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Original Article

A New Look into the Generating Function for the Partition Function $\overline{\mathcal{EO}}(n)$

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Abstract *- Let $\mathcal{EO}(n)$, denote the number of partitions of n where every even part is less than each odd part and $\overline{\mathcal{EO}}(n)$, denote the number of partitions counted by $\mathcal{EO}(n)$, in which only the largest even part appears an odd number of times. In our work, the author uses 5-dissections of q-products and some identities for the Rogers-Ramanujan continued fraction to obtain the exact generating function for $\overline{\mathcal{EO}}(10n+8)$.*

Keywords *- Partitions, Partition congruences, Rogers-Ramanujan continued fraction.*

1. Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$, of a positive integer n is a finite sequence of non-increasing positive integer parts λ_i such that $n = \sum_{i=1}^{k} \lambda_i$. In 1919, Ramanujan [7] proved that

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6}, \tag{1.1}$$

Where, for any complex number a and $|q| < 1$, the q-product is defined as $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$ and for simplicity, here and throughout the paper, the notation $f_j = (q^j;q^j)_\infty$, has been used. Now, (1.1) immediately implies one of his famous partition congruences, namely,

$$p(5n+4) \equiv 0 (\text{mod } 5).$$

Let $\mathcal{EO}(n)$ denote the number of partitions of n where every even part is less than each odd part and $\overline{\mathcal{EO}}(n)$ denote the number of partitions counted by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. Andrews [1] in 2018 studied these two restricted partitions and established that the generating function for $\overline{\mathcal{EO}}(n)$ may be written as

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)\, q^n = \frac{f_4^3}{f_2^2},$$

Furthermore, with the help of mock theta functions, he showed that

$$\overline{\mathcal{EO}}(10n+8) \equiv 0 \ (\text{mod } 5). \tag{1.2}$$

He proposed that a further investigation of the properties of $\overline{\mathcal{EO}}(n)$, may be done to understand the various properties of this interesting function. Following this, many mathematicians have worked on both the functions $\mathcal{EO}(n)$ and $\overline{\mathcal{EO}}(n)$, to study their properties and obtain numerous results. A brief account of the same is provided here.

In 2020, Ray and Barman [8] studied the divisibility properties of $\overline{\mathcal{EO}}(n)$. Using the theory of Hecke eigenforms, they found two infinite families of congruences for $\overline{\mathcal{EO}}(n)$ modulo 2 and 8, and proved that $\overline{\mathcal{EO}}(n)$, is even for almost all n. In their work, they also established that congruence (1.2) is also true modulo 4 if $n \equiv 1,2,3,4$ (mod 5), thereby proving the congruences

$$\overline{\mathcal{EO}}(50n+18) \equiv 0 \ (\text{mod } 20),$$

$$\overline{\mathcal{EO}}(50n+28) \equiv 0 \pmod{20},$$
$$\overline{\mathcal{EO}}(50n+38) \equiv 0 \pmod{20},$$
$$\overline{\mathcal{EO}}(50n+48) \equiv 0 \pmod{20}.$$

Pore and Fathima [6] also studied the function $\overline{\mathcal{EO}}(n)$, and obtained similar congruences modulo 2, 4, 10, and 20. They had also conjectured the above congruences in their work.

Chen and Chen [3] in 2023 proved the same congruence $\overline{\mathcal{EO}}(10n+8) \equiv 0 \pmod{4}$, for almost all n.

Very recently, Garvan and Morrow [4] in 2025 extended the congruences for $\overline{\mathcal{EO}}(n)$ to modulo higher powers of 2 via Fricke involutions and some results of Newman [5]. These rely on modular form techniques, leaving a gap for Ramanujan-style q-series dissections that directly yield the generating function for $\overline{\mathcal{EO}}(10n+8)$ contrasting Andrews' mock theta route by emphasizing continued fraction identities for a cleaner algebraic path. Simplicity here means fewer transcendental elements and direct q-product manipulation, offering fresh combinatorial insight beyond congruence proofs.

Sarma [9] proved several congruences modulo 16, 10, and 40 for $\overline{\mathcal{EO}}(n)$ and that $\overline{\mathcal{EO}}(10n+8)$ and $\overline{\mathcal{EO}}(40n+38)$ are almost always divisible by 10 and 40, respectively. He also showed that considering the third-order mock theta function

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}},$$

and

$$\nu(-q) = \sum_{n=0}^{\infty} p_\nu(n) q^n,$$

and following the works of Andrews [1], Baruah and Begum [4], and Xia [12], it follows that

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8)\, q^n = \sum_{n=0}^{\infty} p_\nu(10n+8) q^n = 5\frac{f_2^2 f_5^2 f_{10}}{f_1^4}.$$

Unlike Garvan-Morrow's [4] eta-quotient focus for 2-adic properties, the current approach dissects via Rogers-Ramanujan fractions, avoiding half-integer weights and yielding the exact generating function— a gap in elementary derivations absent in these modular-heavy methods. Baruah-Begum connected mock thetas to partitions, yet this work advances by contrasting their third-order mock theta reliance with continued fraction lemmas for broader q-series applicability.

The result that is obtained in the current work is stated in the form of the following theorem.

Theorem 1.1 For any $n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(10n+8)\, q^n = 5\frac{f_2^2 f_5^2 f_{10}}{f_1^4}.$$

As a corollary, Andrews' congruence (1.2) follows immediately from the above theorem.

