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Original Article

Fermatean Picture Fuzzy Connectedness and Its Compactness

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Abstract *- This paper aims to introduce and explore the concepts of connectedness and compactness within Fermatean Picture fuzzy topological spaces. We will examine some fundamental properties and provide characterization theorems for these newly defined notions of connectedness and compactness in this specific type of topological space.*

Keywords *- Fermatean Picture fuzzy set (FPF), Fermatean Picture fuzzy topological spaces (FPFTS), FPF-continuous, FPF-connectedness.*

1. Introduction

Fuzzy set theory, first introduced by Zadeh[20] in 1965, revolutionized the approach to uncertainty in Mathematics and Engineering by allowing partial membership rather than relying on binary classification. This development laid the groundwork for handling imprecision in real-world problems. Building on this idea, Atanassov[1,2] introduced intuitionistic fuzzy sets (IFS), which incorporate both membership and non-membership degrees, thus improving decision-making under uncertainty. IFS found applications in areas such as expert systems and medical diagnostics. Further advancing this field, Atanassov and Gargov[3] developed interval-valued intuitionistic fuzzy sets, providing a more refined way to manage uncertainty.

The introduction of neutrosophic sets took the concept of fuzziness even further by treating truth, indeterminacy, and falsity as independent components. Broumi and Smarandache[4,5] made significant contributions by defining correlation coefficients and similarity measures for neutrosophic sets, expanding their use in practical domains like pattern recognition and decision analysis. Neutrosophic structures have also been applied in algebra, with Jun and colleagues[11,12] exploring cubic structures in BCK/BCI algebras.

Interval neutrosophic sets, as proposed by Wang et al. [15], merge theoretical progress with computational applications, particularly in artificial intelligence and machine learning. Similarly, Pythagorean fuzzy sets, introduced by Yager[17], offer a broader framework by allowing the square sum of membership and non-membership degrees to be less than or equal to one. Xindong Peng and Yong Yang[16,19] further developed this concept, producing novel results that strengthened the mathematical foundation of Pythagorean fuzzy sets.

Extending the work on fuzzy and neutrosophic sets, picture fuzzy sets were introduced by Bui Cong Cuong[6,7,18] as a new approach to tackling computational intelligence challenges. These sets expand intuitionistic fuzzy logic by adding a fourth parameter, neutrality, to better model real-world situations. Recent advancements have focused on refining the arithmetic operations and aggregation operators for picture fuzzy sets, highlighting their usefulness in multi-criteria decision-making[8,19].

Fermatean fuzzy sets and their applications in topology represent another important step forward in fuzzy logic research. Ibrahim[9,10] explored Fermatean fuzzy topological spaces, providing valuable theoretical insights and practical applications in optimization and topology. In our current research[13,14], we introduce Fermatean Picture fuzzy sets, which combine the strengths of both Fermatean and Picture fuzzy frameworks to address nonlinear variational problems.

The aim of this paper is to introduce and explore the concepts of connectedness and compactness in Fermatean Picture fuzzy topological spaces. It also seeks to examine some fundamental properties and provide characterization theorems for these newly defined concepts within this specific type of topological space. It provides fresh insights and methods for analyzing the behavior of Fermatean Picture fuzzy systems within a topological framework. This study connects the theoretical and practical aspects of fuzzy topology, paving the way for future developments in the field.

2. Preliminaries

Definition: 2.1

Let X be a universe of discourse, then a fuzzy set A is an object having the following formulation:
$A = \{\langle x, \mu_A(x)\rangle | x \in X\}$ where $\mu_A: X \to [0, 1]$ and $\mu_A(x)$ is called the membership degree of x in X.

Definition: 2.2.[13]

A Fermatean Picture Fuzzy (FPF) set $\mathcal{A}$ in a universe U is an object of the form, $\mathcal{A} = \{\langle x, \langle x, \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x)\rangle | x \in U\rangle\}$ where $\alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of X in $\mathcal{A}$ and the following conditions are satisfied

$$0 \leq \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x) \leq 1,$$

$$0 \leq {\alpha_{\mathcal{A}}}^3(x) + {\beta_{\mathcal{A}}}^3(x) + {\gamma_{\mathcal{A}}}^3(x) \leq 1, \forall x \in U.$$

Then $\forall x \in U, \pi_{\mathcal{A}}(x) = 1 - {\alpha_{\mathcal{A}}}^3(x) + {\beta_{\mathcal{A}}}^3(x) + {\gamma_{\mathcal{A}}}^3(x)$ is called the degree of refusal membership of X in $\mathcal{A}$.

