International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V71I2P104

Volume 71 Issue 2, 30-36, February 2025

Original Article

Generator Implementations of Pythagorean Power Uncertainty Commutative Group Structures

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Received: 22 December 2024 Revised: 27 January 2025 Accepted: 13 February 2025 Published: 28 February 2025

Abstract - *In this paper, we study the concept of Pythagorean uncertainty set (PUS) to introduce the concepts of Pythagorean power uncertainty abelian subgroups (PPUAS), Pythagorean power uncertainty normal subgroups (PPUNS) and its properties. Also, we study those concepts in terms of the Cartesian product of Pythagorean power uncertainty sets. Finally, homomorphic images and pre-images of Pythagorean uncertainty sets are established.*

Index Terms *- Uncertainty set, Pythagorean uncertainty set, Power subgroup, Pythagorean power uncertainty set, Commutative group, Homomorphism, Pre-image, Support, Normal subgroup.*

1. Introduction

Uncertainty Set Theory (UST) is the concept and techniques that apply a form of mathematical precision to human thought processes that, in many ways, are imprecise and ambiguous by the standards of classical mathematics, uncertainty set, intuitionistic uncertainty sets, interval-valued sets, bipolar uncertainty sets and other mathematical tools are often useful approaches to deal with uncertainties. In 1965, Zadeh [21] introduced the notion of uncertainty sets. At present, this concept has been applied to many mathematical branches. In uncertainty set theory, several types of extensions exist, such as intuitionistic interval-valued uncertainty set, vague set, etc. Jun et al. established an extension of the bipolar valued uncertainty set, which Yager introduces, launched a non-standard uncertainty set referred to as the Pythagorean uncertainty set, which is the generalization of intuitionistic uncertainty set, the new Pythagorean uncertainty sets called (3,2)-uncertainty sets introduced by [[17], [18]]. Yager and Abbasov [18] studied the Pythagorean membership grades (PMGs) and the considerations related to Pythagorean uncertainty collections and presented the association between the PMGs and the imaginary numbers. Reformat and Yager [11] applied the PFSs in dealing with the communitarian with respect to the recommender system. Garg. H et al. originated a few Pythagorean uncertainty mappings and investigated their preliminary properties like derivability, continuity and differentiability in detail. In 2020, Fermatean uncertainty sets proposed by Senapati and Yager [13] can handle uncertain information more easily in the process of decision-making. In recent years, many researchers have studied various properties of uncertainty subgroups. In 2015, Tarnauceanu [16] classified uncertainty as a normal subgroup of finite groups. In 2016, Onasanya [8] reviewed some anti-uncertainty properties of fuzzy subgroups. Shuaib [14] and Shaheryar [15] studied the properties of the omicron uncertainty subgroup and omicron anti-uncertainty subgroup. In 2018, Addis [1] developed uncertainty homomorphism theorems on groups. In 1986, Atanassov [2] invented intuitionistic uncertainty set. In 1996, Biswas first studied the intuitionistic uncertainty subgroup [4]. Zhan and Tan [20] introduced intuitionistic uncertainty M-group. Furthermore, researchers developed intuitionistic fuzzy subgroups in many ways [[4], [5]].

In this paper, we apply the concept of Pythagorean uncertainty set (PUS) to introduce the concepts of Pythagorean power uncertainty abelian subgroups (PPUAS), Pythagorean power uncertainty normal subgroups (PPUNS) and their properties. Also, we study those concepts in terms of the Cartesian product of Pythagorean power uncertainty sets. Finally, homomorphic images and pre-images of Pythagorean uncertainty sets are established.

2. Preliminaries and Various Basic Concepts

Definition 2.1: **Uncertainty Set (US):** Let 'X' be a non-empty set. An uncertainty set 'A' drawn from X is defined as $A = \{x, \mu_A(x) : x \in X\}$, where $\mu_A(x) : X \to [0, 1]$ is the membership function of the fuzzy set A.

