International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V71I3P101

Volume 71 Issue 3, 1-15, March 2025

Original Article

Two Effective Methods for Solution of the (2+1)-Dimensional Zakharov-Kuznetsov Equation

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Received: 05 January 2025 Revised: 14 February 2025 Accepted: 02 March 2025 Published: 15 March 2025

Abstract *- This article discusses the (2+1)-dimensional case of the Zakharov-Kuznetsov (ZK) equation. The (2+1)-dimensional ZK equation is primarily used to describe wave propagation phenomena in multi-dimensional media. In plasma, liquids, or gases, waves may be influenced by multiple elements. Due to nonlinear effects, the propagation speed, shape and interactions of these waves become complex. We have obtained a variety of exact solutions of the (2+1) dimensional ZK equation by using two effective methods: the improved extended tanh function method and the modified Kudryashov method. The forms of the solutions include exponential solutions, logarithmic solutions, hyperbolic solutions and trigonometric solutions. In addition, by selecting appropriate parameter values,we have plotted three-dimensional and two-dimensional images to illustrate the physical behavior of the exact solutions.*

Keywords *- (2+1)-dimensional ZK equation. Wave solution. Modified extended tanh-function method. Modified generalized Kudryashov method.*

1. Introduction

The Korteweg de-Vries (KdV) equation is provided by

$$u_t + auu_x + u_{xxx} = 0, \tag{1.1}$$

Where a is an arbitrary constant and the commonly used constants are $a = \pm 1$ and $a = \pm 6$. The KdV equation[1-4] models various nonlinear phenomena, including ion acoustic waves in plasma and shallow water waves. Many equations describing water waves have been derived from the KdV equation, which we refer to as a family of KdV-type equations. For example, the modified KdV (mKdV) equation[5,6] serves as a model for the evolution of nonlinear plasma waves, the Kadomtsev-Petviashvili (KP) equation[7-9] is used to study small amplitude long ion acoustic waves, and the Benjamin-Bona-Mahony (BBM) equation[10-12] describes the unidirectional propagation of weak long dispersive waves in inviscid fluids.

It is essential to study the dynamic processes and solution forms of such nonlinear evolution equations (NLEEs), as they can be applied not only in oceanography, nonlinear optics and fluid mechanics, but also in solid-state physics, geology, thermodynamics, and more. Common methods include: the inverse scattering method [13,14], the Hirota’s bilinear method [15-18], the Jacobi elliptic functions method [19-22], the extended (G'/G)-expansion method [23], the Sinh-Gordon expansion method [24,25], the F-expansion method [26,27], the $\exp(-(\phi(\xi)))$ expansion method [28-30], the generalized Kudryashov method [31-34], the new ϕ^6-model expansion method [35], the sine-cosine method [36,37], the first integral method[38] and so on [39-42]. These methods transform the NLEEs into ordinary differential equations (ODE) and numerical computations using tools like Maple or Mathematica can also assist in the solutions.

This paper focuses on the Zakharov-Kuznetsov (ZK) equation [43,44], written as

$$u_t + auu_x + b(\nabla^2 u)_x = 0, \tag{1.2}$$

Where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the isotropic Laplace operator. Eq (1.2) is an extension of the KdV equation. The ZK equation governs the behavior of weakly nonlinear ion acoustic waves in a plasma composed of cold ions and thermally isothermal electrons under uniform magnetic field. The ZK equation is a more isotropic equation, originally derived to describe weakly nonlinear ion acoustic waves in two-dimensional strongly magnetized lossless plasma. Unlike the KP equation, the ZK equation cannot be integrated using the inverse scattering transform method.

The (2 + 1)-dimensional ZK equation is given by

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0. \tag{1.3}$$

Where the coefficients *a* and *b* are nonzero constants and are related to the physical parameters of plasma, typically associated with temperature, density and other state variables. Specifically, the derivative u_t characterizes the time evolution of the wave propagating in one direction, the nonlinear term uu_x describes the steepening of the wave, bu_{xxx} represents spatial dispersion and bu_{yyx} denotes the cross-dispersion effect. The (2+1)-dimensional ZK equation has wide applications in many fields, such as analyzing multidimensional plasma wave phenomena in plasma physics, studying multidimensional fluid waves and vortices in fluid mechanics, modeling wave propagation in oceans and assisting in predicting wave behavior in ocean engineering, and describing the propagation of sound waves in complex environments in acoustics. The (2+1)-dimensional ZK equation is an important component of nonlinear wave theory. By studying this equation in detail, a deeper understanding of the characteristics and behaviors of multidimensional waves can be achieved, which is significant for relevant wide scientific research and engineering applications.

The rest of this paper is organized as follows. Section 2 introduces the modified extended tanh-function method [45-47]; Section 3 introduces the modified generalized Kudryashov method [48]; Section 4 presents a variety of new exact solutions for the (2+1)-dimensional ZK equation using the methods mentioned above; Section 5 provides graphical representations of the obtained solutions and their physical interpretations; Section 6 offers a brief conclusion.

