International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V71I12P105

Volume 71 Issue 12, 40-45, December 2025

Original Article

A Goat Problem and An Integer Sequence

Sanjay M Chinchole

Department of Mathematics, Arts, Science and Commerce College, Manmad, Maharashtra, India.

Corresponding Author : mr.sanjaymaths@gmail.com

Received: 12 October 2025 Revised: 23 November 2025 Accepted: 14 December 2025 Published: 30 December 2025

Abstract - *This paper introduces a goat problem and the resulting integer sequence. The new integer sequence introduced is structured like Narayana's cow sequence. This new sequence investigated in the paper has a fourth-order recurrence relation. The sequence of the ratios of the terms of this sequence with the preceding terms is convergent. The limit of this sequence of ratios is investigated as 1.56638327726619. The characteristic equation and generating function of this sequence are also derived.*

Keywords - *Goat Problem, Integer Sequence, Recurrence Relation, Characteristic Equation, Generating Function*.

1. Introduction

From ancient times, the sequences of integers have been studied. The Fibonacci sequence [1-4] is a sequence in which each element is the sum of the two elements that precede it. Lucassequence [1-4] is an integer sequence similar to the Fibonacci sequence, where each number is the sum of the two preceding numbers, but they start with different initial values. The initial terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … while the initial terms of the Lucas sequence are 1, 3, 4, 7, 11, 18, 29, 47, 76, … The Fibonacci sequence is associated with the famous problem of rabbits. From a problem of a herd of cows and calves, proposed by the Indian mathematician Narayana Pandit, Narayana's sequence [2-3] originated. The initial terms of this sequence are 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, …. This research study introduces a new problem of the goat and kids. Thereafter, a new integer sequence has been formed.

2. Preliminaries

2.1.Fibonacci, Lucas, and Narayana Sequences

2.1.1.Recursive Definitions

A recurrence relation [10] for a sequence $a_1, a_2, a_3,$ ... is a formula that relates each term a_n to certain of its predecessors$a_{n-1}, a_{n-2}, \dots, a_{n-i}$, where i is an integer with $n - i > 0$. If i is a fixed integer, the initial conditions for such a recurrence relation specify the values of $a_1, a_2, \dots, a_i$. If i depends on k, the initial conditions specify the values of $a_1, a_2, \dots, a_m$, where m is an integer with $m > 0$.

The recursive definition of the nth Fibonacci number [1],F_n is
Initial conditions: $F_1 = F_2 = 1$ and
Recurrence Relations: $F_n = F_{n-1} + F_{n-2},\ n \geq 3$.

The recursive definition of the nth Lucas number [1],L_n is
Initial conditions: $L_1 = 1, L_2 = 3$ and
Recurrence Relations: $L_n = L_{n-1} + L_{n-2},\ n \geq 3$.

The recursive definition of the nth Narayana Cows number [5],S_n is
Initial conditions: $S_1 = 1, S_2 = 1, S_3 = 1$ and
Recurrence Relations: $S_n = S_{n-1} + S_{n-3},\ n \geq 4$.

2.2. Characteristic Equation and Generating Function

2.2.1. Characteristic Equations:

A characteristic equation [10] of an integer sequence is an equation in $1, t, t^2$, ... satisfying the recurrence relation of an integer sequence.

The characteristic equation of Fibonacci numbers [6, 11],F_n is

$$t^2 - t - 1 = 0.$$

The characteristic equation of Lucas numbers [7],L_n is

$$t^2 - t - 1 = 0.$$

The characteristic equation of Narayananumbers [8],S_n is

$$t^3 - t^2 - 1 = 0.$$

2.2.2. Generating Functions:

Let $a_1 a_2, a_3, ...$ be a sequence of real numbers. Then the function

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{,,} x'' + \cdots$$

is called the generating function [1] for the sequence $\{a_n\}$.