Definition 2.2: If μ is an uncertainty subset of P, for $\alpha \in [0,1]$, then the set $\mu_\alpha = \{x \in P / \mu(x) \geq \alpha\}$ is called a level subset of P with respect to an uncertainty subset μ.

Definition 2.3: A fuzzy subset $\mu: P \to [0,1]$ is a non-empty uncertainty subset if μ is not a constant function.

***Definition 2.4: Intuitionistic uncertainty set (IUS)*:** Let 'X' be a non-empty set. An intuitionistic uncertainty set 'A' in X is an object having the form $A = \{x, \mu_A(x), v_A(x) : x \in X\}$, where the functions $\mu_A(x),\ v_A(x) : X \to [0,1]$ define, respectively, the degree of membership and degree of non-membership of the element x ∈ X to the set A, which is a subset of X and for every element x ∈ X, $0 \leq \mu_A(\mathrm{x}) + v_A(\mathrm{x}) \leq 1$. Furthermore, we have $\pi_A(\mathrm{x}) = 1 - \mu_A(\mathrm{x}) - v_A(\mathrm{x})$ called the intuitionistic uncertainty set index or hesitation margin of x ∈ A. $\pi_A(\mathrm{x})$ is the degree of indeterminacy of x ∈ X to the IFS A and $\pi_A(\mathrm{x}) \in [0,1]$, i.e., $\pi_A(\mathrm{x})\colon \mathrm{X} \longrightarrow [0,1]$ and $0 \leq \pi_A(\mathrm{x}) \leq 1,\ \forall\, \mathrm{x} \in \mathrm{X}$. $\pi_A(\mathrm{x})$ expresses the lack of knowledge of whether x belongs to IUS 'A' or not.

Definition 2.5: Pythagorean uncertainty set (PUS): A Pythagorean uncertainty set (PUS) on a non-empty set X is defined as a structure $C := \{(x, \alpha_A(x), \beta_A(x)\ /\ x \in X\}$,→ (1)where $\alpha_A : X \to [0,1]$ and $\beta_A : X \to [0,1]$, such that $0 \leq \alpha_A^2(x) + \beta_A^2(x) \leq 1$.

The notations $\alpha_A^2(x)$ and $\beta_A^2(x)$ are used listed of $(\alpha_A(x))^2\ and\ (\beta_A(x))^2$, respectively and the Pythagorean uncertainty set (1) simply indicated by $C := (X, \alpha_A, \beta_A)$.

***Definition 2.6: Power Group*:** A group 'G' is cyclic if it contains 'a' is a generator of G, then for every $x \in G$, there is a unique integer $n \in Z$ such that $x = a^n$. Then, 'n' is called the discrete logarithm of 'x' to base 'a', and we denoted it by DLOGG $_a(x)$.
$G = \{a^n/\ n \in Z\}$. Here, 'a' is called a generator of G, and it is denoted by $< a >$.

***Definition 2.7: Pythagorean power Fuzzy Subgroup*:**
Let $A = (G, \alpha_A, \beta_A)$ be a Pythagorean uncertainty set of a group G. Then 'A' is called a Pythagorean power uncertainty subgroup (PPUS) of G if the following conditions hold,
PPUS$_1$: $\alpha_A^2(a^n * a^m) \geq\ min\,\{\alpha_A^2(a^n), \alpha_A^2(a^m)\}$ and
$$\beta_A^2(a^n * a^m) \leq\ max\,\{\beta_A^2(a^n), \beta_A^2(a^m)\}$$
PPUS$_2$: $\alpha_A^2(a^{-n}) \geq \alpha_A^2(a^n)$ and
$$\beta_A^2(a^{-n}) \leq \beta_A^2(a^n),\ \forall\, n, m \in Z\ and\ a \in G.$$
Equivalently, a Pythagorean power uncertainty set $\mathrm{A} = (\mathrm{X}, \alpha_A,\ \beta_A)$ of G is a Pythagorean uncertainty set of G if and only if $\alpha_A^2(a^n a^{-m}) \geq min\,\{\alpha_A^2(a^n), \alpha_A^2(a^m)\}$ and
$$\beta_A^2(a^n a^{-m}) \leq max\,\{\beta_A^2(a^n), \beta_A^2(a^m)\},\ \forall\, n, m \in Z_+.$$

3. Pythagorean Power Uncertainty Subgroup
This section will discuss the concept of the Pythagorean power uncertainty subgroup and its properties.

