International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V71I2P103
Volume 71 Issue 2, 15-29, February 2025

Original Article

The Application of Factor Analysis in Comprehensive Evaluation of Student Achievement

Hang Zhang

School of Mathematics and Information Science, Henan Polytechnic University, Henan, China.

Corresponding Author : 15538907580@126.com

Received: 20 December 2024 Revised: 26 January 2025 Accepted: 11 February 2025 Published: 28 February 2025

Abstract *- As an important method in multivariate statistics, factor analysis has been used in many fields, but it is less used in education, but the effect is better. In education, there is a need for a rational, comprehensive analysis of student achievement, and factor analysis can achieve this goal. This paper uses the factor analysis method to complete the comprehensive evaluation of the student's performance, introduces the theory, basic concepts and methods of the factor analysis method in detail, and then uses the method to complete the comprehensive evaluation of the student's performance. In factor analysis, we use R to implement and get the output results of factor analysis and then comprehensively analyze and evaluate students' scores according to factor scores. Finally, it compares with the commonly used grade point method to verify the rationality of the factor analysis method and obtains the advantages and disadvantages of the two methods through comparison. The results show that the factor analysis method is reasonable and effective in comprehensively evaluating students' performance and can be used as a method.*

Keywords *- Comprehensive assessment, Factor analysis, Factor loading, Factor rotation, Factor score.*

1. Introduction

In scientific research, it is often necessary to judge the degree of superiority or inferiority of a certain thing among its counterparts and its law of development. Since there are many factors affecting the characteristics of a thing and its laws of development, in order to reflect its characteristics and laws of development more comprehensively and accurately, when conducting a study on the thing, it should not be evaluated only from a single indicator or unilaterally but should take into account the many factors related to it. That is to say, more variables related to the thing should be introduced into the study to analyze and evaluate it comprehensively. In addition, multivariate large sample data can undoubtedly provide researchers or decision-makers with a lot of valuable information, but when analyzing and dealing with multivariate problems, there is often a certain degree of correlation between the many variables, resulting in the overlap of information reflected in the observed data. [1] Therefore, to minimize overlapping information and reduce workload, it is often desirable to find a few unrelated composite variables that reflect as much as possible the majority of the information contained in the original data. [2] Factor analysis is one of the multivariate statistical methods developed to solve this problem.

Factor analysis was developed by psychologists Charles Spearman and Karl Pearson in 1904. [3] They used this method to statistically analyze IQ scores. It has since been used to solve problems in psychology and education, and it was developed in the 1960s due to the application of computer technology. Currently, the method is widely used in many fields, including economic, social, educational, and technological.

The basic idea of factor analysis is to find a few random variables that control all the variables to describe the correlation between multiple variables by studying the internal structure of the correlation coefficient matrix of the variables. [4] In other words, the observed variables are classified, and the variables with a high correlation and the variables with a close connection are divided into the same class so that the correlation between variables of different classes is low. Then, each category of variables actually represents an essential factor or a basic structure, and factor analysis is the search for such unobservable factors or structures in the system. It is a multivariate statistical analysis that simplifies the observing system by reducing multiple indicators to a small number of unobservable composite indicators. It ultimately provides a quantitative evaluation of the sample points.

In fact, factor analysis can be seen as a generalization of principal component analysis. Both principal component analysis and factor analysis aim to simplify the algorithm by converting high dimensions into low dimensions by reducing multiple variables into a few composite data or factors. [5] A feature of principal component analysis is that several composite indicators are extracted that reflect as much information as possible from the original data to avoid duplicating information and complicating the algorithm. The composite data are, as far as possible, unrelated to each other. [6] Factor analysis focuses on the internal dependence of correlation and covariance matrices, transforming multiple variables into several factors to achieve the goal of reproducing the relationship between the original data and the factors. [7]

At present, one of the key references for the education of students in colleges and universities is their grades during their school years, which is, at the same time, an important criterion for evaluating students. Based on the students' grades, we can get a general idea of their learning status and professional standard during their school years and a general idea of the student's ability level. In general, the main factor that colleges and universities now take into account in the evaluation of scholarships and the creation of excellence is the student's performance during the school year. The average score method is simple and easy to complete for the evaluation method, but it is too one-sided, and the effect is not good. The grade point average method is one of the more commonly used methods in colleges and universities, and it is based on the importance of the courses in finding the average. Although it is intuitive and easy to understand, different courses are affected by a variety of factors, and there is no uniform standard, and thus the comparability is relatively poor. [8] These methods cannot reflect the characteristics of each aspect of students' abilities and conceal their individuality, nor can they make a comprehensive and holistic evaluation of students to reflect their professional abilities, and they can only obtain students' comprehensive grades and rankings, which have obvious limitations. These methods can neither reflect the characteristics of students' abilities in every aspect nor make a comprehensive and holistic evaluation of students and can only obtain students' comprehensive grades and rankings, which have obvious limitations.

Factor analysis addresses the shortcomings of the above methods by analyzing the data to more accurately and comprehensively assess the student's performance and complete a comprehensive assessment of the student's performance. This paper uses factor analysis to find out the public factors affecting students' performance and realize a comprehensive analysis of students' performance. In this way, it finds out the special characteristics of students and makes a comprehensive assessment and comprehensive ranking of students' performance.

The rest of the paper is organized as follows. Section 2 introduces the factor analysis method, mainly the theoretical knowledge related to factor analysis, to provide theoretical support for this paper. In Section 3, the process of realizing a comprehensive assessment of student performance using factor analysis is described in detail. Section 6 concludes the paper with a brief discussion.

2. Basic Theory of Factor Analysis

2.1. Models for Factor Analysis

Charles Spearman introduced the idea of factor analysis in 1904 while studying student test scores. [9] He obtained the following correlation matrix:

$$\begin{matrix} C \\ \mathrm{F} \\ \mathrm{E} \\ \mathrm{M} \\ \mathrm{D} \\ \mathrm{Mu} \end{matrix} \begin{bmatrix} 1.00 & 0.83 & 0.78 & 0.70 & 0.66 & 0.63 \\ 0.83 & 1.00 & 0.67 & 0.67 & 0.65 & 0.57 \\ 0.78 & 0.67 & 1.00 & 0.64 & 0.54 & 0.51 \\ 0.70 & 0.67 & 0.64 & 1.00 & 0.45 & 0.51 \\ 0.66 & 0.65 & 0.54 & 0.45 & 1.00 & 0.40 \\ 0.63 & 0.57 & 0.51 & 0.51 & 0.40 & 1.00 \end{bmatrix}.$$

Where the symbols denote Classical (C), French (F), English (E), Mathematics (M), Discernment (D), and Music (Mu).