The generating function [1- 9]of theFibonacci sequenceF_n is

$$g(x) = \frac{x}{1 - x - x^2}$$

The generating function [1] of the LucassequenceL_n is

$$g(x) = \frac{2 - x}{1 - x - x^2}.$$

The generating function [8] of the Narayana sequence is

$$g(x) = \frac{1}{1 - x - x^3}.$$

3. A Goat and Kid Problem

3.1. Description of a goat and kid problem

The goat and kids problem is designed as follows:

Suppose there is a newborn female goat. Find the number of female goats in the herd when this newborn female goat reaches 15 years of age, if:

1) Each newborn female goat reaches the juvenile stage at the end of the first year.
2) Each newborn female goat reaches the adult stage at the end of the second year.
3) Each newborn female goat gives birth to one or two female kids from the age of three years onwards (i. e. the female goat gives birth to 1 female kid at the end of every odd year, such as the third, fifth, seventh, etc., and 2 female kids at the end of every even year, such as the fourth, sixth, eighth, etc.); and
4) All female goats have the same characteristics and can all live up to 15 years.

The number of female goats at the different stages from the start of 1^{st} year to the end of 15^{th} year is shown in Table 1.

Table 1. Population of Female Goats at Different Stages in the First 15 Years

Number of Female Goats/Year	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}	8^{th}	9^{th}	10^{th}	11^{th}	12^{th}	13^{th}	14^{th}	15^{th}
Adults	0	0	1	1	1	2	4	5	8	13	21	31	50	78	123
Juveniles	0	1	0	0	1	2	1	3	5	8	10	19	28	45	67
Babies born at an odd age	1	0	0	1	0	1	1	3	2	6	7	14	17	33	45
Babies born at an even age	0	0	0	0	2	0	2	2	6	4	12	14	28	34	66
Total	1	1	1	2	4	5	8	13	21	31	50	78	123	190	301

Thus, from the above table, the number of female goats in the herd when the first newborn female goat reaches 15 years of age is 301.

3.2. The New Integer Sequence and Its Properties

3.2.1. Formation of New Integer Sequence

The goat and kid problem described above generates a new integer sequence.$\{C_n\}$. The nth term of the integer sequence$\{C_n\}$ Is the total number of female goats at the end of nth year. Thus, the first fifteen terms of this integer sequence are $1, 1, 1, 2, 4, 5, 8, 13, 21, 31, 50, 78, 123, 190, 301$.

If we continue the above-mentioned goat and kid problem without considering any death, the first 100 numbers in this sequence$\{C_n\}$ Are obtained. These 100 terms of sequence are tabulated as follows:

Table 2. First 100 Terms of the New Integer Sequence $\{C_n\}$
((Table captions should be placed above the tables.))

n	C_n	n	C_n
1	1	51	3113512305
2	1	52	4876948743
3	1	53	7639176934
4	2	54	11965871650
5	4	55	18743150287
6	5	56	29358946070
7	8	57	45987375805
8	13	58	72033839657
9	21	59	112832622449
10	31	60	176739107602
11	50	61	276841213716
12	78	62	433639409365
13	123	63	679245566216
14	190	64	1063958838285
15	301	65	1666567403013
16	469	66	2610483223231
17	737	67	4089017373730
18	1150	68	6404968302814
19	1808	69	10032635402987
20	2825	70	15714952123006
21	4432	71	24615638453261
22	6933	72	38557524131621
23	10873	73	60395861382241
24	17015	74	94603066830894
25	26670	75	148184662420384
26	41754	76	232113976476377
27	65431	77	363579452015760
28	102454	78	569504772558549
29	160525	79	892062753332905
30	251393	80	1397312177527060
31	393841	81	2188726429922970
32	616826	82	3428384475977070
33	966284	83	5370164114115850
34	1513453	84	8411735260954170
35	2370792	85	13176001449938900
36	3713389	86	20638668327024100
37	5816813	87	32328064939124700
38	9111087	88	50638140298871300
39	14271786	89	79318736166026600
40	22354678	90	124243541892044000
41	35016499	91	194613006343147000
42	54848638	92	304838558655814000

43	85914749		93	477494020567245000
44	134574493		94	747938648783050000
45	210796385		95	1171558591909350000
46	330186518		96	1835109786661920000
47	517200376		97	2874485281826890000
48	810131889		98	4502545676137370000
49	1268979664		99	7052712252307520000
50	1987705301		100	11047250531288100000

3.2.2.Recursive Relation of New Integer Sequence

Further examination of the numbers in the sequence listed above reveals that the numbers satisfy the recursion relation. $C_n = C_{n-2} + C_{n-3} + 2C_{n-4}, n \geq 5$ subject to the initial conditions$C_1 = C_2 = C_3 = 1$, and $C_4 = 2$. Thus, the recursive definition of this sequence is
Initial conditions: $C_1 = C_2 = C_3 = 1, C_4 = 2$, and
Recurrence Relations: $C_n = C_{n-2} + C_{n-3} + 2C_{n-4}, n \geq 5$.