Theorem 3.1: Let $A = (\mathrm{X}, \alpha_\mathrm{A}, \beta_\mathrm{A})$ be a Pythagorean power uncertainty set of a group G. Then we have the following
(i) N(A) is a subgroup of G.
(ii) A is a Pythagorean power uncertainty normal subgroup (PPUNS) of G if and only if N(A) = G.
(iii) A is a Pythagorean power uncertainty normal subgroup (PPUNS) of N(A).

Proof: Let $\mathrm{a^n}, \mathrm{a^m} \in \mathrm{N(A)}$, Then
1) $\alpha_\mathrm{A}^2(\mathrm{a^{-n} x\, a^n}) = \alpha_\mathrm{A}^2(\mathrm{x})$ and $\beta_\mathrm{A}^2(\mathrm{a^{-n} x\, a^n}) = \beta_\mathrm{A}^2(\mathrm{x}),\ \forall\, \mathrm{x} \in \mathrm{G}$
2) $\alpha_\mathrm{A}^2(\mathrm{a^{-m} y\, a^m}) = \alpha_\mathrm{A}^2(\mathrm{y})$ and $\beta_\mathrm{A}^2(\mathrm{a^{-m} y\, a^m}) = \beta_\mathrm{A}^2(\mathrm{y}),\ \forall\ \mathrm{y} \in \mathrm{G}.$

Put $y = a^{-n} x\, a^n$ in (2)
and using (1), we get
$$\alpha_A^2(a^{-m} a^{-n} x\, a^n\, a^m) = \alpha_A^2(a^{-n} x\, a^n) = \alpha_A^2(x) \text{ and}$$
$$\beta_A^2(a^{-m} a^{-n} x\, a^n\, a^m) = \beta_A^2(a^{-n} x\, a^n) = \beta_A^2(x).$$
That is,
$$\alpha_A^2((a^n a^m)^{-1} x\, (a^n a^m)) = \alpha_A^2(x) \text{ and } \beta_A^2((a^n a^m)^{-1} x\, (a^n a^m)) = \beta_A^2(x).$$
Thus, $a^n a^m = a^{n+m} \in N(A)$
Next, Change x to x^{-1} in (1)
We get, $\alpha_A^2(a^{-n} x^{-1}\, a^n) = \alpha_A^2(x^{-1}) = \alpha_A^2(x)$ and
$$\beta_A^2(a^{-n} x^{-1}\, a^n) = \beta_A^2(x^{-1}) = \beta_A^2(x).$$
that is
$$\alpha_A^2((a^n x\, a^{-n})^{-1}) = \alpha_A^2(a^n x\, a^{-n}) = \alpha_A^2(x) \text{ and}$$
$$\beta_A^2((a^n x\, a^{-n})^{-1}) = \beta_A^2(a^n x\, a^{-n}) = \beta_A^2(x).$$
that is
$$\alpha_A^2((a^{-n})^{-1} x (a^{-n})) = \alpha_A^2(a^n x\, a^{-n}) = \alpha_A^2(x) \text{ and}$$
$$\beta_A^2\big((a^{-n})^{-1} x (a^{-n})\big) = \beta_A^2(a^n x\, a^{-n}) = \beta_A^2(x) \in N(A).$$
Hence, N(A) is a subgroup of G.