2. The Modified Extended Tanh-Function Method

In this section, the following general NLEE is written as

$$F(u, u_t, u_x, u_y, u_{xx}, \cdots) = 0, \tag{2.1}$$

Where *F* is a function of $u(x, y, t)$ and its own derivatives. Using the wave transformation

$$g(\xi) = u(x, y, t), \ \xi = x + my + ct, \tag{2.2}$$

Eq. (2.1) can be converted into an ordinary differential equation (ODE), given as

$$F(g, g_{\xi}, g_{\xi\xi}, g_{\xi\xi\xi}, \cdots) = 0. \tag{2.3}$$

Based upon the modified extended tanh-function method, the solution of Eq. (2.3) is supposed having the form of

$$g(\xi) = a_0 + \sum_{i=1}^{M} \left(a_i H^i(\xi) + b_i H^{-i}(\xi) \right). \tag{2.4}$$

Where a_i and b_i are constants to be determined later and the M is a positive integer obtained by the balance principle. The $H(\xi)$ satisfies the following ODE

$$H'(\xi) - H^2(\xi) - \lambda = 0, \tag{2.5}$$

Where λ is a constant to be determined later. The solution of Eq. (2.5) is written as follows according to the sign of the parameter λ.

When $\lambda < 0,$ we have

$$H(\xi) = -\sqrt{-\lambda} \tanh(\sqrt{-\lambda}\xi), \tag{2.6}$$

$$H(\xi) = -\sqrt{-\lambda} \coth(\sqrt{-\lambda}\xi). \tag{2.7}$$

When $\lambda > 0,$ we have

$$H(\xi) = \sqrt{\lambda} \tan(\sqrt{\lambda}\xi), \tag{2.8}$$

$$H(\xi) = -\sqrt{\lambda} \cot(\sqrt{\lambda}\xi). \tag{2.9}$$

When $\lambda = 0,$ we have

$$H(\xi) = -\frac{1}{\xi}. \tag{2.10}$$

Inserting Eq. (2.4) into Eq. (2.3) and rearranging the terms, and further setting all coefficients of the same $H^j(\xi),\ (j = 0,1,2,\cdots,M)$ power to zero. Then by using Maple program, the algebraic equations are solved to obtain the values of the unknowns, the exact solutions of Eq. (1.3) can be obtained.

3. The Modified Generalized Kudryashov Method

As in the previous section, a wave transformation is performed to convert the NLEE into an ODE. The modified generalized Kudryashov method assumes the solution has the following form

$$g(\xi) = \sum_{j=0}^{M} \frac{s_j}{(1 + H(\xi))^j}, \tag{3.1}$$

Where $s_0, s_1, \cdots s_M$ are constants to be determined later, and the M is a positive integer obtained by the balance principle. Furthermore, the function $H(\xi)$ satisfies the following ODE

$$H'(\xi) = \sigma + \delta H(\xi) + \tau H^2(\xi), \tag{3.2}$$

Where σ, δ and τ are real constants. Solving the Eq. (3.2) yields the following three cases:
When σ, δ are arbitrary constants and $\tau \neq 0$, the following solution is obtained,written as

$$H(\xi) = \frac{\sqrt{4\tau\sigma - \delta^2} \tan\left(\frac{1}{2}(E+\xi)\sqrt{4\tau\sigma - \delta^2}\right) - \delta}{2\tau}. \tag{3.3}$$

When $\sigma = 0, \ \delta \neq 0$ and τ is an arbitrary constant, the solution of Eq.(3.2) is given as

$$H(\xi) = -\frac{\delta e^{\delta(E+\xi)}}{\tau e^{\delta(E+\xi)} - 1}. \tag{3.4}$$

When σ is an arbitrary constant, $\delta \neq 0$ and $\tau = 0$, the solution of Eq.(3.2) is written as

$$H(\xi) = \frac{e^{\delta(E+\xi)} - \sigma}{\delta}. \tag{3.5}$$

In the expressions mentioned above E is a constant of integration.

Inserting Eq. (3.1) into Eq. (2.3) and rearranging the terms, and furthermore setting all coefficients of the same $H(\xi)$ power to zero. Then by using Maple program, the algebraic polynomial is solved to obtain the values of the unknowns, the exact solutions to Eq. (1.3) are obtained.

4. Application of the Above Methods

For the (2+1)-dimensional ZK equation, the following wave transformation is used and written as

$$g(\xi) = u(x, y, t), \ \xi = x + my + ct, \tag{4.1}$$

which convert Eq. (1.3) into the ODE

$$cg + \frac{1}{2}ag^2 + b(1+m^2)g'' = 0. \tag{4.2}$$

Balancing $g^2 = 2M$ with $g'' = M + 2$ gives $M = 2,$ the following exact solutions are derived.

4.1. Using the Modified Extended Tanh-Function Method

By taking $M = 2,$ Eq. (2.4) is written in the following form

$$g(\xi) = a_0 + a_1 H(\xi) + a_2 H(\xi)^2 + \frac{b_1}{H(\xi)} + \frac{b_2}{H(\xi)^2}. \tag{4.3}$$

Substituting Eq. (4.3) into Eq. (4.2) and using the Eq. (2.5), collecting the coefficients of the same powers of $H(\xi)$ and making them equal to zero, the following system of algebraic equations is obtained, written as

$$0 = 6bm^2 a_2 + \frac{1}{2}aa_2^2 + 6ba_2$$

$$0 = 2bm^2 a_1 + aa_1 a_2 + 2ba_1$$

$$0 = 8\lambda bm^2 a_2 + aa_0 a_2 + \frac{1}{2}aa_1^2 + 8\lambda ba_2 + ca_2$$

$$0 = 2\lambda bm^2 a_1 + aa_0 a_1 + aa_2 b_1 + 2\lambda ba_1 + ca_1$$

$$0 = 2\lambda^2 bm^2 a_2 + 2\lambda^2 ba_2 + 2bm^2 b_2 + \frac{1}{2}aa_0^2 + aa_1 b_1 + aa_2 b_2 + 2bb_2 + ca_0$$

$$0 = 2\lambda bm^2 b_1 + aa_0 b_1 + aa_1 b_2 + 2\lambda bb_1 + cb_1$$

$$0 = 8\lambda bm^2 b_2 + aa_0 b_2 + \frac{1}{2} ab_1^2 + 8\lambda bb_2 + cb_2$$

$$0 = 2\lambda^2 bm^2 b_1 + 2\lambda^2 bb_1 + ab_1 b_2$$

$$0 = 6\lambda^2 bm^2 b_2 + 6\lambda^2 bb_2 + \frac{1}{2} ab_2^2$$

Here the unknowns are a_0, a_1, a_2, b_1, b_2 and c. Using Maple program, the following six cases are obtained.