When dealing with human opinions that involve multiple types of responses such as "yes," "abstain," "no," and "refusal," Fermatean Picture Fuzzy Sets offer a suitable mathematical framework to handle the complexity and uncertainty inherent in such scenarios. For an example, in feedback mechanisms for products or services, users might express satisfaction (yes), dissatisfaction (no), neutrality (abstain) or refuse to provide feedback.

Definition: 2.3.[13]

Let X be a non-empty set, and the FPF sets A and B be in the form
$A = \{\langle x, \langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle | x \in X\rangle\}$ and $B = \{\langle x, \langle x, \alpha_B(x), \beta_B(x), \gamma_B(x)\rangle | x \in X\rangle\}$

1. $(A) \subseteq (B)$ iff $\alpha_A(x) \leq \alpha_B(x), \beta_A(x) \leq \beta_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$
2. $(A) = (B)$ iff $\alpha_A(x) = \alpha_B(x), \beta_A(x) = \beta_B(x)$ and $\gamma_A(x) = \gamma_B(x) \forall x \in X$
3. $A \cap B = \{\langle x, \alpha_{AB}(x), \beta_{AB}(x), \gamma_{AB}(x)\rangle | x \in X\}$ where
   I. $\alpha_{A\cap B}(x) = min\{\alpha_A(x), \alpha_B(x)\}$
   II. $\beta_{A\cap B}(x) = min\{\beta_A(x), \beta_B(x)\}$
   III. $\gamma_{A\cap B}(x) = max\{\gamma_A(x), \gamma_B(x)\}$
4. $A \cup B = \{\langle x, \alpha_{AB}(x), \beta_{AB}(x), \gamma_{AB}(x)\rangle | x \in X\}$ where
   I. $\alpha_{A\cup B}(x) = max\{\alpha_A(x), \alpha_B(x)\}$
   II. $\beta_{A\cup B}(x) = min\{\beta_A(x), \beta_B(x)\}$
   III. $\gamma_{A\cup B}(x) = min\{\gamma_A(x), \gamma_B(x)\}$

Definition: 2.4. Images of Fermatean Picture Fuzzy Set [13]

Let X and Y be two non-empty sets, and $f: X \longrightarrow Y$ be a function. If $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle | x \in X\}$ is an FPF set in X, then the image of A under f denoted by $f(A)$ is the FPF set in Y defined by

$$f(A) = \{\langle y, f(\alpha_A)(y), f(\beta_A)(y), 1 - f(1 - (\gamma_A)(y))\rangle | y \in Y\}$$

$$f(\alpha_A)(y) = \begin{cases} sup_{x \epsilon f^{-1}(y)} \alpha_A(x) & if\ f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

$$f(\beta_A)(y) = \begin{cases} inf_{x \epsilon f^{-1}(y)} \beta_A(x) & if\ f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

$$1 - f\big(1 - (\gamma_A)(y)\big) = \begin{cases} inf_{x\epsilon f^{-1}(y)}\gamma_A(x) & if\ f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

Definition: 2.5.[13]

Let X be a non-empty set and τ be a family of Fermatean Picture Fuzzy ($\mathcal{F}$) subset of X. If

1. $1_X, 0_X \in \tau$
2. for any $\mathcal{F}_1, \mathcal{F}_2 \in \tau$, we have $\mathcal{F}_1 \cap \mathcal{F}_2 \in \tau$,
3. for any $\{\mathcal{F}_i\}_{i\in I} \subset \tau$, we have $\bigcup_{i\in I}\mathcal{F}_i \in \tau$ where I is an arbitrary index set, then τ is called a Fermatean Picture Fuzzy topology on X.

Fermatean Picture Fuzzy topological space (FPFTS) is defined as the pair (X, τ). Every element in τ is referred to as an open Fermatean Picture Fuzzy subset. A closed Fermatean Picture Fuzzy set is the complement of an open Fermatean Picture Fuzzy subset.