(2) Obvious, when N(A) = G, then
$\alpha_A^2(a^{-n}x\, a^n) = \alpha_A^2(x)$ and
$$\beta_A^2(a^{-n}x\, a^n) = \beta_A^2(x), \quad \forall\, x, a \in G \text{ and } n \in Z.$$
Hence, 'A' is a Pythagorean power uncertainty in the normal subgroup of G.
Conversely, when 'A' is a Pythagorean power uncertainty normal subgroup of G, then
$\alpha_A^2(a^{-n}x\, a^n) = \alpha_A^2(x)$ and
$$\beta_A^2(a^{-n}\mathrm{x}\, a^n) = \beta_A^2(x), \quad \forall\, x, a \in G \text{ and } n \in Z.$$
That is, the set $\alpha_A^2(a^{-n}\, x\, a^n) = \alpha_A^2(x)$ and the set $\beta_A^2(a^{-n}\, x\, a^n) = \beta_A^2(x), \forall\, x \in G$ = G.
That is N(A) = G.
(3) Let $a^n,\ a^m \in N(A). Then$
$\alpha_A^2(a^{-n}x\, a^n) = \alpha_A^2(x)$ and
$$\beta_A^2(a^{-n}x\, a^n) = \beta_A^2(x), \quad \forall\, x \in G.$$
Put $x = a^{n+m}$, we get
$$\alpha_A^2(\, a^{n+m}\,) = \alpha_A^2(a^{-n}\, a^{n+m}\, a^n)$$
$$= \alpha_A^2(\, a^{m+n}\,) = \alpha_A^2(\, a^m \cdot a^n\,) \text{ and}$$
$$\beta_A^2(\, a^{n+m}\,) = \beta_A^2(a^{-n}\, a^{n+m}\, a^n)$$
$$= \beta_A^2(\, a^{m+n}\,) = \beta_A^2(\, a^m \cdot a^n\,).$$
Hence, 'A' is a Pythagorean power uncertainty normal subgroup of N(A).

***Definition 3.2*:** Let 'A' be a Pythagorean power uncertainty subgroup of a group G. Then 'A' is called a Pythagorean power uncertainty abelian subgroup of G if $C_{r,s}(A)$ is an abelian subgroup of G for all r, s $\in [0,1]$ with $0 \le r + s \le 1$.

Remark 3.3: If G is an abelian group, then every Pythagorean power uncertainty subgroup (PPUS) of G is a Pythagorean power uncertainty abelian subgroup of G (PPUAS).

Theorem 3.4: Let 'A' be a Pythagorean power uncertainty abelian subgroup (PPUAS) of group G. Then the set $H = \{a^n \in G \,/\, \alpha_A^2(a^{n+m}) = \alpha_A^2(a^{m+n})$ and $\beta_A^2(a^{n+m}) = \beta_A^2(a^{m+n}),\ \forall\, a^m \in G\}$ is an abelian subgroup of G.
Proof: Since $A = (G, \alpha_A, \beta_A)$is a Pythagorean uncertainty abelian subgroup of G, $C_{r,s}$(A) is an abelian subgroup of G, for all r, s $\in [0,1]$ with $0 \le r + s \le 1$.
Clearly, $H \neq \phi$, for e $\in H$.
Let $a^n, a^m \in H$. Then
$\alpha_A^2(a^n\, x) = \alpha_A^2(x\, a^n)$,
$\beta_A^2(a^n\, x) = \beta_A^2(x\, a^n)$ and
$$\alpha_A^2\big((a^{n+m})x\big) = \alpha_A^2\big(a^n\,(a^m\, x)\big)$$
$$= \alpha_A^2((a^m\, x)a^n)$$
$$= \alpha_A^2\big(a^m\,(xa^n)\big)$$
$$= \alpha_A^2\big((xa^n)a^m\big)$$
$$= \alpha_A^2\big(x(a^{n+m})\big) \quad \text{and}$$
$$\beta_A^2\big((a^{n+m})x\big) = \beta_A^2\big(a^n\,(a^m\, x)\big)$$
$$= \beta_A^2((a^m\, x)a^n)$$
$$= \beta_A^2\big(a^m\,(xa^n)\big)$$
$$= \beta_A^2\big((xa^n)a^m\big)$$
$$= \beta_A^2\big(x(a^{n+m})\big), \forall\, x \in G$$
Therefore, $a^{n+m} \in H$.
Also, Let $a^n \in H$.
Now, $a^n \in H \Rightarrow \alpha_A^2(a^n\, x) = \alpha_A^2(x\, a^n)$,
$$\beta_A^2(a^n\, x) = \beta_A^2(x\, a^n), \forall\, x \in G. \quad \to (1)$$
Put $x = y^{-1}$ in (1), we get,
$\alpha_A^2(a^n\, y^{-1}) = \alpha_A^2(y^{-1}a^n)$,
$\beta_A^2(a^n\, y^{-1}) = \beta_A^2(y^{-1}a^n)$.
Now, $\alpha_A^2(a^{-n}\, y) = \alpha_A^2(a^{-n}\, y)^{-1}) = \alpha_A^2(y^{-1}a^n)$
$= \alpha_A^2(a^n\, y^{-1})$
$$= \alpha_A^2((a^n\, y^{-1})^{-1})$$
$$= \alpha_A^2(y\, a^{-n}).$$
Similarly $\beta_A^2(a^{-n}\, y) = \beta_A^2(y\, a^{-n}),\ \forall\, y \in G$.
Thus, $a^{-n} \in H$.
So, H is a subgroup of G.