It can be found that if we exclude the correlation between the scores of each subject and itself, any two columns of the correlation matrix are basically proportional. For example, for columns 1 and 3 of the correlation matrix, there are

$$\frac{0.83}{0.67} \approx \frac{0.70}{0.64} \approx \frac{0.66}{0.54} \approx \frac{0.63}{0.51} \approx 1.2.$$

According to this, Spearman proposes that grades in each subject follow:

$$X_i = a_i F + e_i,$$

Where X_i is the standardized grade for course i, F is a common factor with mean 0 and variance 1. e_i is a special factor and is independent from F. That is, each course's exam grade can be viewed as consisting of the sum of a general factor and a special factor. Moreover,

$$cov(X_i, X_j) = E\left[(a_iF + e_i), (a_jF + e_j)\right] = a_ia_jD(F) = a_ia_j,$$

Thus it is possible to obtain

$$\frac{cov(X_i, X_j)}{cov(X_i, X_k)} = \frac{a_j}{a_k}.$$

This equation is independent of *i* and has the same proportions as described previously.

In addition to this, we get the following relational equation:

$$\begin{aligned} Var(X_i) &= Var(a_iF + e_i) = Var(a_iF) + Var(e_i) \\ &= a_i^2 Var(F) + Var(e_i) = a_i^2 + Var(e_i). \end{aligned}$$

Since a_i is constant, F and e_i are independent of each other, and $Var(F) = 1, Var(X_i) = 1$. So

$$1 = a_i^2 + Var(e_i),$$

Where a_i^2 denotes the proportion of $Var(X_i)$ explained by factor F, call a_i the factor loading, and call a_i^2 the communality.

The following is a general model. If there are *n* sample and *p* variables, then we have

(1) There is a p-dimensional observable random vector $X = (x_1, x_2, \cdots, x_p)^T$, which has expectation $E(X) = \mu = (\mu_1, \mu_2, \cdots, \mu_p)^T$, and covariance matrix

$$cov(X) = \Sigma = \begin{bmatrix} \sigma_{11}^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{pp}^2 \end{bmatrix}.$$

(2) $F = (f_1, f_2, \cdots, f_m)^T (m < p)$ is an unobservable variable, and $E(F) = 0$, $cov(F) = I$.

(3) $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots \varepsilon_p)^T$ and F are independent of each other, and $E(\varepsilon) = 0$, ε has a diagonal covariance matrix Σ_ε

$$cov(\varepsilon) = \Sigma_\varepsilon = \begin{bmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p^2 \end{bmatrix}.$$

That is, the components of ε are independent of each other.

Based on the above assumptions, the model

$$\begin{cases} x_1 = \mu_1 + a_{11}f_1 + a_{12}f_2 + \cdots + a_{1m}f_m + \varepsilon_1, \\ x_2 = \mu_2 + a_{21}f_1 + a_{22}f_2 + \cdots + a_{2m}f_m + \varepsilon_2, \\ \vdots \\ x_p = \mu_p + a_{p1}f_1 + a_{p2}f_2 + \cdots + a_{pm}f_m + \varepsilon_p. \end{cases}$$

is called the orthogonal factor model.

The matrix form of the model is

$$X = \mu + AF + \varepsilon,$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pm} \end{bmatrix}, F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}, \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_p \end{bmatrix}.$$

In this model, the following conditions are present.

(i) $m \leq p$.

(ii) $cov(F, \varepsilon) = 0$. That is, F and ε are irrelevant.

(iii) $Var(F) = I_m$. That is, $f_1, f_2, \cdots, f_m$ are uncorrelated, and all have variance 1.

(iv) $Var(\varepsilon) = \begin{bmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p^2 \end{bmatrix}$ is a diagonal matrix. That is, ε_1, ε_2, $\cdots \varepsilon_p$ are uncorrelated and have different variances.

Where $f_1, f_2, \cdots, f_m$ are common factors, they are independent of each other, and in general, they cannot be observed. ε_1,

ε_2, $\cdots\varepsilon_p$ are special factors which act only on the corresponding x_i, each x_i corresponds to an ε_i. And ε is independent of ε and F. $A=(a_{ij})$ is the factor loading matrix, a_{ij} is the factor loading, which means the loading of the *i*th variable on the *j*th factor and represents the degree of influence of the variable x_i on the common factor f_j. The dependence of $|a_{ij}|$ and x_i on f_j is positively correlated.

2.2. Significance of Statistics

2.2.1. Factor loading a_{ij}

For the factor model

$$x_i=\mu_i+a_{i1}f_1+a_{i2}f_2+\cdots+a_{im}f_m+\varepsilon_i, i=1,2,\cdots,p.$$

It can be obtained that

$$cov(X,F)=E[(X-\mu)F^T]=E[(AF+\varepsilon)F^T]=E(AFF^T+\varepsilon F^T)$$
$$=AE(FF^T)+E(\varepsilon F^T)=A.$$

That is, the loading matrix A is the covariance matrix of X and F.

So

$$cov(x_i,f_j)=E\left[\left(\sum_{k=1}^{m}a_{ik}f_k+\mu_i+\varepsilon_i-\mu_i\right)f_j\right]=E\left[\left(\sum_{k=1}^{m}a_{ik}f_k+\varepsilon_i\right)f_j\right]$$
$$=E\left[\left(\sum_{k=1}^{m}a_{ik}f_k\right)f_j\right]+E(\varepsilon_i f_j)=a_{ij}.$$

Thus a_{ij} is the loading of the *i*th variable on the *j*th factor. Since $A=(a_{ij})$, it is the covariance matrix of X and F.