Definition: 2.6.[13]

$A = \{x, \alpha_{\mathcal{F}}(x), \beta_{\mathcal{F}}(x), (\gamma_{\mathcal{F}}(x)) : x \in X\}$ be an FPFS in X and (X, τ) be an FPFTS. Fermatean Picture fuzzy closure and interior are defined by

1. $FPFcl(A) = \cap \{H: H\ is\ closed\ Fermatean\ Picture\ fuzzy\ set\ in\ X\ and\ \mathcal{F} \subset H\}$.
2. $FPFint(A) = \cup \{G: G\ is\ open\ Fermatean\ Picture\ fuzzy\ set\ in\ X\ and\ G \subset \mathcal{F}\}$.

Definition: 2.7.[13]

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Fermatean Picture fuzzy subsets in an FPFTS. Then, if there is an open Fermatean Picture fuzzy subset A, such as $\mathcal{F}_1 \subset A \subset \mathcal{F}_2$, then $\mathcal{F}_2$ is said to be a neighbourhood of $\mathcal{F}_1$.

Definition: 2.8.[13]

Let $g : X \to Y$ be a function and (X, τ_1), (Y, τ_2) two Fermatean Picture fuzzy topological spaces. If there exists a neighbourhood U of $\mathcal{F}_1$ such that $g[U] \subset V$ for every neighbourhood V of $g[\mathcal{F}_1]$ and for any Fermatean Picture fuzzy subset $\mathcal{F}_1$ of X, then g is said to be Fermatean Picture fuzzy continuous.

Definition: 2.9.[14]

Let (X, $\mathscr{F}_p$) be a Fermatean Picture fuzzy topological space (FPFTS) and a Fermatean Picture fuzzy (FPF) subset A on a non-empty set X is called Fermatean Picture fuzzy open (FPF-O) set if, for each element $x \in A$, there exists an open neighborhood around x within A, where:

- The degrees of positive membership $(m(x))$, neutral membership $(n(x))$, and negative membership $(l(x))$ are consistently high within this neighborhood for each element.
- The hesitation degree $h(x)$ is also appropriately accounted for, ensuring the Fermatean condition $m^3 + n^3 + l^3 + h^3 \leq 1$.

Definition: 2.10.[14]

An FPF subset A on a non-empty set X is called an FPF pre-open (FPF-PO) set if there exists an FPF subset $B \subseteq X$ such that B is FPF-O set in X and $B \subseteq A \subseteq FPFcl(B)$.

Definition: 2.11.[14]

A subset $A \subseteq X$ in an FPFTS is called Fermatean Picture fuzzy semi-open (FPF-SO) set if $A \subseteq FPFcl(FPFint(A))$.

Definition: 2.12.[14]

A subset $A \subseteq X$ in an FPFTS is called Fermatean Picture fuzzy β-open (FPF- βO) set if there exists an FPF-O set O such that $O \subseteq A \subseteq FPFcl(O)$.

Theorem: 2.13.[14]

For any FPFTS (X, $\mathscr{F}_p$), we have the following,

1. Every FPF-O set is a FPF-PO set.
2. Every FPF-SO set is a FPF-PO set.
3. Every FPF-O set is a FPF-SO set.
4. Every FPF-βO set is a FPF-PO set.

The converse of the above statements need not be true, as can be seen from the following examples.

Definition: 2.14.[14]
Let (X, $\mathscr{F}_p$) and (Y, $\mathscr{H}_p$) be two FPFTS. Then, a bijective mapping f : (X, $\mathscr{F}_p$) →(Y, $\mathscr{H}_p$) is called a

(i) Fermatean Picture Fuzzy continuous mapping (in short FPF-C mapping) if and only if f^{-1} (L) is a FPF-O set in X, whenever L is a FPF-O set in Y.
(ii) Fermatean Picture Fuzzy semi continuous mapping (in short FPF-SC mapping) if and only if f^{-1}(L) is a FPF-SO set in X, whenever L is a FPF-O set in Y.
(iii) Fermatean Picture Fuzzy pre continuous mapping (in short FPF-PC mapping) if and only if f^{-1}(L) is a FPF-PO set in X, whenever L is FPF-O set in Y.
(iv) Fermatean Picture Fuzzy β-continuous mapping (in short FPF-βC mapping) if and only if f^{-1} (L) is a FPF-βO set in X, whenever L is a FPF-O set in Y.