Theorem 3.1: The characteristic equation of the new integer sequence is

$$r^4 - r^2 - r - 2 = 0.$$

Proof. The recurrence relation of the new integer sequence is

$$C_n = C_{n-2} + C_{n-3} + 2C_{n-4}.$$

This homogeneous linear recurrence relation may have a solution $C_n = r^n$. If we put $C_n = r^n$ In the above recurrence relation, we have

$$r^n = r^{n-2} + r^{n-3} + 2r^{n-4}.$$

Solving this, we have

$$r^{n-4}(r^4 - r^2 - r - 2) = 0.$$

Thus either $r = 0$ or$r^4 - r^2 - r - 2 = 0$.
Hence, the equation $r^4 - r^2 - r - 2 = 0$ is the characteristic equation of the new integer sequence.

Theorem 3.2: The generating function for the integer sequence $\{C_n\}$ is

$$g_n(x) = \frac{1+x}{1 - x^2 - x^3 - 2x^4}.$$

Proof. Let the generating function for the integer sequence. $\{C_n\}$ be

$$g(x) = C_1 + C_2x + C_3x^2 + C_4x^3 + \cdots + C_nx^{n-1} + \cdots.$$

Then, we have
$x^2g(x) = C_1x^2 + C_2x^3 + \cdots + C_{n-1}x^n + C_nx^{n+1} + \cdots.$
$x^3g(x) = C_1x^3 + C_2x^4 + \cdots + C_{n-1}x^{n+1} + C_nx^{n+2} + \cdots.$
$2x^4g(x) = 2C_1x^4 + 2C_2x^5 + \cdots + 2C_{n-1}x^{n+2} + 2C_nx^{n+3} + \cdots.$

Hence,

$$g(x) - x^2g(x) - x^3g(x) - 2x^4g(x) = C_1 + C_2x + (C_3 - C_1)x^2 + (C_4 - C_2 - C_1)x^3 + (C_5 - C_3 - C_2 - 2C_1)x^4 + \cdots + (C_n - C_{n-2} - C_{n-3} - 2C_{n-4})x^{n-1} + \cdots.$$

Since $C_1 = C_2 = C_3 = 1, C_4 = 2$, and $C_n = C_{n-2} + C_{n-3} + 2C_{n-4}, n \geq 5$, we have
$g(x) - x^2g(x) - x^3g(x) - 2x^4g(x) = 1 + x + (0)x^2 + (0)x^3 + (0)x^4 + \cdots + (0)x^{n-1} + \cdots.$
$\Rightarrow (1 - x^2 - x^3 - 2x^4)g(x) = 1 + x.$

Thus, the generating function for the integer sequence $\{C_n\}$ is

$$g(x) = \frac{1+x}{1 - x^2 - x^3 - 2x^4}.$$

4. Behavior of an Integer Sequence

Looking at the ratio of consecutive numbers of the new integer sequence, we observe that.

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n}$$

Exists. This suggests that $\{C_n\}$Behaves in some sense like a geometric sequence. To find the *common ratio*, we let

$$L = \lim_{n\to\infty} \frac{C_{n+1}}{C_n}.$$

Then,

$$L = \lim_{n\to\infty} \frac{C_{n-1} + C_{n-2} + 2C_{n-3}}{C_n}.$$

$$\Rightarrow L = \lim_{n\to\infty} \left[\frac{C_{n-1}}{C_n} + \frac{C_{n-2}}{C_n} + 2\frac{C_{n-3}}{C_n}\right].$$