If the standardized variable x_i, i.e. x_i has a mean of 0 and a variance of 1, and the variance of f_j is also 1, then

$$r_{x_i,f_j}=\frac{cov(x_i,f_j)}{\sqrt{Var(x_i)Var(f_j)}}=cov(x_i,f_j)=a_{ij}.$$

Therefore, for the standardized variables, a_{ij} is the correlation coefficient between x_i and f_j. From the above analysis, it is clear that the dependence of x_i on f_j can be determined by the magnitude of the value of $|a_{ij}|$. In other words, we can also use a_{ij} to reflect the role of x_i for f_j.

2.2.2. Variable Communality h_i^2

The covariance matrix of X can be obtained from the above orthogonal factor model

$$\Sigma=\mathrm{cov}(X)=E[(X-\mu)(X-\mu)^T]=AE(FF^T)A^T+E(\varepsilon F^T)A^T+AE(F\varepsilon^T)+E(\varepsilon\varepsilon^T)=AA^T+\Sigma_\varepsilon.$$

From the above equation, the variance of x_i is

$$Var(x_i)=a_{i1}^2Var(f_1)+a_{i2}^2Var(f_2)+\cdots+a_{im}^2Var(f_m)+Var(\varepsilon_i)$$
$$=a_{i1}^2+a_{i2}^2+\cdots+a_{im}^2+\sigma_i^2.$$

This formula states that the variance of x_i is obtained from the sum of the variance of *m* factors and ε_i. $a_{ij}^2(j=1,2,\cdots,m)$ denotes the contribution of the *j*th factor to the variance of x_i. σ_i^2 is the contribution of the *i*th special factor to the variance of x_i, called uniqueness. The uniqueness of variable x_i can be expressed as $h_i^2=\sum_{j=1}^{m}a_{ij}^2$, $i=1,2,\cdots,p$.

Since x_i has been standardized, the variance is 1, and thus there is

$$h_i^2+\sigma_i^2=1.$$

This also accounts for the variance structure of x_i. The communality h_i^2 is the sum of the variances of the *m* common factors, which reflects the total variance contribution made by the *m* common factors to each x_i. It is an important statistic in practice. The closer the value of h_i^2 is to 1, the more the variance of the variable x_i is represented by the common factor, i.e., the common factor basically represents most of the information in the data. In practice, we would like to have a larger value of h_i^2 so that the analysis can be carried out. When the value of h_i^2 is small, the common factor represents less information, which is not conducive to factor analysis.

2.2.3. Variance Contribution of the Common Factor

The above communality reflects the effect of all factors on one variable. Below, we introduce the effect of a common factor on all of the variables.

We call g_j^2 the contribution of factor f_j to the variance of p variables, i.e.,

$$g_j^2=\sum_{i=1}^{p} a_{ij}^2, j=1,2,\cdots,m.$$

It denotes the sum of the variance contributions of the jth factor to x_i and is called the variance contribution of the jth factor. The larger g_j^2 is, the more information about the variable represented by factor f_j, indicating that the factor has a large role to play, or it can be shown that it contributes a large amount to the total variable. Using g_j^2, we can judge the importance of each common factor.

2.3. Solution of the Factor Loading Matrix

For the above factor model, it is very important to estimate A and the special variance $\sigma_i(i=1,2,\cdots,p)$. Commonly used estimation methods include principal component method, principal factor method, maximum likelihood method and so on. This paper introduces the principal component method.

The principal component method of determining factor loading involves performing a principal component analysis on the data before performing the factor analysis and then using the first few principal components as common factors. [10] Here is a brief introduction of the method.

Let the eigenvalues of the covariance matrix Σ of X be $\lambda_1,\lambda_2,\cdots,\lambda_p\left(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_p\geq 0\right)$, $e_1,e_2,\cdots,e_p$ are the corresponding normalized eigenvectors, then Σ can be written as:

$$\Sigma=\lambda_1 e_1 e_1^T+\lambda_2 e_2 e_2^T+\cdots+\lambda_p e_p e_p^T=\left(\sqrt{\lambda_1}e_1,\sqrt{\lambda_2}e_2,\cdots\sqrt{\lambda_p}e_p\right)\begin{pmatrix}\sqrt{\lambda_1}e_1^T\\ \sqrt{\lambda_2}e_2^T\\ \vdots\\ \sqrt{\lambda_p}e_p^T\end{pmatrix}.$$

The above equation is a special case that shows that only the common factor works in A, while the special factor does not work, and its variance is 0, i.e.,

$$\Sigma=\mathrm{A}A^T+0=\mathrm{A}A^T.$$

Note that we have given an expression for Σ in the above equation and that the Σ obtained using the above equation is accurate. However, it can only be used in theory; in practice, it is meaningless. The above method is the conclusion obtained by applying all the eigenvalues of Σ. If the last $p-m$ eigenvalues are small, the contribution of $\lambda_{m+1}e_{m+1}e_{m+1}^T+\cdots+\lambda_p e_p e_p^T$ to Σ is omitted, and we get

$$\Sigma\approx\lambda_1 e_1 e_1^T+\lambda_2 e_2 e_2^T+\cdots+\lambda_m e_m e_m^T=\mathrm{A}A^T.$$

Here we have assumed that ε in $X=\mu+AF+\varepsilon$ can be omitted in the decomposition of Σ. If not, the variance of ε can be taken as the diagonal element of $\Sigma-\mathrm{A}A^T$, so

$$\Sigma\approx\mathrm{A}A^T+diag(\Sigma-\mathrm{A}A^T)=\mathrm{A}A^T+\Sigma_\varepsilon,$$

where

$$\Sigma_\varepsilon=\begin{bmatrix}\sigma_1^2&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&\sigma_p^2\end{bmatrix},\sigma_i^2=\sigma_{ii}^2-\sum_{j=1}^{m}a_{ij}^2\,(i=1,2,\cdots,p).$$

In fact Σ is unknown, and thus the covariance matrix S is usually used in practical calculations. In actuality, the individual variables are standardized so that the sample correlation coefficient matrix R of the standardized data is equal to the sample covariance matrix S, so that the principal components are not affected by dimension or order of magnitude. [11] Thus we can complete the solution with R.