3. Fermatean Picture fuzzy connectedness and compactness

Definition: 3.1.
An FPFTS (X, $\mathscr{F}_p$), where X is the universe of discourse and $\mathscr{F}_p$ is the FPF topology, is said to be FPF-connected if there exist two non-empty disjoint FPF open sets $A, B \in \mathscr{F}_p$ such that:
$A \cap B = \emptyset$, and
$A \cup B = X$, where $A \neq \emptyset, B \neq \emptyset$

Example: 3.2.
Let $X = \{x_1, x_2\}$, where the Fermatean picture fuzzy degrees of x_1 and x_2 are as follows:
For x_1: $\mu = 0.6, \nu = 0.3, \pi = 0.1$
For x_2: $\mu = 0.5, \nu = 0.4, \pi = 0.1$
We cannot be partitioned X into two disjoint FPF open sets and so it is FPF-connected.

Theorem: 3.3.
If {(X, $\mathscr{F}_{pi}$), $i \in I$} is a family of connected FPFTS and $\bigcap_{i\in I} X_i \neq \emptyset$, then the union $\bigcup_{i\in I} X_i$ is also FPF-connected in the FPF topology.
Proof: Let us assume $\bigcup_{i\in I} X_i$ is not FPF-connected. Then there exist two disjoint FPF open sets A and B such that $A \cap B = \emptyset$ and A ∪ B = $\bigcup_{i\in I} X_i$. Since $\bigcap_{i\in I} X_i \neq \emptyset$, there is at least one common point x belonging to all X_i, which implies x cannot belong exclusively to A or B.
This contradiction implies $\bigcup_{i\in I} X_i$ is FPF-connected.

Example: 3.4.
Consider the FPFTS (X, $\mathscr{F}_p$), where:
$X = \{x_1, x_2, x_3, x_4\}$,
$\mathscr{F}_p = \{\emptyset, X, \{x_1, x_2\}, \{x_3, x_4\}, \{x_1, x_2, x_3\}\}$.

The Fermatean values for the elements are as follows:

Element	Membership(μ)	Non- membership(γ)	Hesitation(π)
x_1	0.6	0.3	0.1
x_2	0.7	0.2	0.1
x_3	0.5	0.4	0.1
x_4	0.4	0.5	0.1

Let $X_1 = \{x_1, x_2\}$ and $X_2 = \{x_3, x_4\}$. Assume X_1 and X_2 are FPF-connected.
X_1 cannot be separated into disjoint FPF open sets because there are no subsets of X_1 in $\mathscr{F}_p$ that are both disjoint and open. Similarly, X_2 is FPF-connected for the same reason.
$If\ X_1 \cap X_2 = \emptyset$, the union $X_1 \cup X_2 = X$ is FPF-connected because X cannot be partitioned into disjoint FPF open sets.

Theorem: 3.5.
Let f : (X, $\mathscr{F}_p$) →(Y, $\mathscr{H}_p$) be a FPF-continuous function. If (X, $\mathscr{F}_p$) is FPF-connected, then $f(X)$ is also FPF-connected in (Y, $\mathscr{H}_p$).
Proof: Let us assume that $f(X)$ is not FPF-connected. Then $f(X)$ can be expressed as the union of two disjoint non-empty FPF open sets A and B in (Y, $\mathscr{H}_p$) : A ∪ B = f(X) and $A \cap B = \emptyset$. Since f is FPF-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are FPF open sets in (X, $\mathscr{F}_p$) that are disjoint and cover X.

This contradicts the connectedness of (X, $\mathscr{F}_p$).
Hence, $f(X)$ is FPF-connected.

Example: 3.6.
By using example 3.4, consider a FPF-continuous function f: (X, $\mathscr{F}_p$) →(Y, $\mathscr{H}_p$),
where $Y = \{y_1, y_2\}$ and $f(x_1) = y_1, f(x_2) = y_1, f(x_3) = y_2, f(x_4) = y_2$
Let $\mathscr{H}_p$= $\{\emptyset, Y, \{y_1\}, \{y_2\}\}$.
Since X is FPF-connected, $f(X) = \{y_1, y_2\}$ is also FPF-connected because $f^{-1}(y_1) = \{x_1, x_2\}$ and $f^{-1}(y_2) = \{x_3, x_4\}$ are open in FPFTS, satisfying continuity.

Theorem: 3.7.
A FPFTS (X, $\mathscr{F}_p$) is FPF-connected if and only if X cannot be expressed as the union of two non-empty disjoint FPF closed sets.
Proof:
Necessity:
Assume X is FPF-connected. If $X = A \cup B$, where A and B are non-empty disjoint FPF closed sets, then their complements $X - A$ and $X - B$ would form non-empty disjoint FPF open sets covering X. This violates the connectedness of X.
Sufficiency:
Assume X cannot be decomposed into two non-empty disjoint FPF closed sets. If $X = A \cup B$, where A and B are disjoint FPF open sets, their complements would form disjoint FPF closed sets, which contradicts the assumption.
Thus, X is FPF-connected.