Case 1

$$a_0 = -\frac{4b\lambda(m^2+1)}{a}, a_1 = 0, a_2 = -\frac{12b(m^2+1)}{a}, b_1 = 0, b_2 = 0, c = -4b\lambda(m^2+1).$$

For $\lambda < 0, a \neq 0$, the solution of Eq.(1.3) has the following forms

$$g_{1.1} = -\frac{4b\lambda(m^2+1)}{a} + \frac{12b\lambda(m^2+1)}{a} \tanh^2(\sqrt{-\lambda}(x+my-4b\lambda(m^2+1)t)), \quad (4.4)$$

$$g_{1.2} = -\frac{4b\lambda(m^2+1)}{a} + \frac{12b\lambda(m^2+1)}{a} \coth^2(\sqrt{-\lambda}(x+my-4b\lambda(m^2+1)t)), \quad (4.5)$$

For $\lambda > 0, a \neq 0$, the solutions of Eq.(1.3) are written as

$$g_{1.3} = -\frac{4b\lambda(m^2+1)}{a} - \frac{12b\lambda(m^2+1)}{a} \tan^2(\sqrt{\lambda}(x+my-4b\lambda(m^2+1)t)), \quad (4.6)$$

$$g_{1.4} = -\frac{4b\lambda(m^2+1)}{a} - \frac{12b\lambda(m^2+1)}{a} \cot^2(\sqrt{\lambda}(x+my-4b\lambda(m^2+1)t)), \quad (4.7)$$

Case 2

$$a_0 = -\frac{12b\lambda(m^2+1)}{a}, a_1 = 0, a_2 = -\frac{12b(m^2+1)}{a}, b_1 = 0, b_2 = 0, c = 4b\lambda(m^2+1).$$

For $\lambda < 0, a \neq 0$, the solutions of Eq.(1.3) are obtained and given as

$$g_{2.1} = -\frac{12b\lambda(m^2+1)}{a} + \frac{12b\lambda(m^2+1)}{a} \tanh^2(\sqrt{-\lambda}(x+my+4b\lambda(m^2+1)t)), \quad (4.8)$$

$$g_{2.2} = -\frac{12b\lambda(m^2+1)}{a} + \frac{12b\lambda(m^2+1)}{a} \coth^2(\sqrt{-\lambda}(x+my+4b\lambda(m^2+1)t)), \quad (4.9)$$

For $\lambda > 0, a \neq 0$, the solutions of Eq.(1.3) are written as follows

$$g_{2.3} = -\frac{12b\lambda(m^2+1)}{a} - \frac{12b\lambda(m^2+1)}{a} \tan^2(\sqrt{\lambda}(x+my+4b\lambda(m^2+1)t)), \quad (4.10)$$

$$g_{2.4} = -\frac{12b\lambda(m^2+1)}{a} - \frac{12b\lambda(m^2+1)}{a} \cot^2(\sqrt{\lambda}(x+my+4b\lambda(m^2+1)t)), \quad (4.11)$$

Case 3

$a_0=\frac{8b\lambda(m^2+1)}{a}, a_1=0, a_2=-\frac{12b(m^2+1)}{a}, b_1=0, b_2=-\frac{12b\lambda^2(m^2+1)}{a}, c=-16b\lambda(m^2+1).$ For $\lambda<0, a\neq 0$, the solution of Eq.(1.3) is given as follows

$$\begin{aligned} g_{3.1}=&\frac{8b\lambda(m^2+1)}{a}+\frac{12b\lambda(m^2+1)}{a}\tanh^2(\sqrt{-\lambda}(x+my-16b\lambda(m^2+1)t)) \\ &+\frac{12b\lambda(m^2+1)}{a}\coth^2(\sqrt{-\lambda}(x+my-16b\lambda(m^2+1)t)), \end{aligned} \tag{4.12}$$

For $\lambda>0, a\neq 0$, the solution of Eq.(1.3) is obtained, written as

$$g_{3.2}=\frac{8b\lambda(m^2+1)}{a}-\frac{24b\lambda(m^2+1)}{a}\tan^2(\sqrt{\lambda}(x+my-16b\lambda(m^2+1)t)), \tag{4.13}$$

Case 4

$$a_0=-\frac{24b\lambda(m^2+1)}{a}, a_1=0, a_2=-\frac{12b(m^2+1)}{a}, b_1=0, b_2=-\frac{12b\lambda^2(m^2+1)}{a}, c=16b\lambda(m^2+1).$$

For $\lambda<0, a\neq 0$, the solution of Eq.(1.3) is given as

$$\begin{aligned} g_{4.1}=&-\frac{24b\lambda(m^2+1)}{a}+\frac{12b\lambda(m^2+1)}{a}\tanh^2(\sqrt{-\lambda}(x+my+16b\lambda(m^2+1)t)) \\ &+\frac{12b\lambda(m^2+1)}{a}\coth^2(\sqrt{-\lambda}(x+my+16b\lambda(m^2+1)t)), \end{aligned} \tag{4.14}$$