Let $a, b \in H$. Without loss of generality,
let $\alpha_A^2(a^n) = r, \beta_A^2(a^n) \leq 1 - r$ and
$\alpha_A^2(a^m) = r_1, \beta_A^2(a^m) \leq 1 - r_1$.
Then, $a^n \in C_{r,1-r}(A)$, $a^m \in C_{r_1,1-r_1}(A)$.
Let $r < r_1$. Then $\alpha_A^2(a^m) = r_1 > r$ and
$$\beta_A^2(a^m) \leq 1 - r_1 < 1 - r.$$
Implies, $a^m \in C_{r,1-r}(A)$.
Thus $a^n, a^m \in C_{r,1-r}(A)$ and so, $a^{n+m} = a^{m+n}$.
Hence, H is an abelian subgroup of G.

Proposition 3.5: (1) If $A = (G, \alpha_A, \beta_A)$is a Pythagorean power uncertainty abelian subgroup (PPAUAS) of a group G, then C(A) is a Pythagorean power uncertainty abelian subgroup of G. (2) The Sets H and C(A) is the same, that is, C(A) = H.
Proof: $C(A) = \{a^n \in G \,/\, \alpha_A^2(a^n.\, x) = \alpha_A^2(e)\ and\ \beta_A^2(a^n.\, x) = \beta_A^2(e),\ \forall\, x \in G\}$
$$= \{a^n \in G \,/\, \alpha_A^2(a^{-n} x^{-1} a^n x) = \alpha_A^2(e)\ and\ \beta_A^2(a^{-n} x^{-1} a^n x) = \beta_A^2(e), \forall\, x \in G\}$$
$$= \{a^n \in G/\alpha_A^2((xa^n)^{-1}a^n x) = \alpha_A^2(e)\ and\ \beta_A^2((xa^n)^{-1}a^n x) = \beta_A^2(e), \forall\, x \in G\}$$
$$= \left\{a^n \in G \,/\, \alpha_A^2\big((xa^n)\big) = \alpha_A^2(a^n x)\ and\ \beta_A^2\big((xa^n)\big) = \beta_A^2(a^n x), \quad \forall\, x \in G\right\} = H.$$