Example: 3.8.
By using example 3.4, In (X, $\mathscr{F}_p$), consider the sets:
$A = \{x_1, x_2\}$
$B = \{x_3, x_4\}$
Both A and B are FPF closed sets because their complements $X \backslash A = \{x_3, x_4\}$ and $X \backslash B = \{x_1, x_2\}$ are FPF open sets in $\mathscr{F}_p$.
If $X = A \cup B$, $A \cap B = \emptyset$, then X is not FPF-connected. Hence, X being connected means it cannot be partitioned this way.

Theorem: 3.9.
Let (X, $\mathscr{F}_p$) be a FPF-connected FPFTS. If $A \subseteq X$ is an FPF subspace such that $A \cap B \neq \emptyset$ for any FPF closed set $B \subseteq X$, then A is FPF-connected.
Proof: Suppose A is not FPF-connected. Then A can be partitioned into two disjoint FPF open subsets A_1 and A_2 such that $A_1 \cup A_2 = A$. Extend A_1 and A_2 to FPF open sets in X, which would imply X is not connected.
This contradiction proves that A is FPF-connected.

Example: 3.10.
By using example 3.4, Let $A = \{x_1, x_2, x_3\}$, which is a subspace of X. The Fermatean Picture fuzzy topology on A is $\mathscr{F}_p$= $\{\emptyset, A, \{x_1, x_2\}, \{x_3\}\}$.
Here, A is FPF-connected because there are no disjoint Fermatean Picture fuzzy open sets in $\mathscr{F}_p$ that partition A.

Theorem: 3.11.
If (X, $\mathscr{F}_p$) and (Y, $\mathscr{H}_p$) are connected FPFTS, then their product $X \times Y$ is also FPF-connected.
Proof: Assume $(X \times Y)$ is not FPF-connected. Then, there exist two disjoint FPF open sets A and B in $X \times Y$ such that $A \cup B = X \times Y$. The projections onto X and Y would lead to disjoint Fermatean picture fuzzy open sets in X and Y, contradicting the connectedness of X and Y.
Hence, $X \times Y$ is FPF-connected.

Example: 3.12.
By using example 3.4, Let (X, $\mathscr{F}_p$) and (Y, $\mathscr{H}_p$) be FPF-connected in FPFTS, where:
$X = \{x_1, x_2\}$ and $\mathscr{F}_p = \{\emptyset, X, \{x_1\}, \{x_2\}\}$
$Y = \{y_1, y_2\}$ and $\mathscr{H}_p = \{\emptyset, Y, \{y_1\}, \{y_2\}\}$
The product space $X \times Y = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$ with the product topology is connected because any attempt to partition $X \times Y$ into disjoint Fermatean Picture fuzzy open sets will violate the connectedness of X or Y.

Theorem: 3.13.
If (X, $\mathscr{F}_p$) is an FPF-connected and $f: X \to \mathbb{R}$ is an FPF-continuous function, then f has the intermediate value property.
Proof: Let us assume f does not have the intermediate value property. Then there exist $a, b \in X$ such that $f(a) < c < f(b)$, but $f(x) \neq c$ for all $x \in X$. Define $A = \{x \in X: f(x) < c\}$ and $B = \{x \in X: f(x) > c\}$. Both are disjoint FPF open sets that cover X, contradicting its connectedness.
Hence, f has the intermediate value property.

Definition: 3.14.
A subset B is called a FPF-β disconnected subset of a FPFTS (X, $\mathscr{F}_p$) if there exist FPF-β open sets M, N such that M ∩ B ≠ φ ≠ N ∩ B , M ∩ N ∩ B = φ and B ⊆ M∪N. Otherwise B is called an FPF- βconnected subset.