If the eigenvalues of R are $\hat{\lambda}_1,\hat{\lambda}_2,\cdots,\hat{\lambda}_p\left(\hat{\lambda}_1\geq\hat{\lambda}_2\geq\hat{\lambda}_p\geq 0\right)$, $\hat{e}_1,\hat{e}_2,\cdots,\hat{e}_p$ are the corresponding orthonormal eigenvectors, an estimate of A is obtained

$$\hat{A} = \left(\sqrt{\hat{\lambda}_1}\hat{e}_1, \sqrt{\hat{\lambda}_2}\hat{e}_2, \cdots, \sqrt{\hat{\lambda}_m}\hat{e}_m\right) = (\hat{a}_{ij}).$$

The variance σ_i^2 of the special factor is estimated as $\hat{\sigma}_i^2 = 1 - \sum_{j=1}^{m} \hat{a}_{ij}^2$. And the communality h_i^2 is estimated as $\hat{h}_i^2 = \sum_{j=1}^{m} \hat{a}_{ij}^2$, $i = 1,2,\cdots,p$.

2.4. Factor Rotation

It is easy to know that the factor model satisfying the variance structure $\Sigma = AA^T + \Sigma_\varepsilon$ is not unique, nor are the common factor and loading matrix of the model unique. If F is a common factor of the model, A is the corresponding loading matrix, and T is an m-order orthogonal matrix, then $F^* = T^T F$ is also a common factor, and the corresponding loading matrix is $A^* = AT$. A^* also satisfies $\Sigma = AA^T + \Sigma_\varepsilon$, which suggests that orthogonal transformations of the common factor and loading matrix do not change the communality. We denote the orthogonal transformation of factor loading as factor orthogonal rotation.

We should not only find out the factors but also know what they mean. If it is difficult to see the meaning of the factors obtained by the model, it is necessary to implement factor rotation to obtain more appropriate factors. The meaning of the factor can only be explained if its meaning is clear.

The purpose of factor rotation is to simplify the structure of the factor loading matrix so that each variable has a large loading on only one common factor. [12] For A, there should be very few loading values in the columns that are very large; the rest should be small, and the location of the large loading values in the columns should be different in each column. That is, the loading on each common factor should be polarized.

2.5. Factor Score

2.5.1. Factor Score

For the model $X = \mu + AF + \varepsilon$, if we ignore the effect from ε, when A is invertible, we can derive F = AX from the value of $F = A^{-1}X$. In this way, we can calculate the factor score.

However, this method cannot be used in practice; it assumes that the number of common factors is *p*, but in reality, the number of common factors is small, just a few, and far from *p*. For this reason, we can't exactly calculate the score of $f_j (j = 1,2,\cdots m)$, but we can estimate it. Methods for calculating factor scores include weighted least squares, regression methods, etc. This paper describes the regression method.

We assume that the common factor can be obtained from the variables by regression, i.e.,

$$\hat{F}_j = b_{j0} + b_{j1}x_1 + \cdots + b_{jp}x_p, j = 1,2,\cdots,m.$$

If each of the above quantities has been normalized, then $b_{j0} = 0$.

From equation $r_{x_i,f_j} = cov(x_i, f_j) = a_{ij}$, it follows that for any $i = 1,2,\cdots,p, j = 1,2,\cdots,m$, there are $a_{ij} = r_{x_i,f_j} = E(x_i, f_j) = E[x_i(b_{j1}x_1 + \cdots + b_{jp}x_p)] = b_{j1}E(x_i x_1) + \cdots + b_{jp}E(x_i x_p) = b_{j1}r_{i1} + \cdots + b_{jp}r_{ip}$.

Denote

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix},$$

Then, the above formula can be written as the matrix form $A = RB^T$ or $B = A^T R^{-1}$. So $\hat{F} = BX = A^T R^{-1} X$, where R is the correlation coefficient matrix of X.

2.5.2. Composite Factor Score

The comprehensive score is the total score obtained by combining the scores of each common factor. It is essentially a weighted average, and the weight is the contribution rate of each factor. This method is widely used at present. Specifically, we can calculate the composite factor score using the following equation.

$$f = \frac{\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m}{\lambda_1 + \lambda_2 + \cdots \lambda_m} = \sum_{i=1}^{m} \omega_i f_i,$$

Where $f_1, f_2, \cdots f_m$ are the scores of each common factor, ω_i is the contribution rate of each factor.

2.6. Correlation Test for Variables

From the previous section, we can see that $x_1, x_2, \cdots, x_p$ are correlated with each other, which is the prerequisite for us to do the process of factor analysis. Therefore, before that, we first need to make sure that the variables are correlated with each other. Intuitively, if the correlation coefficient between each variable is mostly small, it indicates that the correlation between the variables is weak. At this time, the accuracy and credibility of the factor analysis are very low, and its effect cannot be guaranteed. Therefore, before doing the factor analysis, we should ensure that the problem is suitable for the analysis. This correlation test is essential to determine whether factor analysis can be done. Common methods include Bartlett's test of sphericity, [13] KMO test [14] and so on.

2.6.1. Bartlett's Test of Sphericity

Bartlett's test of sphericity takes the correlation coefficient matrix of the original variables as the starting point and its null hypothesis. $H_0: R$ is the identity matrix. The test statistics are calculated using the following approximate formula:

$$\chi^2 = \frac{(11 + 2p - 6n)}{6} \ln |R|,$$

where p is the number of variables, n is the number of samples, $|R|$ is the determinant of R. and $\chi^2 \sim \chi^2\big(p(1-p)\big)$, it is a chi-square distribution with $p(1-p)$ degrees of freedom.

If the value of the statistic is so large that the significance probability is less than α, the null hypothesis is rejected, and it is considered suitable for factor analysis. The reverse is not suitable.

2.6.2. KMO Test

The value of the KMO statistics can be calculated as follows:

$$KMO = \frac{\sum\sum_{i \neq j} r_{ij}^2}{\sum\sum_{i \neq j} r_{ij}^2 + \sum\sum_{i \neq j} p_{ij}^2},$$

where r_{ij} is the simple correlation coefficient of the variables and p_{ij} is the partial correlation coefficient.

The KMO statistic takes values between 0 and 1. When the sum of the squares of the simple correlation coefficients between all variables is much greater than the sum of the squares of the partial correlation coefficients, the KMO value is close to 1. The closer the KMO value is to 1, the stronger the correlation between the variables, and the more suitable the original variables are for factor analysis. When the sum of the squares of the simple correlation coefficients between all the variables is close to 0, the KMO value is close to 0. The closer the KMO value is to 0, the weaker the correlation between the variables, and the less suitable the original variables are for factor analysis.

