi\{\int_0^{\mu_{(x,x,y)}(t)} \varphi(t)dt\}$$

Where

$$\mu_{(x,x,y)}(t) = \min\left\{\frac{S_{b(y,y,f(y))}(t)[1+S_{b(x,x,f(y)}(t)]}{[1+S_{b(x,x,y)}(t)]}, S_{b(x,x,y)}(t)\right\}$$

For all $x, y \in X$, $\varphi \in \psi$, $q \in \left(0, \frac{1}{3k}\right)$ and $\phi \in \Phi$. Then f has a fixed point.

Proof. Since $\phi(a) > a$ for each $0 < a < 1$,
therefore, the result follows immediately from Theorem (4.1).

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References

[1] Akansha Sharma, Mahesh Tiwari, and Ramakant Bhardwaj, "Pseudo-S-Metric Spaces and Pseudo-S-Metric Product Spaces," *International Journal of Theoretical and Applied Sciences*, vol. 8, no. 1, pp. 139-141, 2016. [Google Scholar]

[2] Yumnam Rohen, Tatjana Došenović, and Stojan Radenović, "A Note on the Paper "A Fixed-Point Theorems in S_b-Metric Spaces,"" *Filomat*, vol. 31, no. 11, pp. 3335-3346, 2017. [CrossRef] [Google Scholar] [Publisher Link]

[3] Berthold Schweizer, and Abe Sklar, "Statistical Metric Spaces," *Pacific Journal of Mathematics*, vol. 10, no. 1, pp. 313-334, 1960. [Google Scholar] [Publisher Link]

[4] Sh. Sedghi et al., "Common Fixed Point of Four Maps in Sb-Metric Spaces," *Journal of Linear and Topological Algebra*, vol. 5, no. 2, pp. 93-104, 2016. [Google Scholar]

[5] N. Souayah, and N. Mlaiki, "A Fixed-Point Theorem in S_b-Metric Spaces," *Journal of Mathematics and Computer Science*, vol. 16, no. 2, pp. 131-139, 2016. [CrossRef] [Google Scholar] [Publisher Link]

[6] Shaban Sedghi, Nabi Shobe, and Abdelkrim Aliouche, "A Generalization of Fixed-Point Theorems in *S*-Metric Spaces," *Matematički Vesnik*, vol. 64, no. 3, pp. 258-266, 2012. [Google Scholar] [Publisher Link]

[7] Jerolina Fernandez et al., "N_b-Fuzzy Metric Spaces with Topological Properties and Applications," *AIMS Math*, vol. 8, no. 3, pp. 5879-5898, 2023. [CrossRef] [Google Scholar] [Publisher Link]

[8] Vishal Gupta, Naveen Mani, and Anil Sain, "Fixed Point Theorems and Its Applications in Fuzzy Metric Spaces," *Proceedings of the Conference AEMDS*, pp. 961-964, 2013. [Google Scholar]