Theorem: 3.15.
The union of any family of FPF-β connected sets with a nonempty intersection is FPF-β connected.
Proof: Take P = $\cup_{i\in I}$P$_i$, where each P_i is FPF-β connected with ∩P_i≠ ∅. Suppose that P is not FPF-β connected. Then P = R∪S, where R and S are two nonempty disjoint sets such that (R∩FPF-βcl(S)) ∪ (FPF-βcl(R)∩S) = ∅. Since P_i is FPF-β connected and P_i ⊆ P, we have P_i ⊆ R or P_i ⊆ S. Therefore, ∪P_i ⊆ R or, ∪P_i ⊆ S. Since ∩P_i≠ ∅, there exist atleast one element x∈∩P_i. Therefore, x∈ P_i for all i. So, x∈ P. Therefore, x∈ R or x∈ S. Suppose x∈ R, since R∩S = ∅, we have x ∉ R. Therefore R ⊄ S. Thus P ⊆ R. This contradicts P = R∪S. Thus, P is FPF – βconnected.

Example: 3.16.
Let $X = \{x_1, x_2, x_3\}$ and define FPF-β connected subsets A and B of X are
$A = \{< x_1, 0.85, 0.1, 0.05 >, < x_2, 0.8, 0.15, 0.05 >\}$
$B = \{< x_2, 0.8, 0.15, 0.05 >, < x_3, 0.75, 0.2, 0.05 >\}$
$A \cup B = \{< x_1, 0.85, 0.1, 0.05 >, < x_2, 0.8, 0.15, 0.05 >, < x_3, 0.75, 0.2, 0.05 >\}$
Since $A \cap B \neq \emptyset$(i.e., A and B share x_2), their union $A \cup B$ is also FPF-β connected.

Definition: 3.17.
A subset A of a $FPFTS$ (X, $\mathscr{F}_p$) is called compact if for every collection of FPF-O sets $\{U_\alpha\}_{\alpha\in I}$ in $\mathscr{F}_p$ that covers A (i.e.,$A \subseteq \cup_{\alpha\in I} U_\alpha$), there exists a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ that also covers A.

Theorem: 3.18.
In a $FPFTS$ (X, $\mathscr{F}_p$), a subset $A \subseteq X$ is compact if and only if every collection of FPF-O sets $\{U_\alpha\}_{\alpha\in I}$ covering A (i.e.,$A \subseteq \cup_{\alpha\in I} U_\alpha$), there exists a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ that also covers A.

Proof:
Necessity:
Let us assume that A is compact in the FPFTS (X, $\mathscr{F}_p$). By using the definition, which means for every collection of FPF-O sets $\{U_\alpha\}_{\alpha\in I}$ in $\mathscr{F}_p$ that covers A (i.e.,$A \subseteq \cup_{\alpha\in I} U_\alpha$), there must exist a finite subcollection that also covers A.
Hence, for any cover $\{U_\alpha\}_{\alpha\in I}$ of A, we can find the finite subset $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ such that $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.
Sufficiency:
Assume that every collection of FPF-O sets that covers A has a finite subcover. Let us take any arbitrary cover $\{U_\alpha\}_{\alpha\in I}$ of A by FPF-O sets. By assumption, there exists a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ such that $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. This finite subcover verifies the compactness of A in the FPFTS.

Example: 3.19.
Consider a $FPFTS$ (X, $\mathscr{F}_p$), where $X = \{p, q, r\}$ and the FPF topology $\mathscr{F}_p$ consists of the following open sets: $\mathscr{F}_p = \{\emptyset, \{p\}, \{p, q\}, X\}$
Each element in X has associated membership μ, non membership ν and indeterminate γ values with the condition $\mu(x)^3 + \nu(x)^3 + \gamma(x)^3 \leq 1$ for FPF sets.
Assume the following:

- For p: $\mu(p) = 0.5, \nu(p) = 0.2, \gamma(p) = 0.3$
- For q: $\mu(q) = 0.6, \nu(q) = 0.3, \gamma(q) = 0.1$
- For r: $\mu(r) = 0.5, \nu(r) = 0.2, \gamma(r) = 0.3$

Let $A = \{p, q\}$, to cover A, we need to find a collection of FPF-O sets that include A.

Consider the collection $\{\{p\},\{p,q\},X\}$:

- $\{p\}$ only contains p,
- $\{p,q\}$ contains both p and q, which covers A,
- X contains p,q and r, also covering$A$.

Therefore $\{\{p\},\{p,q\},X\}$ is an open cover of A.
From the collection$\{\{p\},\{p,q\},X\}$, the set $\{p,q\}$ alone is sufficient to cover A, because $A \subseteq \{p,q\}$.
Thus $\{\{p,q\}\}$ is a finite subcover of A.
Since we have found the finite subcover for every open cover of A, we conclude that $A = \{p,q\}$ is compact in the FPFTS (X, $\mathscr{F}_{\mathrm{p}}$).