3. Factor analysis of student achievement

3.1. Data Description

This chapter is the actual process of factor analysis, using R software to complete the factor analysis of 54 students' grades, thus completing a comprehensive analysis of the student's grades. In this paper, the grades of 54 students in this major were selected, and some rounding was done to select a representative number of 9 courses, i.e., the grades of nine courses of 54 students. These include Ordinary Differential Equations (x_1), Physics (x_2), Probability Theory (x_3), Advanced Algebra (x_4), Mathematical Analysis (x_5), Physical Education (x_6), Basic Principles of Marxism (x_7), The Introduction to Mao Zedong Thought and the Theoretical System of Socialism with Chinese Characteristics (x_8), and College English (x_9). The following factor analyses were conducted.

3.2. Correlation Test

According to the previous paragraph, we first need to ensure that there is a correlation between the variables, and we need to use the above two methods to test. The test results are shown in the following table:

Table 1. Bartlett's test of sphericity and KMO test

Kaiser-Meyer-Olkin Measure of Sampling Adequacy		0.9
Bartlett's test of sphericity	Approx. Chi-Square	33.217
	df	8
	Sig.	0.000

According to the table, the value of KMO is 0.9, and the p-value of Bartlett's test of sphericity test is less than the significance level of 0.05, so we believe that factor analysis can be performed.

3.3. Solution of the Factor Analysis Model

3.3.1. Selection of the Number of Factors

First, we give the correlation coefficient matrix of the data

Table 2. Correlation coefficient matrix

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1	1.00								
x_2	0.80	1.00							
x_3	0.65	0.73	1.00						
x_4	0.62	0.69	0.60	1.00					
x_5	0.86	0.79	0.68	0.63	1.00				
x_6	0.11	0.24	0.24	0.12	0.07	1.00			
x_7	0.57	0.62	0.40	0.52	0.60	0.06	1.00		
x_8	0.71	0.67	0.46	0.64	0.71	-0.06	0.61	1.00	
x_9	0.56	0.69	0.53	0.61	0.64	0.100	0.64	0.64	1.00

The table also shows that the correlation between the variables is strong enough to do a factor analysis.

We also need to determine the number of factors before the actual analysis. The eigenvalues, proportion variance and cumulative proportion of the correlation coefficient matrix are first given in Table 3. Secondly a scree plot of this data can be drawn in the R software based on the eigenvalues. The results are shown in the Figure 1.

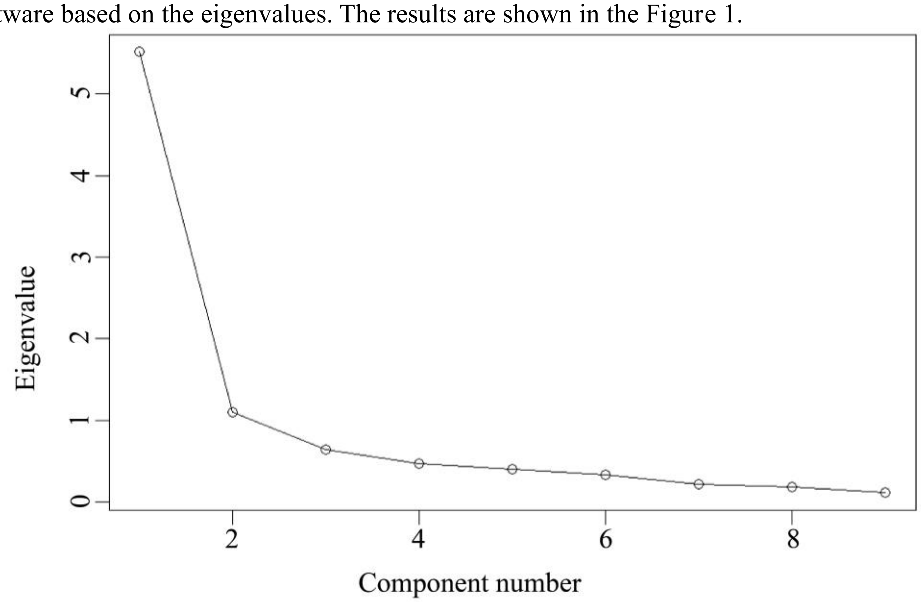

Fig.1 Common factor scree plot

From the figure, it is known that the first two eigenvalues are larger and the rest of the eigenvalues are smaller. This suggests that the first two factors contribute significantly and summarise most of the information from the data. The figure shows that there is an turning point at the second factor, after which the figure is basically smooth and no longer has any major fluctuations. Therefore we use 2 factors.

Table 3. Total variance explained

Factors	Eigenvalues	Proportion variance	Cumulative proportion
F1	5.51	0.61	0.61
F2	1.10	0.12	0.74
F3	0.65	0.07	0.81
F4	0.47	0.05	0.86

F5	0.41	0.05	0.90
F6	0.34	0.04	0.94
F7	0.22	0.02	0.97
F8	0.18	0.02	0.99
F9	0.12	0.01	1.00

It can also be seen from the table that the cumulative contribution rate of the first two factors is 74%, indicating that they can summarize most of the information. This further indicates that the selection of the number of factors is appropriate.

3.3.2. Principal Component Method Factor Analysis

Based on the above preparation, the following factor analysis was carried out by applying the principal component method to obtain the factor loading matrix, and the results are as follows:

Table 4. Factor loading matrix

	Common factor F1	Common factor F2	Communaliy	Uniqueness
x_1	0.88	-0.01	0.77	0.23
x_2	0.91	0.14	0.85	0.15
x_3	0.77	0.28	0.67	0.33
x_4	0.80	0.01	0.64	0.36
x_5	0.90	-0.06	0.81	1.19
x_6	0.16	0.93	0.90	0.10
x_7	0.74	-0.17	0.58	0.42
x_8	0.82	-0.31	0.76	0.24
x_9	0.80	-0.08	0.64	0.36

In the table, second and third columns are the loading of the variables on the common factors F1 and F2, fourth column is the communaliy of the variables, and the last column is the uniqueness. The larger communaliy is, the more information is demonstrated that can be represented by the factor. Uniqueness is the contribution of a special factor to the variance of a variable. As can be seen from the table, the values of the communaliy are all large, while the values of the uniqueness are generally small, indicating that the common factor represents more information and basically reflects most of the information of the original variables.