$$\Rightarrow L = \lim_{n\to\infty} \frac{C_{n-1}}{C_n} + \lim_{n\to\infty} \frac{C_{n-2}}{C_n} + 2\lim_{n\to\infty} \frac{C_{n-3}}{C_n}.$$

$$\Rightarrow L = \frac{1}{\lim_{n\to\infty} \frac{C_n}{C_{n-1}}} + \frac{1}{\lim_{n\to\infty} \left[\frac{C_n}{C_{n-1}} \times \frac{C_{n-1}}{C_{n-2}}\right]} + 2\frac{1}{\lim_{n\to\infty} \left[\frac{C_n}{C_{n-1}} \times \frac{C_{n-1}}{C_{n-2}} \times \frac{C_{n-2}}{C_{n-3}}\right]}.$$

$$\Rightarrow L = \frac{1}{L} + \frac{1}{L \times L} + 2\frac{1}{L \times L \times L}.$$

$$\Rightarrow L = \frac{1}{L} + \frac{1}{L^2} + 2\frac{1}{L^3}.$$

$$\Rightarrow L^4 = L^2 + L + 2.$$

$$\Rightarrow L^4 - L^2 - L - 2 = 0.$$

The approximate value of the positive root of this equation is 1.56638327726619. Thus, the sequence of ratio consecutive terms of the new integer sequence is convergent, and the approximate value of this limit of convergence is 1.56638327726619.

5. Conclusion

This study of research has introduced a goat problem and hence a new integer sequence obtained from the cumulative number of goats in the successive number of years. The new integer sequence has been generalized and expressed in the form of a recursive definition. The characteristic equation and generating function are also derived for the new integer sequence. The sequence of the ratios of the terms of this sequence with the preceding terms is convergent.

References

[1] Thomas Koshy, *Fibonacci and Lucas Numbers with Applications, Volume 2*, John Wiley & Sons, 2019. [Google Scholar] [Publisher Link]

[2] Jhon J. Bravo, Pranabesh Das, and Sergio Guzman, "Repdigits in Narayana's Cows Sequence and Their Consequences," *arXiv preprint arXiv:2007.12797,* 2020. [CrossRef] [Google Scholar] [Publisher Link]

[3] Kisan Bhoi, and Prasanta Kumar Ray, "Fermat Numbers in Narayana's Cows Sequence," *Integers: Electronic Journal of Combinatorial Number Theory*, vol. 22, 2022. [Google Scholar] [Publisher Link]

[4] Masum Billal, and Samin Riasat, *Integer Sequences*, Springer, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[5] Jean-Paul Allouche, and Tom Johnson, "Narayana's Cows and Delayed Morphisms," *Journées d'Informatique Musicale*, 1996. [Google Scholar] [Publisher Link]

[6] E. Kilic, and A.P. Stakhov, "On the Fibonacci and Lucas p-numbers, Their Sums, Families of Bipartite Graphs and Permanents of Certain Matrices," *Chaos, Solitons & Fractals,* vol. 40, no. 5, pp. 2210–2221, 2009. [CrossRef] [Google Scholar] [Publisher Link]

[7] Alexey Stakhov, and Boris Rozin "Theory of Binet Formulas for Fibonacci and Lucas p-numbers," *Chaos, Solitons & Fractals,* vol. 27, no. 5, pp. 1162–1177, 2006. [CrossRef] [Google Scholar] [Publisher Link]

[8] Xin Lin, "On the Recurrence Properties of Narayana's Cows Sequence," *Symmetry,* vol. 13, no. 1, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[9] Jeffrey R. Chasnov, *Fibonacci Numbers and The Golden Ratio*, The Hong Kong University of Science and Technology, 2016. [Google Scholar] [Publisher Link]

[10] Susanna S. Epp, *Discrete Mathematics with Applications*, Brooks/Cole Publishing Co., 2010. [Google Scholar][Publisher Link]

[11] Can Kızılateş et al., "On Higher-Order Generalized Fibonacci Hybrid Numbers with q-Integer Components: New Properties, Recurrence Relations, and Matrix Representations," *Symmetry*, vol. 17, no. 4, 2025. [CrossRef] [Google Scholar] [Publisher Link]