Theorem: 3.20.
A subset A of a FPFTS (X, $\mathscr{F}_{\mathrm{p}}$) is compact if and only if every collection of FPF-C sets $\{F_\alpha\}_{\alpha\in I}$ with the finite intersection property (i.e., the intersection of any finite subcollection is non-empty) has a non-empty intersection.
Proof:
Necessity:
Suppose A is compact, and $\{F_\alpha\}_{\alpha\in I}$ is a collection of FPF-C sets with the finite intersection property.
Let us assume the contradiction, that $\bigcap_{\alpha\in I} F_\alpha = \emptyset$. Then $\{X\backslash F_\alpha\}_{\alpha\in I}$ is an open cover of A. Since A is compact, there is a finite subcover $\{X\backslash F_{\alpha_{1,\dots,}} X\backslash F_{\alpha_n}\}$. This implies that $\bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$, contradicting the finite intersection property.
Sufficiency:
Assume the finite intersection property holds for closed sets.
Given an open cover $\{U_\alpha\}_{\alpha\in I}$ of A, take $F_\alpha = X\backslash U_\alpha$. Then the collection $\{F_\alpha\}_{\alpha\in I}$ has the finite intersection property, implying A is compact.

Theorem: 3.21.
Let f: (X, $\mathscr{F}_{\mathrm{p}}$) →(Y, $\mathscr{H}_{\mathrm{p}}$) be a continuous mapping between two FPFTS. If $A \subseteq X$ is compact in (X, $\mathscr{F}_{\mathrm{p}}$), then $f(A)$ is compact in (Y, $\mathscr{H}_{\mathrm{p}}$).
Proof: Assume A is compact in (X, $\mathscr{F}_{\mathrm{p}}$). Consider an open cover $\{V_\beta\}_{\beta\in J}$ of $f(A)$ in Y.
By the continuity of f, the inverse images $\{f^{-1}(V_\beta)\}_{\beta\in J}$ from an open cover of A in X. Since A is compact, there exists a finite sub cover $\{f^{-1}(V_{\beta_1}),\dots,f^{-1}(V_{\beta_n})\}$. Therefore $\{V_{\beta_1},\dots V_{\beta_n}\}$ is a finite cover of $f(A)$, proving that $f(A)$ is compact.

Example: 3.22.
Let (X, $\mathscr{F}_{\mathrm{p}}$) be a FPFTS, where $X = \{p,q\}$ and $\mathscr{F}_{\mathrm{p}} = \{\emptyset,\{p\},X\}$.
Let (Y, $\mathscr{H}_{\mathrm{p}}$), be another space, where Y={x,y,z} and $\mathscr{H}_{\mathrm{p}} = \{\emptyset,\{x\},\{x,y\},Y\}$.
Let us define a continuous function $f: X \to Y$ by $f(p) = x, f(q) = y$.
Assume $A = \{p,q\} \subset X$ is compact in (X, $\mathscr{F}_{\mathrm{p}}$). The image $f(A) = \{x,y\}$. The open set $\{x,y\} \in \mathscr{F}_{\mathrm{p}}$ covers $f(A)$ alone, showing that $f(A)$ is compact in (Y, $\mathscr{H}_{\mathrm{p}}$).

Theorem: 3.23.
In a FPFTS (X, $\mathscr{F}_{\mathrm{p}}$), if $A \subseteq X$ is compact and $B \subseteq A$ is a closed subset, then B is also compact.
Proof: Assume $A \subseteq X$ is compact, and $B \subseteq A$ is closed. Consider an open cover $\{U_\alpha\}_{\alpha\in I}$ of B in X. By adding $X \setminus B$ to this collection, we get an open cover $\{U_\alpha\}_{\alpha\in I} \cup \{X\backslash B\}$ of A. Since A is compact, there exists a finite subcover for A. Removing $X \setminus B$ from this subcover gives a finite subcover for B, proving that B is compact.

4. Conclusion

In this paper, we present and investigate the concepts of connectedness and compactness within Fermatean Picture fuzzy topological spaces. We explore key properties and offer characterization theorems for these newly defined concepts in this particular class of topological spaces. Additionally, we provide illustrative examples to highlight the significance of the results obtained.

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