3.4. Factor Rotation

To better interpret the meaning of the factors, we perform factor rotation. The method used in this paper is maximal rotation of variance, the results are as follows. Table 5 shows the rotated common factor contribution and Table 6 shows the rotated factor loading matrix.

Table 5. Rotated common factor contribution

Factors	Variance	Proportion variance	Cumulative proportion
F1	5.41	0.60	0.60
F2	1.20	0.13	0.74

As can be seen from Table 6, the first common factor has a large loading in subjects except physical education (x_6), which indicates that the above subjects are the most important aspects to be considered in evaluating students' achievements. After a detailed analysis of these indicators, we find that they are some courses that need to be learned, representing the students' academic performance. They reflect the students' learning ability. Therefore, the first common factor can be regarded as the "intellectual learning factor". The second common factor has a large loading on physical education (x_6), which shows that physical education also occupies a certain importance in students' achievements and is a part of students' achievements that cannot be ignored. The second factor reflects the physical condition of students, from which the physical quality of students can be seen, so the second common factor can be regarded as "physical quality factor".

Table 6. Rotated factor loading matrix

	Common factor F1	Common factor F2	Communaliy	Uniqueness
x_1	0.87	0.12	0.77	0.23
x_2	0.88	0.27	0.85	0.15
x_3	0.72	0.40	0.67	0.33
x_4	0.79	0.13	0.64	0.36

x_5	0.90	0.08	0.81	1.19
x_6	0.02	0.95	0.90	0.10
x_7	0.76	-0.06	0.58	0.42
x_8	0.85	-0.18	0.76	0.24
x_9	0.80	0.05	0.64	0.36

In order to better see the quantitative relationship between the variables on the common factor loading and to better identify the factor structure, we draw a plot of the factor loading after factor rotation.

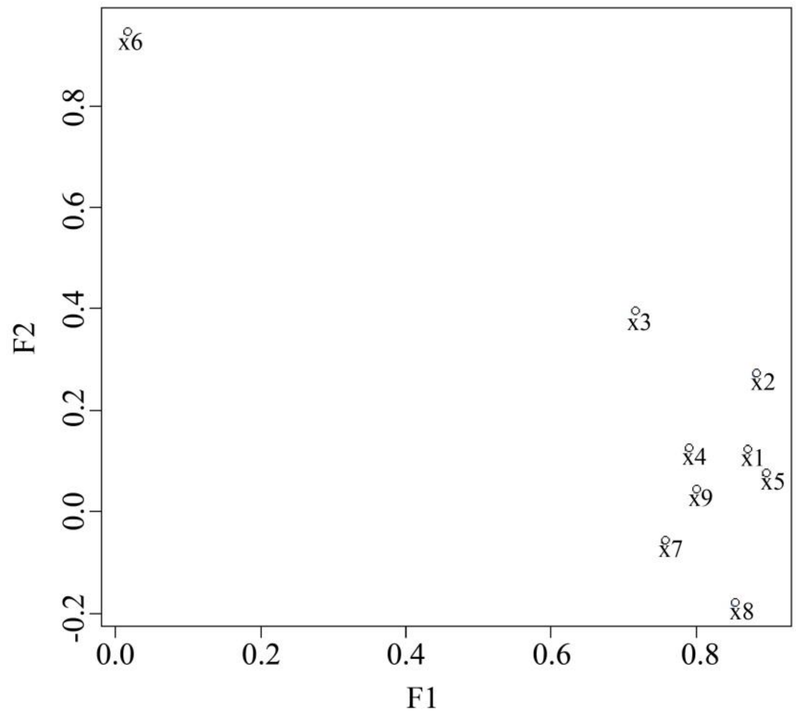

Fig. 2 Factor loading plot

It is clear from the figure that the variables x_1, x_2, x_3, x_4, x_5, x_7, x_8, and x_9 are on the lower right side of the figure, closer to the horizontal axis F1, and close to the right side of the horizontal axis, which indicates that these variables have a larger loading on F1. While variable x_6 is on the upper left side of the graph, it is closer to the vertical axis F2 and close to the upper side of the vertical axis, which indicates that the variable has a larger loading on F2. According to the loading diagram, the factor structure can be found out more easily and accurately. The specific factor structure is shown in the table below:

Table 7. Factor structure table

Factors	Course name	Latent variable
F1	Ordinary Differential Equations (x_1), Physics (x_2), Probability Theory (x_3), Advanced Algebra (x_4), Mathematical Analysis (x_5), Basic Principles of Marxism (x_7), The Introduction to Mao Zedong Thought and the Theoretical System of Socialism with Chinese Characteristics (x_8), College English (x_9).	learning ability
F2	Physical Education (x_6)	physical quality

From the factor structure table we can clearly see the main variables represented by each common factor and the specific meaning it represents, which have been described earlier and will not be recounted here.

3.5. Factor Score

3.5.1. Factor Score Coefficient

Before calculating the factor score, we first get the factor score coefficient, through which we calculate the factor score. The following table is the factor score coefficient table.

Table 8. Factor score coefficient

	F1	F2
x_1	0.159	0.016
x_2	0.145	0.147
x_3	0.099	0.276
x_4	0.142	0.027
x_5	0.169	-0.029
x_6	-0.099	0.841
x_7	0.156	-0.132
x_8	0.188	-0.251
x_9	0.153	-0.046

The factor score coefficients give the linear relationship between the two common factors with respect to the variables, and the table allows us to know how to calculate the factor scores, as shown in the following equation:
F1$=0.159x_1^* +0.145x_2^* +0.099x_3^* +0.142x_4^* +0.169x_5^* -0.099x_6^* +0.156x_7^* +0.188x_8^* +0.153x_9^*$,
F2$=0.016x_1^* +0.147x_2^* +0.276x_3^* +0.027x_4^* -0.029x_5^* +0.841x_6^* -0.132x_7^* -0.251x_8^* -0.046x_9^*$.
Where x_1^*, x_2^*......x_9^* is normalized to the original data x_1,x_2,......x_9, which are normalized variables.