For $\lambda>0, a\neq 0$, the solution of Eq.(1.3) is written as

$$g_{4.2}=-\frac{24b\lambda(m^2+1)}{a}-\frac{24b\lambda(m^2+1)}{a}\tan^2(\sqrt{\lambda}(x+my+16b\lambda(m^2+1)t)), \tag{4.15}$$

Case 5

$$a_0=-\frac{4b\lambda(m^2+1)}{a}, a_1=0, a_2=0, b_1=0, b_2=-\frac{12b\lambda^2(m^2+1)}{a}, c=-4b\lambda(m^2+1).$$

For $\lambda<0, a\neq 0$, the solutions of Eq.(1.3) are given as

$$g_{5.1}=-\frac{4b\lambda(m^2+1)}{a}+\frac{12b\lambda(m^2+1)}{a}\coth^2(\sqrt{-\lambda}(x+my-4b\lambda(m^2+1)t)), \tag{4.16}$$

$$g_{5.2}=-\frac{4b\lambda(m^2+1)}{a}+\frac{12b\lambda(m^2+1)}{a}\tanh^2(\sqrt{-\lambda}(x+my-4b\lambda(m^2+1)t)), \tag{4.17}$$

For $\lambda>0, a\neq 0$,the solutions of Eq.(1.3) are written as

$$g_{5.3}=-\frac{4b\lambda(m^2+1)}{a}-\frac{12b\lambda(m^2+1)}{a}\cot^2(\sqrt{\lambda}(x+my-4b\lambda(m^2+1)t)), \tag{4.18}$$

$$g_{5.4}=-\frac{4b\lambda(m^2+1)}{a}-\frac{12b\lambda(m^2+1)}{a}\tan^2(\sqrt{\lambda}(x+my-4b\lambda(m^2+1)t)), \quad (4.19)$$

For $\lambda=0$, the solution of Eq.(1.3) is

$$g_{5.5}=0. \quad (4.20)$$

Case 6

$$a_0=-\frac{12b\lambda(m^2+1)}{a}, a_1=0, a_2=0, b_1=0, b_2=-\frac{12b\lambda^2(m^2+1)}{a}, c=4b\lambda(m^2+1).$$

For $\lambda<0, a\neq 0$, the solutions of Eq.(1.3) are given as

$$g_{6.1}=-\frac{12b\lambda(m^2+1)}{a}+\frac{12b\lambda(m^2+1)}{a}\coth^2(\sqrt{-\lambda}(x+my+4b\lambda(m^2+1)t)), \quad (4.21)$$

$$g_{6.2}=-\frac{12b\lambda(m^2+1)}{a}+\frac{12b\lambda(m^2+1)}{a}\tanh^2(\sqrt{-\lambda}(x+my+4b\lambda(m^2+1)t)), \quad (4.22)$$

For $\lambda>0, a\neq 0$, the solutions of Eq.(1.3) are written as

$$g_{6.3}=-\frac{12b\lambda(m^2+1)}{a}-\frac{12b\lambda(m^2+1)}{a}\cot^2(\sqrt{\lambda}(x+my+4b\lambda(m^2+1)t)), \quad (4.23)$$

$$g_{6.4}=-\frac{12b\lambda(m^2+1)}{a}-\frac{12b\lambda(m^2+1)}{a}\tan^2(\sqrt{\lambda}(x+my+4b\lambda(m^2+1)t)), \quad (4.24)$$

For $\lambda=0$, the solution of Eq.(1.3) is

$$g_{6.5}=0. \quad (4.25)$$

4.2. Using the Modified Generalized Kudryashov Method

By taking $M=2$, Eq. (3.1) becomes

$$g(\xi)=s_0+\frac{s_1}{1+H(\xi)}+\frac{s_2}{(1+H(\xi))^2}, \quad (4.26)$$

Substituting Eq. (4.26) into Eq. (4.2) and using the Eq. (3.2), and furthermore collecting the coefficients of the same powers of $H(\xi)$ and making them equal to zero, the following systems of algebraic equations are obtained and written as

$$0=2b\delta m^2\tau s_1-4bm^2\tau^2 s_1+4bm^2\tau^2 s_2+2b\delta\tau s_1-4b\tau^2 s_2+as_0^2+2cs_0$$

$$\begin{aligned}0=&-8b\tau^2 s_2-4b\delta m^2\tau s_1+12b\delta m^2\tau s_2+2(\delta^2+2\tau\sigma)bs_1+2as_0s_1-4b\tau^2 s_1+4as_0^2+8cs_0\\&+2cs_1+12bs_2\tau\delta-4b\delta\tau s_1+2(\delta^2+2\tau\sigma)bm^2 s_1-4bm^2\tau^2 s_1-8bm^2\tau^2 s_2\end{aligned}$$

$$\begin{aligned}0=&-12bs_2\tau\delta+8(\delta^2+2\tau\sigma)bm^2 s_2+6bs_1\delta\sigma+2as_0s_2+6as_0s_1-6bs_1\tau\delta+6as_0^2+as_1^2\\&+12cs_0+6cs_1+2cs_2+8(\delta^2+2\tau\sigma)bs_2+6bm^2 s_1\delta\sigma-6bm^2 s_1\delta\tau-12bm^2 s_2\delta\tau\end{aligned}$$