Theorem 3.6**:** Let $A = (G_1, \alpha_A, \beta_A)$ and $B = (G_2, \alpha_B, \beta_B)$ be a Pythagorean power uncertainty abelian subgroups (PPUAS) of G_1 and G_2 respectively. Then A × B is a Pythagorean power uncertainty abelian subgroup (PPUAS) of $G_1 \times G_2$ if and only if both A and B are Pythagorean power uncertainty abelian subgroups (PPUAS) of G_1 and G_2 respectively.
Proof: First, let A and B be Pythagorean power uncertainty abelian subgroups (PPUAS) of G_1 and G_2 respectively. Then, $C_{r,s}(A)\ and\ C_{r,s}(B)$ are abelian subgroups of G_1 and G_2 respectively $\forall r, s \in [0,1]$ with $0 < r + s \leq 1 \Rightarrow$
$C_{r,s}(A) \times C_{r,s}(B)$ is an abelian subgroup of $G_1 \times G_2$.
But, $C_{r,s}(A \times B) = C_{r,s}(A) \times C_{r,s}(B)$ by definition.
Therefore, $C_{r,s}(A \times B)$ is an abelian subgroup of $G_1 \times G_2$, $\forall r, s \in [0,1]$ with $0 < r + s \leq 1$
$\Rightarrow A \times B$ is Pythagorean power uncertainty abelian subgroup of $G_1 \times G_2$.
Conversely, Let A × B be a Pythagorean power uncertainty abelian subgroup of $G_1 \times G_2$.
Then, $C_{r,s}(A \times B)$ is an abelian subgroup of $G_1 \times G_2$.
that is $C_{r,s}(A) \times C_{r,s}(B)$ is an abelian subgroup of $G_1 \times G_2$
implies $C_{r,s}(A)$ and $C_{r,s}(B)$ are abelian subgroups of G_1 and G_2 respectively.
Therefore, A and B are Pythagorean power uncertainty abelian subgroups of G_1 and G_2 respectively.

Definition 3.7*:* Let $A = (G, \alpha_A, \beta_A)$be a Pythagorean uncertainty set (PUS) of a group G. Then 'A' is called a Pythagorean power uncertainty subgroup of G, $\forall\ r, s \in [0,1]$ with $0 \leq r + s \leq 1$.

Proposition 3.8: If 'G' is a power group, then every Pythagorean uncertainty set of G is a Pythagorean power uncertainty subgroup (PPUS) of G.
Proof: Let $G = < a >$ be a power group and let 'A' be the Pythagorean uncertainty set of G.
Then, $\alpha_A^2(a^n) \geq \alpha_A^2(a^{n-1}) \geq \alpha_A^2(a^{n-2}) \ldots \geq \alpha_A^2(a^2)$ and
$$\beta_A^2(a^n) \geq \beta_A^2(a^{n-1}) \geq \beta_A^2(a^{n-2}) \ \ldots\ \geq\ \beta_A^2(a^2), \qquad \text{for all } n \in Z.$$
Therefore, $a^m \in C_{r,s}(A)$, for some $m \in Z$, then $a^m, a^{m+1}, a^{m+2}, \ldots \in C_{r,s}(A)$.
That is $C_{r,s}(A) = < a^{-n} >$, which is a power subgroup of G for all r, s $\in$ [0,1] with $0 \leq r + s \leq 1$. Hence, 'A' is a Pythagorean power uncertainty subgroup of G.

Theorem 3.9: Let $f : G_1 \to G_2$ be a homomorphism of a group G_1 into a group G_2. Let 'B' be a Pythagorean power uncertainty abelian subgroup of G_2. Then f^{-1}(B) is a Pythagorean power uncertainty abelian subgroup of G_1.
Proof: Let 'B' be a Pythagorean power uncertainty abelian subgroup of G_2.
Therefore, $C_{r,s}(B)$ is the abelian subgroup of G_2 . For all r, s ∈ [0,1] with $0 < r + s \leq 1$.
It follows that, $C_{r,s}\big(f^{-1}(B)\big) = f^{-1}(C_{r,s}(B)) = \{x \in G_1 \,/\, f(x) \in C_{r,s}(B)\}$
is an abelian subgroup of G_2. Therefore, $f(a^n) = f(a^m) = f(a^m) \cdot f(a^n) \Rightarrow f(a^n \cdot a^m) = f(a^m \cdot a^n)$
$$\Rightarrow f(a^{n+m}) = f(a^{m+n})$$
And so, $\alpha_B^2(f(a^n \cdot a^m)) = \alpha_B^2(f(a^n\ a^m))$ and $\beta_B^2(f(a^n \cdot a^m)) = \beta_B^2(f(a^n\ a^m))$
$\Rightarrow \alpha_{f^{-1}(B)}^2(a^n\ a^m) = \alpha_{f^{-1}(B)}^2(a^m\ a^n)$ and $\beta_{f^{-1}(B)}^2(a^n\ a^m) = \beta_{f^{-1}(B)}^2(a^m\ a^n)$
Thus, $C_{r,s}(f^{-1}(B))$ is an abelian subgroup of G_1. For all r, s $\in$ [0,1] with $0 < r + s \leq 1$.
Hence, $f^{-1}(B)$ is a Pythagorean power uncertainty abelian subgroup of G_1.
Theorem 3.10: Let f: $G_1 \to G_2$ be a surjective homomorphism of a group G_1 into a group G_2 and 'A' a Pythagorean power uncertainty abelian subgroup of G_1, then f(A) is a Pythagorean power uncertainty abelian subgroup of G_2 .
Proof: Since 'A' is a Pythagorean power uncertainty abelian subgroup of G_1, for all r, s $\in$ [0,1] with $0 < r + s \leq 1$.