3.5.2. Factor Score

Based on the above two equations, we can find the factor scores FS1,FS2 for factors F1,F2. The factor score table is shown in Table 9. In order to observe the factor score more clearly and explicitly, we drew the factor score chart with the score of common factor F1 as the X-axis and the factor score of F2 as the Y-axis, as shown in Figure 3.

In addition, we calculated the comprehensive score F. At the same time, we can use the credit performance point average (GPA) method (Σgrade per subject*credits for that subject / total credits) to get the comprehensive score of each student and rank them, and the two comparisons are more reflective of the results. Specific results are shown in Table 9.

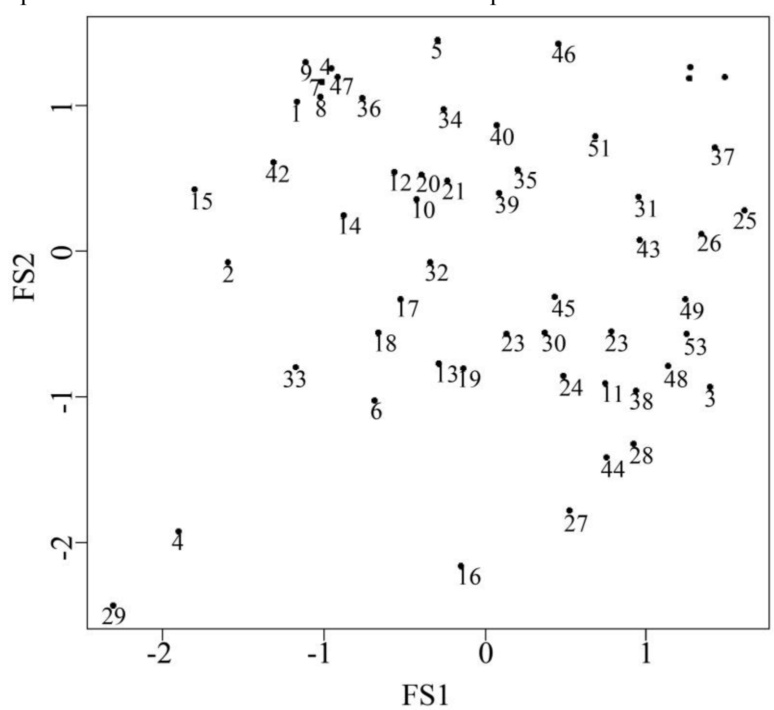

Fig. 3 Factor score plot

Table 9. Factor score and comprehensive score

Students	FS1	Ranking	FS2	Ranking	F	Comprehensive ranking	GPA ranking
1	-1.170	7	1.026	43	-0.768	7	11

2	-1.598	4	-0.076	25	-1.309	4	4
3	1.392	51	-0.935	9	0.964	48	46
4	-1.902	2	-1.925	3	-1.880	2	2
5	-0.300	22	1.452	54	0.013	28	28
6	-0.684	15	-1.030	7	-0.736	8	7
7	-1.015	10	1.164	46	-0.619	13	8
8	-1.027	9	1.059	45	-0.646	11	10
9	-1.114	8	1.291	52	-0.676	9	12
10	-0.429	19	0.353	30	-0.286	22	22
11	0.746	38	-0.904	10	0.446	37	38
12	-0.564	17	0.537	36	-0.363	20	19
13	-0.285	23	-0.771	15	-0.367	19	18
14	-0.876	13	0.244	28	-0.668	10	15
15	-1.801	3	0.421	33	-1.386	3	3
16	-0.147	26	-2.165	2	-0.499	16	20
17	-0.527	18	-0.331	22	-0.485	17	16
18	-0.662	16	-0.563	18	-0.635	12	9
19	-0.138	27	-0.806	12	-0.254	23	25
20	-0.394	20	0.527	35	-0.227	24	24
21	-0.238	25	0.484	34	-0.108	25	23
22	0.781	40	-0.549	20	0.537	39	39
23	0.130	30	-0.571	17	0.005	27	27
24	0.481	35	-0.861	11	0.240	33	34
25	1.607	54	0.278	29	1.352	53	53
26	1.340	50	0.118	27	1.107	49	49
27	0.525	36	-1.784	4	0.112	29	30
28	0.922	41	-1.326	6	0.515	38	37
29	-2.311	1	-2.435	1	-2.302	1	1
30	0.367	32	-0.560	19	0.199	31	32
31	0.952	43	0.374	31	0.837	45	45
32	-0.341	21	-0.078	24	-0.292	21	21
33	-1.174	6	-0.799	13	-1.092	5	6
34	-0.259	24	0.970	42	-0.040	26	26
35	0.203	31	0.554	37	0.262	34	33
36	-0.762	14	1.050	44	-0.433	18	17
37	1.423	52	0.711	39	1.278	52	51
38	0.934	42	-0.958	8	0.589	40	40
39	0.090	29	0.397	32	0.142	30	29
40	0.071	28	0.860	41	0.209	32	31
41	-0.954	11	1.250	50	-0.554	14	13
42	-1.318	5	0.607	38	-0.962	6	5
43	0.962	44	0.074	26	0.793	44	44
44	0.750	39	-1.418	5	0.359	36	36
45	0.431	33	-0.314	23	0.294	35	35
46	0.455	34	1.421	53	0.618	41	41
47	-0.917	12	1.197	49	-0.533	15	14
48	1.138	45	-0.787	14	0.785	43	43
49	1.245	46	-0.336	21	0.950	47	48
50	1.276	49	1.262	51	1.256	51	52
51	0.681	37	0.788	40	0.691	42	42
52	1.268	48	1.189	47	1.237	50	50
53	1.246	47	-0.572	16	0.910	46	47
54	1.491	53	1.194	48	1.418	54	54

A higher score on the "intellectual learning factor" F1 corresponds to better academic performance, indicating that the student is more capable of learning, while a higher score on the 'Physical Fitness Factor' F2 corresponds to higher performance in physical education, indicating that the student is more physically fit. The higher the score of "physical quality factor" F2, the higher the corresponding student's performance in physical education, indicating the better the student's physical quality. The factor score chart in Figure 3 shows the position of the factor scores of the common factors F1 and F2 on the axis. In this figure, FS1 is the X-axis and FS2 is the Y-axis. The closer the FS1 axis is to the right, the higher the score of F1 is, the better the learning ability is; otherwise, the worse the learning ability is. And the closer to the upper side of the FS2 axis, it means that the larger the F2 score, the better the physical fitness and vice versa. Through the factor score plot, we can clearly see the score situation and the approximate size and ranking of the two factors, which is more clear and vivid, and can help us to draw a conclusion. Above we do a general description of the output results, the specific analysis of the results in the following section.