$$
\begin{aligned}
0=&-4(\delta^2+2\tau\sigma)bs_2+4bm^2s_1\delta\sigma+20bm^2s_2\delta\sigma+4bs_1\sigma^2+6as_0s_1+4as_0s_2+2as_1s_2\\
&-2(\delta^2+2\tau\sigma)bs_1+4as_0^2+2as_1^2+8cs_0+6cs_1+4cs_2+20bs_2\delta\sigma+4bs_1\delta\sigma+4bm^2s_1\sigma^2\\
&-2(\delta^2+2\tau\sigma)bm^2s_1-4(\delta^2+2\tau\sigma)bm^2s_2
\end{aligned}
$$

$$
\begin{aligned}
0=&-2bm^2s_1\delta\sigma-4bm^2s_2\delta\sigma+4bm^2s_1\sigma^2+12bm^2s_2\sigma^2-2bs_1\delta\sigma-4bs_2\delta\sigma+4bs_1\sigma^2\\
&+12bs_2\sigma^2+as_0^2+2as_0s_1+2as_0s_2+as_1^2+2as_1s_2+as_2^2+2cs_0+2cs_1+2cs_2
\end{aligned}
$$

The unknowns are s_0, s_1, s_2 and c. Using Maple program, the following two cases are obtained.

Case 7

$$
\begin{aligned}
&c=b(\delta^2m^2-4m^2\sigma\tau+\delta^2-4\sigma\tau),\\
&s_0=-\frac{2b(\delta^2m^2-6m^2\delta\tau+2m^2\sigma\tau+6m^2\tau^2+\delta^2-6\delta\tau+2\tau\sigma+6\tau^2)}{a},\\
&s_1=\frac{12b(\delta m^2-2m^2\tau+\delta-2\tau)(\delta-\sigma-\tau)}{a},\\
&s_2=-\frac{12b(m^2+1)(\delta^2-2\delta\sigma-2\tau\delta+\sigma^2+2\tau\sigma+\tau^2)}{a}.
\end{aligned}
$$

Therefore, the solutions of Eq.(1.3) are written as

$$
\begin{aligned}
g_{7.1}(\xi)=&-\frac{2b(\delta^2m^2-6m^2\delta\tau+2m^2\sigma\tau+6m^2\tau^2+\delta^2-6\delta\tau+2\tau\sigma+6\tau^2)}{a}\\
&+\frac{24b\tau(\delta m^2-2m^2\tau+\delta-2\tau)(\delta-\sigma-\tau)}{a\left(\sqrt{4\tau\sigma-\delta^2}\tan\left(\frac{1}{2}\sqrt{4\tau\sigma-\delta^2}(E+x+my+b(m^2+1)(\delta^2-4\sigma\tau)t)\right)-\delta+2\tau\right)}\\
&-\frac{48b\tau^2(m^2+1)(\delta^2-2\delta\sigma-2\tau\delta+\sigma^2+2\tau\sigma+\tau^2)}{a\left(\sqrt{4\tau\sigma-\delta^2}\tan\left(\frac{1}{2}\sqrt{4\tau\sigma-\delta^2}(E+x+my+b(m^2+1)(\delta^2-4\sigma\tau)t)\right)-\delta+2\tau\right)^2}
\end{aligned}
\tag{4.27}
$$

$$
\begin{aligned}
g_{7.2}(\xi)=&-\frac{2b(\delta^2m^2-6m^2\delta\tau+6m^2\tau^2+\delta^2-6\delta\tau+6\tau^2)}{a}\\
&+\frac{12b(\delta m^2-2m^2\tau+\delta-2\tau)(\delta-\tau)(\tau e^{\delta(E+x+my+b\delta^2(m^2+1)t)}-1)}{a((\tau-\delta)e^{\delta(E+x+my+b\delta^2(m^2+1)t)}-1)}\\
&-\frac{12b(\delta^2m^2-2\delta m^2\tau+m^2\tau^2+\delta^2-2\tau\delta+\tau^2)(\tau e^{\delta(E+x+my+b\delta^2(m^2+1)t)}-1)^2}{a((\tau-\delta)e^{\delta(E+x+my+b\delta^2(m^2+1)t)}-1)^2}
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
g_{7.3}(\xi)=&-\frac{2b\delta^2(m^2+1)}{a}+\frac{12b\delta^2(m^2+1)(\delta-\sigma)}{a(e^{\delta(E+x+my+b\delta^2(m^2+1)t)}-\sigma+\delta)}\\
&-\frac{12b\delta^2(\delta^2m^2-2\delta m^2\sigma+m^2\sigma^2+\delta^2-2\delta\sigma+\sigma^2)}{a(e^{\delta(E+x+my+b\delta^2(m^2+1)t)}-\sigma+\delta)^2}
\end{aligned}
\tag{4.29}
$$

Case 8

$$
\begin{aligned}
&c=-b(\delta^2m^2-4m^2\sigma\tau+\delta^2-4\sigma\tau), s_0=\frac{12b\tau(\delta m^2-m^2\sigma-m^2\tau+\delta-\tau-\sigma)}{a},\\
&s_1=\frac{12b(\delta m^2-2m^2\tau+\delta-2\tau)(\delta-\sigma-\tau)}{a},\\
&s_2=-\frac{12b(m^2+1)(\delta^2-2\delta\sigma-2\tau\delta+\sigma^2+2\tau\sigma+\tau^2)}{a}.
\end{aligned}
$$