2. Preliminaries

This section begins with some key relations on the Rogers-Ramanujan continued fraction $R(q)$, defined by

$$R(q) := \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}} = q^{1/5}\frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}, |q|<1.$$

Lemma 2.1. [2, p.165] If $R := \frac{q}{R(q^5)}$, then

$$f_1 = \frac{1}{f_{25}}\left(R - q - \frac{q^2}{R}\right)$$

and $\frac{1}{f_1}=\frac{f_{25}^5}{f_5^6}\left(R^4+qR^3+2q^2R^2+3q^3R+5q^4-\frac{3q^5}{R}+\frac{2q^6}{R^2}-\frac{q^7}{R^3}+\frac{q^8}{R^4}\right)$.

Lemma 2.2. [3, Lemma 1.3] If $x=\frac{q^{1/5}}{R(q)}$ and $y=\frac{q^{2/5}}{R(q^2)}$, then

$$xy^2-\frac{q^2}{xy^2}=K,$$

$$\frac{x^2}{y}-\frac{y}{x^2}=\frac{4q}{K},$$

$$\frac{y^3}{x}+q^2\frac{x}{y^3}=K+\frac{4q^2}{K}-2q,$$

$$x^3y+\frac{q^2}{x^3y}=K+\frac{4q^2}{K}+2q,$$

Where $K=(f_2f_5^5)/(f_1f_{10}^5)$.

Lemma 2.3. The following results hold true.

$$f_2f_5^5+qf_1f_{10}^5=\frac{f_1^4f_5f_{10}^2}{f_2},$$

$$f_2f_5^5+qf_1f_{10}^5=\frac{f_2^4f_5^2f_{10}}{f_1}.$$

For a proof of the above, refer to [3, p. 2000].

3. Proof of Theorem 1.1

The generating function for $\overline{\mathcal{EO}}(n)$, due to Andrews [1] is

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(n)\,q^n=\frac{f_4^3}{f_2^2}.$$

Now, extracting the even powers, it easily follows that.

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(2n)\,q^n=\frac{f_2^3}{f_1^2}.$$

Applying the 5-dissections via Lemma 2.1 in the above, isolating the terms involving q^{5n+4}, dividing both sides of the resulting equation by q^4 and then replacing q^{5n} by q^n the following is obtained.

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathcal{EO}}\big(2(5n+4)\big)\,q^n\\ =\frac{f_5^{10}f_{10}^3}{f_1^{12}}\Bigg(\bigg(20\left(x^4y^3-\frac{q^4}{x^4y^3}\right)-15\left(x^6y^2+\frac{q^4}{x^6y^2}\right)\bigg)\\ +\ q\left(50\left(x^5-\frac{q^2}{x^5}\right)-60\left(xy^2-\frac{q^2}{xy^2}\right)-20\left(\frac{y^3}{x}+q^2\frac{x}{y^3}\right)\right)\\ +q^2\left(75-5\left(\frac{x^6}{y^3}-\frac{y^3}{x^6}\right)-60\left(\frac{x^4}{y^2}+\frac{y^2}{x^4}\right)\right)\Bigg)\end{aligned}$$

where x and y are defined in Lemma 2.2.

Now, one can easily see that.

$$\frac{x^4}{y^2}+\frac{y^2}{x^4}=\left(\frac{x^2}{y}-\frac{y}{x^2}\right)^2+2,$$

$$\frac{x^6}{y^3}-\frac{y^3}{x^6}=\left(\frac{x^2}{y}-\frac{y}{x^2}\right)\left(\left(\frac{x^4}{y^2}+\frac{y^2}{x^4}\right)+1\right),$$

$$x^5-\frac{q^2}{x^5}=\left(\frac{x^2}{y}-\frac{y}{x^2}\right)\left(x^3y+\frac{q^2}{x^3y}\right)+\left(xy^2-\frac{q^2}{xy^2}\right),$$

$$x^4y^3-\frac{q^4}{x^4y^3}=\left(xy^2-\frac{q^2}{xy^2}\right)\left(x^3y+\frac{q^2}{x^3y}\right)+q^2\left(\frac{x^2}{y}-\frac{y}{x^2}\right),$$

$$x^6y^2+\frac{q^4}{x^6y^2}=\left(x^3y+\frac{q^2}{x^3y}\right)^2-2q^2.$$

With the aid of the above identities and Lemma 2.2 in the above equation, and then simplifying stepwise, the generating function takes the following form.

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)\,q^n=5\frac{(f_2f_5^5+qf_1f_{10}^5)^2(f_2f_5^5-4qf_1f_{10}^5)^3}{f_1^{14}f_2^3f_5^5f_{10}^7}.$$

Finally, using Lemma 2.3, it follows that.

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(10n+8)\,q^n=5\frac{f_2^2f_5^2f_{10}}{f_1^4},$$

Which is the desired generating function.

4. Conclusion

The method employed in this paper to obtain the generating function for $\overline{\mathcal{EO}}(10n+8)$ can further be used to obtain other generating functions. For example, using Lemma 2.1 again in Theorem 1.1, extracting the terms involving q^{5n} and then replacing q^{5n} by q^n, the generating function for $\overline{\mathcal{EO}}(50n+8)$ may be obtained as

$$\sum_{n=0}^{\infty}\overline{\mathcal{EO}}(50n+8)\,q^n=5\left(\frac{f_2^{25}f_5^4}{f_1^6f_{10}^2}+160q\frac{f_2^{11}f_5^4}{f_1^{14}}+2000q^2\frac{f_2^{11}f_5^{10}}{f_1^{20}}\right).$$

The fact that this partition function is related to the third-order mock theta function, a number of congruences follow automatically.

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