Let $a^p, a^q \in C_{r,s}(f(A))$. Then there exist $a^n, a^m \in G_1$ such that

$$f(a^n) = a^p, f(a^m) = a^q$$

Then, $f(a^n), f(a^m) \in C_{r,s}(f(A))$ as $C_{r,s}(A)$ is an abelian subgroup of G_1.
Therefore, their exits $C_{u,v}(A)$ such that $a^n, a^m \in C_{u,v}(A)$, where $u, v \in [0,1]$ with $0 < u + v \leq 1$. But, $C_{r,s}(A)$ is an abelian group.
Therefore, $a^{n+m} = a^{m+n} \Rightarrow f(a^n\ a^m) = f(a^m\ a^n) \Rightarrow f(a^n).f(a^m)$

$$= f(a^m) \cdot f(a^n).$$

That is, $a^{p+q} = a^{q+p}$. Thus, $C_{r,s}(f(A))$ is an abelian subgroup of G_2.
Hence, f(A) is a Pythagorean power uncertainty abelian subgroup of G_2.

Theorem 3.11: Let $f: G_1 \rightarrow G_2$ be a homomorphism of a group G_1 into a group G_2 . Let 'B' be a Pythagorean power uncertainty subgroup (PPUS) of G_2. Then $f^{-1}(B)$ is a Pythagorean power uncertainty subgroup of G_1.
Proof: Since 'B' is a Pythagorean uncertainty power subgroup of G_2, $C_{r,s}(B)$ is a power subgroup of G_2, for all $r, s \in [0,1]$ with $0 < r + s \leq 1$.
Let $C_{r,s}(B) = < h_2 >$ *for some* $h_2 \in G_2$.
Now for $h_2 \in G_2$, $\exists h_1 \in G_1$ such that $f(h_1) = h_2$
Thus $C_{r,s}(B) = (\alpha_A(h_1))$ and so $f^{-1}(C_{r,s}) = C_{r,s}(f^{-1}(B)) = h_1$
Hence $f^{-1}(B)$ is a Pythagorean power uncertainty subgroup of G_1.

Theorem 3.12: Let $f: G_1 \rightarrow G_2$ be a surjective homomorphism of a group G_1 into a group G_2 and 'A' a Pythagorean power uncertainty subgroup of G_1. Then f(A) is a Pythagorean power uncertainty subgroup of G_2.
Proof: Let 'A' be a Pythagorean power uncertainty subgroup of G_1. Therefore $C_{r,s}(A)$ is a power subgroup of G_1, for all $r, s \in [0,1]$ with $0 < r + s \leq 1$.
Let $h \in C_{r,s}(\alpha_A(A))$.
As 'f' is surjective, therefore let $h = f(h_1)$, for some $h_1 \in G_1$, we can find one $C_{r,s}$(A) which exists for all $h_1 \in G_1$ (and hence $h \in C_{r,s}(f(A))$ such that $h \in C_{r,s}(A)$).
But, $C_{r,s}(A)$ is a power subgroup of G_1.
Let $C_{r,s}(A) =< h_1 >$. So $h_1 = (h_1)^n$. Thus, h = $f(h) = f(h_1)\, f((h_1)^n) = (f(h_1))^n$.
that is, $C_{r,s}(f(A))$ is a power subgroup of G_2 . Hence $f(A)$ is a Pythagorean power uncertainty subgroup of G_2.