So the solutions of Eq.(1.3) are given as

$$g_{8.1}(\xi)=\frac{12b\tau(\delta m^2-m^2\sigma-m^2\tau+\delta-\tau-\sigma)}{a}$$
$$+\frac{24b\tau(\delta m^2-2m^2\tau+\delta-2\tau)(\delta-\sigma-\tau)}{a\left(\sqrt{4\tau\sigma-\delta^2}\tan\left(\frac{1}{2}\sqrt{4\tau\sigma-\delta^2}(E+x+my-b(m^2+1)(\delta^2-4\sigma\tau)t)\right)-\delta+2\tau\right)} \quad (4.30)$$
$$-\frac{48b\tau^2(m^2+1)(\delta^2-2\delta\sigma-2\tau\delta+\sigma^2+2\tau\sigma+\tau^2)}{a\left(\sqrt{4\tau\sigma-\delta^2}\tan\left(\frac{1}{2}\sqrt{4\tau\sigma-\delta^2}(E+x+my-b(m^2+1)(\delta^2-4\sigma\tau)t)\right)-\delta+2\tau\right)^2}$$

$$g_{8.2}(\xi)=\frac{12b\tau(\delta m^2-m^2\tau+\delta-\tau)}{a}$$
$$+\frac{12b(\delta m^2-2m^2\tau+\delta-2\tau)(\delta-\tau)(\tau e^{\delta(E+x+my-b\delta^2(m^2+1)t)}-1)}{a((\tau-\delta)e^{\delta(E+x+my-b\delta^2(m^2+1)t)}-1)} \quad (4.31)$$
$$-\frac{12b(\delta^2m^2-2\delta m^2\tau+m^2\tau^2+\delta^2-2\tau\delta+\tau^2)(\tau e^{\delta(E+x+my-b\delta^2(m^2+1)t)}-1)^2}{a((\tau-\delta)e^{\delta(E+x+my-b\delta^2(m^2+1)t)}-1)^2}$$

$$g_{8.3}(\xi)=\frac{12b\delta^2(m^2+1)(\delta-\sigma)}{a(e^{\delta(E+x+my-b\delta^2(m^2+1)t)}-\sigma+\delta)}$$
$$-\frac{12b\delta^2(\delta^2m^2-2\delta m^2\sigma+m^2\sigma^2+\delta^2-2\delta\sigma+\sigma^2)}{a(e^{\delta(E+x+my-b\delta^2(m^2+1)t)}-\sigma+\delta)^2} \quad (4.32)$$

4.3. Graphs and Discussion

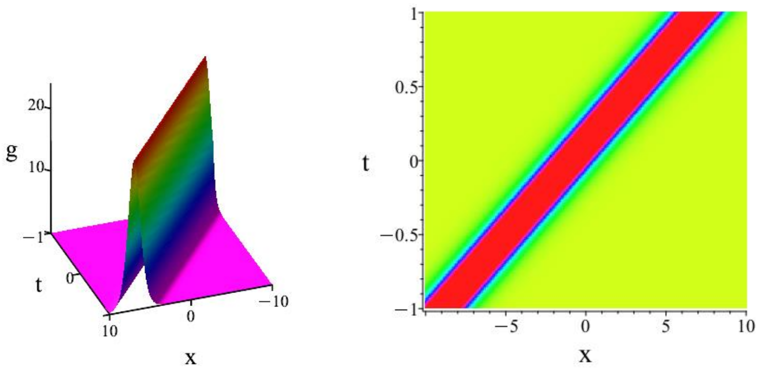
Since the tanh function and the coth function are reciprocals of each other, Eq. (4.8) is the same as Eq. (4.22), and Eq. (4.9) is the same as Eq. (4.21). Since the tan function and the cot function are reciprocals of each other, Eq. (4.10) is the same as Eq. (4.24), and Eq. (4.11) is the same as Eq. (4.23).

There is a relationship between trigonometric functions and hyperbolic functions as follows:

$$\tanh x=-i\tan ix,\ \coth x=i\cot ix. \quad (5.1)$$

Therefore, the forms of the solutions for Eq. (4.8) and Eq. (4.10), Eq. (4.9) and Eq. (4.11), Eq. (4.16) and Eq. (4.18), Eq. (4.17) and Eq. (4.19), Eq. (4.21) and Eq. (4.23), and Eq. (4.22) and Eq. (4.24) are the same.

Therefore, some partial solutions are chose to show the characters of the solutions, such as Eq. (4.8), Eq. (4.10), Eq. (4.16), Eq. (4.27), Eq. (4.29), and Eq. (4.32).



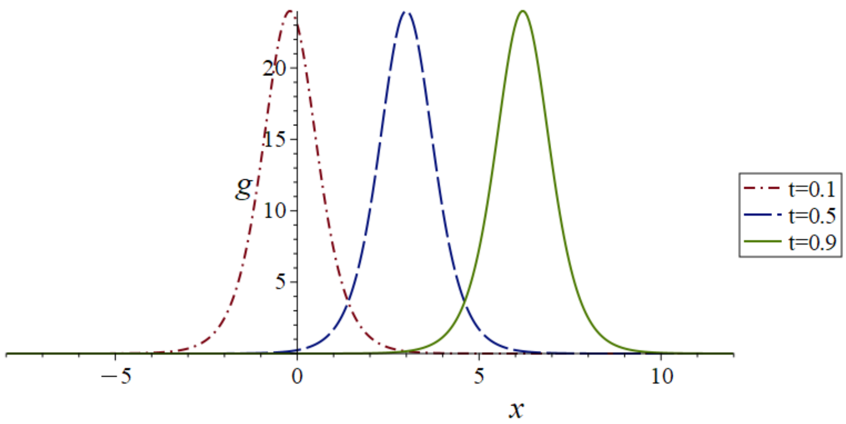


Fig. 1 3D, 2D and density plots of light soliton solution of Eq. (4.8) for $a=1, b=1, m=1, \lambda=-1, y=1.$