3.6. Analysis of the Results

Through the pair factor score table and the factor score plot, we can get:

(1) Student 54 has the second largest common factor F1 score FS1, ranked 53, factor F2 score FS2 is also larger, ranked 48, and the final factor comprehensive score is the largest, ranked 54, which indicates that the student's performance in each course is very good, and usually pay attention to the exercise of the body, the ability to study and physical fitness in the students are at a high level, the ability to study is strong, the physical fitness is also very good, and there is no significant shortcomings, the comprehensive assessment is located in the first.

(2) The common factor F1 score FS1 of student 25 is the largest and ranked 54, the factor F2 score FS2 is ranked 29, while the factor comprehensive score is ranked 53, which indicates that the student has a strong learning ability but the level of physical fitness is average, and is located in the second place in the comprehensive rating. At the same time, according to the factor score, the student's FS1 is the largest, while FS2 ranked medium, for the student, the learning ability has been very strong, but the physical fitness is not too good compared to the learning ability, and is located in the middle among the students, and there is still room for improvement. We can suggest that the student can do more physical exercise in his/her time, which will help to improve the overall rating. The student can refer to Student 54 to achieve strong academic ability and good physical quality.

(3) The common factor F1 score FS1 of student 9 ranked 8, the factor F2 score FS2 ranked 52, and the factor composite score ranked 9, indicating that the development of the student's learning ability and physical fitness is very uneven, with poor academic performance and the need to improve learning ability, but the student is physically fit and is more adept in sports. It is recommended that the student needs to focus on the curriculum in the future to improve his academic performance.

(4) Student 29 has the smallest common factor F1 score FS1 and factor F2 score FS2, and a factor composite score ranking of 1. This indicates that the student is poor in both academics and physical education, and that his academic and physical abilities are low and need to be improved. This indicates that the student has problems in both areas and is at a low level, and we need to pay special attention to it. The student is advised to work hard in both areas in the future and both need to be improved.

For the remaining students, we will not describe them. We know from the above analyses that factor analyses yield results that not only provide a comprehensive ranking of students' performance, but also a comprehensive evaluation of performance. From the factor scores we can see the ranking of the common factor scores, which enables us to analyze the results in a comprehensive manner, to identify the characteristics and deficiencies of each student, and also to see whether the students' development is comprehensive and balanced. As the contribution rate of common factor F1 reaches 60%, it shows that in the comprehensive assessment, academic performance occupies a large proportion, students should still focus on study, but should not ignore other factors, they also play a certain role, and occupy a place in the comprehensive assessment. Usually should not forget to exercise the body, improve physical fitness, for other aspects of the same.

Finally, in order to compare the rationality of factor analysis, we compare it with the results obtained by the commonly used GPA method, and draw the following conclusions.

Comparing the rankings obtained by the comprehensive score and GPA method, we find that the rankings obtained by the two methods are basically the same by and large, without much difference, and there is only a slight discrepancy in the subtle rankings. From the point of view of the students' rankings, the results obtained by the two methods are consistent, which shows that we can use factor analysis to complete the comprehensive assessment of students' performance, and this method has a certain degree of rationality.

For these two methods, the GPA method is based on the weighted average of the importance of each subject, although intuitive and easy to understand, these methods can not reflect the characteristics of each aspect of the student's ability, masking the student's personality, and can not make a comprehensive and comprehensive evaluation of the student, reflecting the student's professional competence, and can only get the student's comprehensive results and rankings, which has obvious limitations.

For the factor analysis method, it not only provides a comprehensive assessment of the results, but also discovers the characteristics of the students and finds their personalities. According to the ranking of factor scores, it can show the differences in different abilities of students, discover the characteristics and deficiencies of students, and provide a comprehensive analysis and assessment of students.

In conclusion, the GPA method is intuitive and reasonable, easy to understand, but it can only get the final ranking and cannot carry out other analyses. The process of factor analysis method is more troublesome and it is necessary to verify its reasonableness, but it gets more information, and it can analyze and evaluate students comprehensively, find out their characteristics and deficiencies, and analyze more comprehensively.

4. Conclusion

This paper mainly describes the theoretical knowledge of the factor analysis method and the process of using the factor analysis method, with the help of R software, to factor analyze the grades of 54 students in 9 courses, and explains the results of the output of the factor analysis, according to the rankings of the scores of the two common factors, to see the differences of the students in different abilities, to find out the characteristics and deficiencies of the students, and to make a comprehensive analysis and assessment of the students. Finally, it is compared with the GPA method commonly used nowadays to verify the reasonableness of the factor analysis method and to get the advantages and shortcomings of these two methods by comparison.

The commonly used GPA method simply assumes that there is only a difference in the weights of the grades of each subject, ignoring the differences in the courses, which is a limitation at this point. Factor analysis solves the shortcomings of the above methods, and through the analysis of the data, it can assess the students‘ grades more accurately and comprehensively, and complete the comprehensive assessment of the students' grades. From the results, the two methods are basically the same, but from factor analysis, we can get more information, discover students‘ strengths and weaknesses, and check whether students' development is balanced and comprehensive. Therefore, the factor analysis method not only solves the shortcomings of the GPA method, but also gets more information, which can be used to make a comprehensive analysis and assessment of students' performance.

Of course, factor analysis has its shortcomings; there is a certain degree of subjectivity in the analysis, and everyone may get different results. It should be used reasonably in response to the circumstances of the problem. In addition, inspired by the above, factor analysis can be used not only for the assessment of grades, but we can also help students to make some choices based on it. For example, in the matter of students' choice of arts and science subjects, we can make favourable choices based on their grades. Factor analysis of the data, with reference to the method of this paper, we can get the factor scores from it, according to which we can analyze, we can know what the students prefer, and provide some help to the students' choices. For factor analysis, we can conduct research to discover more applications.

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