4. Support of Pythagorean Power Uncertainty Set

***Definition 4.1**:* The support of a Pythagorean power uncertainty set A of X is defined to be $Supp_x(A) = \{a^n \in X \,/\, \alpha_A^2(a^n) > 0 \ and\ \beta_A^2(a^n) < 1\}$.
Clearly, $Supp_x(A)$ is $\cup \{C_{r,s}(A);\ \forall\, r, s \in [0,1]\ such\ that\ r + s \leq 1\}$.

***Proposition 4.2**:* For $f:\ X \rightarrow Y$ and Pythagorean power uncertainty set A, B of X and Y, respectively, we have
(1) $f(Supp_x(A)) \subseteq Supp_x(f(A))$, equality holds if '$f$' is bijective.
(2) $f^{-1}(Supp_x(B)) = Supp_x(f^{-1}(B))$.

Proposition 4.3: If 'A' is a non-zero Pythagorean power uncertainty set of a group G, then $Supp_G(A)$ is a subgroup of G.
The following example shows that the converse of the proposition is not true.

Example 4.4: Let G = (R, +) be a group of real numbers under addition.
Define the Pythagorean power uncertainty set A on G by

$$\alpha_A(a^n) = \begin{cases} 0.27 & if\ a^n = 1 \\ 0.73 & if\ a^n \in Q - \{1\} \\ 0 & if\ a^n \in R - Q \end{cases}$$

$$\beta_A(a^n) = \begin{cases} 0.47 & if\ a^n = 1 \\ 0.23 & if\ a^n \in Q - \{1\} \\ 1 & if\ a^n \in R - Q \end{cases}$$

Clearly, A is not a Pythagorean power uncertainty set of G, but $Supp_G(A) = Q$ is a subgroup of G.

***Proposition 4.5**:* Let A $= (G,\ \alpha_A,\ \beta_A)$ be a Pythagorean power uncertainty set of G.
Then, we have the following.
(i) If A $= (G,\ \alpha_A,\ \beta_A)$ be a Pythagorean power uncertainty set of G and N is a subgroup of G, then A/N is also a Pythagorean power uncertainty set of 'N'.

(ii) If A/N is the restriction of the Pythagorean power uncertainty set A of a group on the subgroup N of G, then $Supp_N\left(\frac{A}{N}\right) = Supp_G(A) \cap N$.

(iii) If 'A' is a Pythagorean power uncertainty set of G and N is a subgroup of G, then A/N is also a Pythagorean power uncertainty set of N if and only if N is a power subgroup of G.

Theorem 4.6: Let $f: G_1 \rightarrow G_2$ be a homomorphism of a group G_1 into a group G_2. Then, we have the following.

(i) If A is a Pythagorean power uncertainty abelian subgroup (PPUAS) of G_1, then f(A) is a Pythagorean power uncertainty abelian subgroup of G_2.

(ii) If A is a Pythagorean power uncertainty subgroup (PPUS) of G_1, then f(A) is a Pythagorean power uncertainty subgroup of G_2.

(iii) If A′ is a Pythagorean power uncertainty abelian subgroup (PPUAS) on G_2, then $f^{-1}(A')$ is a Pythagorean power uncertainty abelian subgroup of G_1.

(iv) If A^1 is a Pythagorean power uncertainty subgroup (PPUAS) on G_2, then $f^{-1}(A')$ is a Pythagorean power uncertainty subgroup of G_1.

5. Conclusion

In this work, we have introduced the concept of Pythagorean power uncertainty set equivalent sets and established these properties. We also gave certain counterexamples to prove their properties. As interesting kinds, we have introduced and studied the concepts of Pythagorean power uncertainty abelian subgroups and Pythagorean power uncertainty normal subgroups. Finally, homomorphic images and pre-images of Pythagorean power uncertainty sets are established.

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