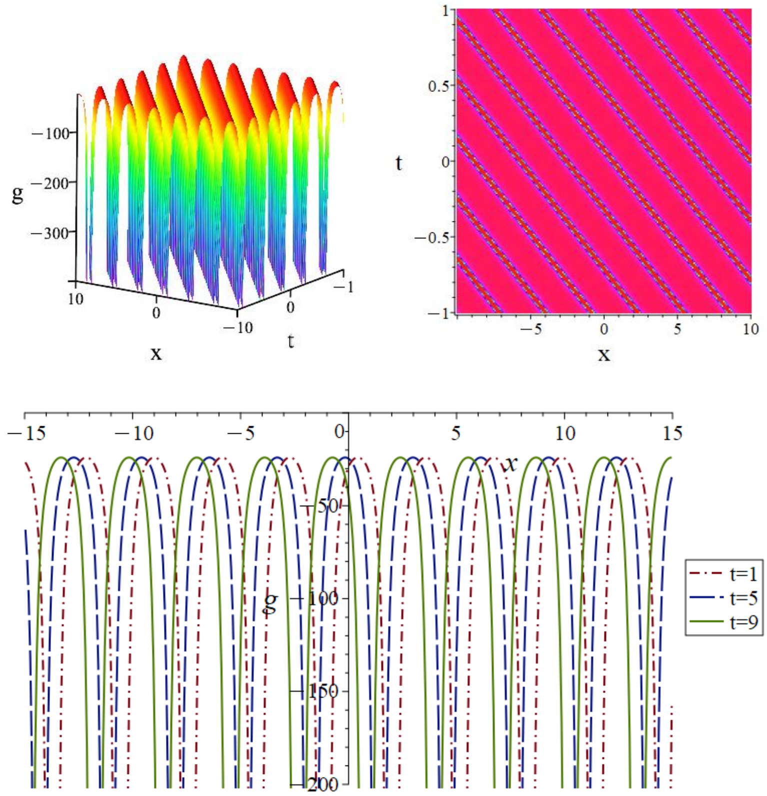
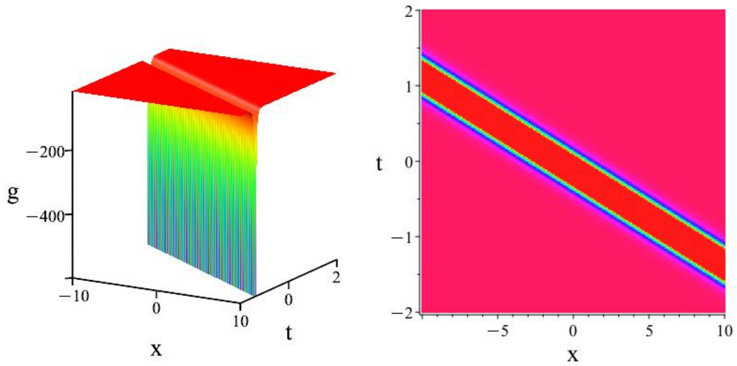


Fig. 2 3D, 2D and density plots of periodic function solution of Eq. (4.10) for $a=1, b=1, m=1, \lambda=1, y=1.$



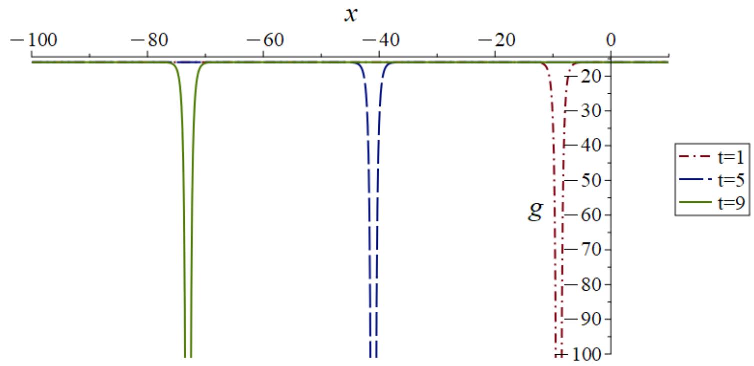


Fig. 3 3D, 2D and density plots of hyperbolic function solution of Eq. (4.16) for $a=1, b=1, m=1, y=1, \lambda=-1.$

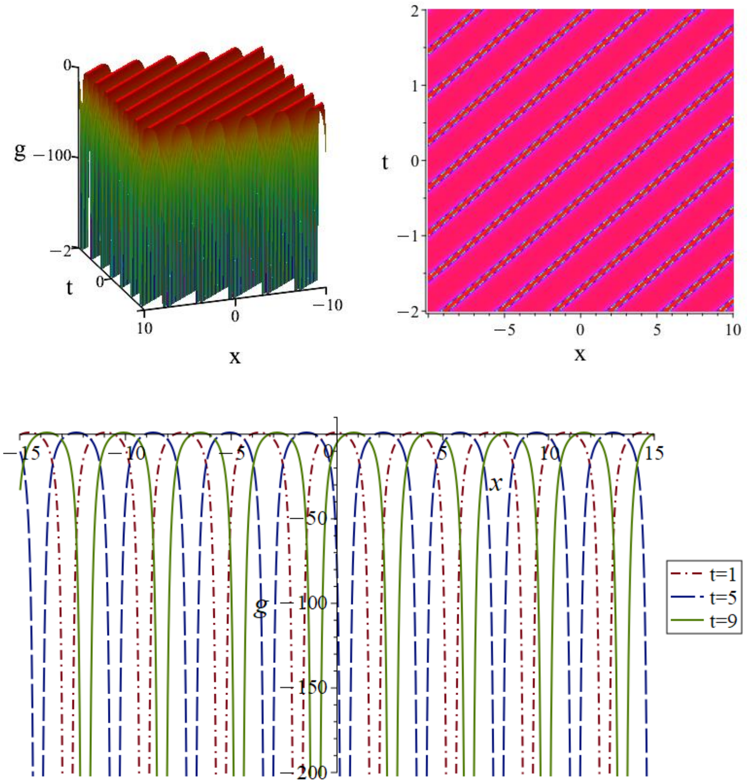
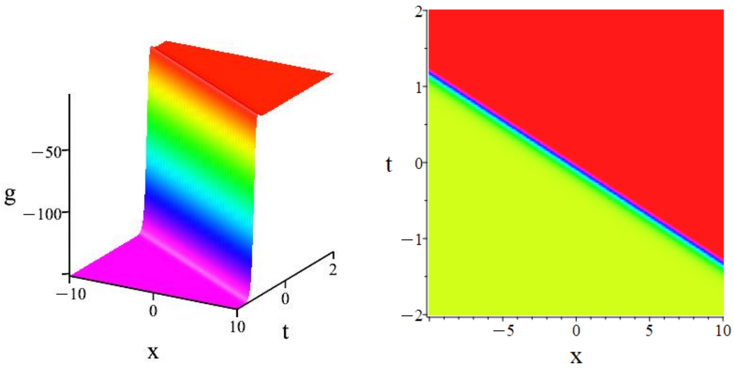


Fig. 4 3D, 2D and density plots of periodic function solution of Eq. (4.27) for $a=2, b=1, m=1, \tau=2, \sigma=1, \delta=1, E=-1, y=1.$



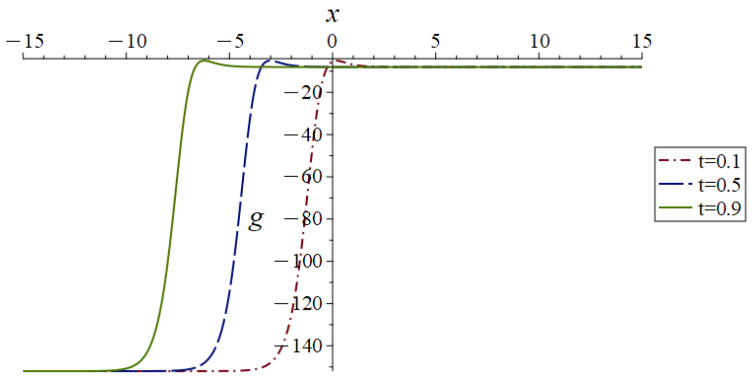


Fig. 5 3D, 2D and density plots of kink solution of Eq. (4.29) for $a = 2, b = 1, m = 1, \sigma = 0, \delta = 2, E = -1, y = 1.$

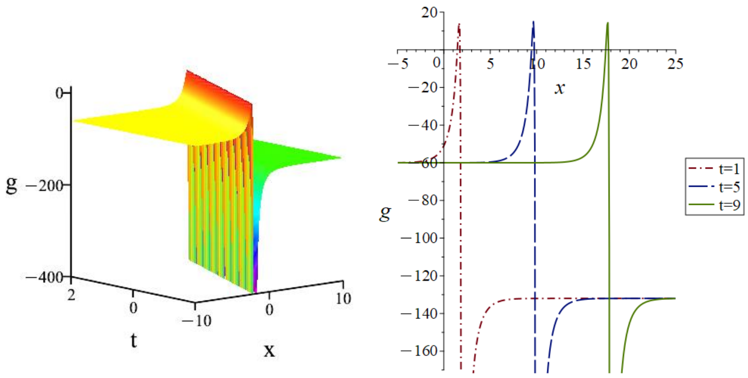


Fig. 6 3D and 2D of singular solution of Eq. (4.32) for $a = 2, b = 1, m = 1, \delta = 1, \tau = 2, \sigma = 0, E = -1, y = 1.$

5. Conclusion

This article provides a detailed introduction to the modified extended tanh-function method and the modified generalized Kudryashov method and menatime utilizes these two methods to obtain a variety of new exact solutions for the (2+1)-dimensional ZK equation, followed by an analysis of the solutions through graphical representations. The solutions to the (2+1)-dimensional ZK equation have profound physical significance and offer an important insights into nonlinear wave phenomena across various disciplines, thereby the work is very important for the theoretical understanding and practical applications of the (2+1)-dimensional ZK equation.

In future work, other mathematical techniques can be selected to find analytical solutions that describe solitary waves and other wave forms; numerical methods can also be employed for simulations to study the characteristics of complex waves. Exploring higher-dimensional ZK equations, such as the (3+1)-dimensional case, and their properties are also a promising research direction.

Funding Statement

The paper is a part of the research done within the project of the Doctoral Fund of Henan Polytechnic University (Grant NO. B2020-35) and is supported by the Fundamental Research Funds for the Universities of Henan Province (Grant NO. NSFRF230430).

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