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Original Article

Some Tri-Parametric Weighted Generalized Information Measures by Employing Two-Dimensional Probability Distribution

Rohit Kumar Verma[1], M. Bhagya Laxmi[2]

[1,2]*Department of Mathematics, Bharti Vishwavidyalaya, Durg, C.G., India.*

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***Abstract** - The fundamental idea in information theory literature, Shannon entropy has several applications in various scientific and technological fields. The difference in this entropy measure has been generalized by researchers using various methodologies. The purpose of this paper is to emphasize how important the concavity characteristic is. Thus, we have presented and examined three new generalized measures of probabilistic entropy with two dimensions, mostly based on the concavity of the entropy function postulate. We have also examined their significant and intriguing aspects.*

***Keywords** - Shannon entropy, Weighted entropy, Useful information measures.*

1. Introduction

The concept of information entropy plays a crucial role in statistical physics. Many issues were resolved using information entropy, a measure of the uncertainty and information provided by a probabilistic experiment. Originating from the well-known Boltzmann H-function, the Shannon entropy quickly gained popularity in a number of fields, particularly in the following: nuclear reaction theory (Bloch), learning theory (Watanabe), measurement theory, statistical physics (Ingarden, [lo], Jaynes), communication theory (Shannon, Feinstein), and mathematical statistics (Kullback), without mentioning other fields like linguistics, music, or social sciences. With no intention of providing an exhaustive list, a number of generalizations of the Shannon entropy were proposed, aside from the applications of classical information theory in each of these domains.

In a physical experiment, it is frequently quite difficult to ignore the subjective elements pertaining to the different objectives of the researcher. However, the perspective of a particular qualitative property may be very different from the various states of a physical system. All elementary events in statistical physics often have the same significance or are physically identical; however, this is not always the case. To explain the latter, each elementary occurrence must be given a qualitative weight in addition to its likelihood of occurring.

It is possible for an event's qualitative weight to be unrelated to its objective probability; for example, an event with a low probability could have a high weight, while an event with a high probability could have a very low weight.

Obviously, giving each primary school event a weight is not something that can be done easily. It is possible for these weights to be objective or subjective. As a result, the weight of one event may convey a qualitative objective feature as well as the subjective usefulness of the event in relation to the experimenter's objective. There may be a relationship between the subjective likelihood that an elementary event will occur and the weight assigned to it.

We shall suppose that these qualitative weight are nonnegative, finite, real numbers, the usual weights in physics or as utility in decision theory. Also, if one event is more relevant, significant, and useful (with respect to a goal or from a given qualitative point of view) than another, the weight of the first event will be greater than that of the second one. How to evaluate the amount of information supplied by a probability space, i.e. by a probabilistic experiment, whose elementary events are characterized both by their probabilities and by some qualitative (objective or subjective) weights? In particular, what is the amount of information supplied by a probabilistic experiment when the probabilities calculated by the experimenter (i.e. the subjective probabilities) do not coincide with the objective probabilities of these random events?

In this work, we will provide a formula for entropy, which is a measure of uncertainty or information provided by a probabilistic experiment that depends on the qualitative (objective or subjective) weights of the potential events and the probabilities of occurrences. Well, refer to this entropy as the weighted entropy. The weighted entropy's attributes, axiomatic treatment, and external properties will all be covered in the following paragraphs.

Examine a probabilistic physical experiment in which the objective probabilities of a finite number of elementary events $\omega_1, \omega_2, \dots, \omega_n$ are given by the numbers in the associated probability space.

$$p_k \geq 0, \forall k = 1,2, \dots, n \; ; \; \sum_{k=1}^{n} p_k = 1$$

The many elementary occurrences have varied (objective or subjective) weights, mostly dependent on the experimenter's objective or a qualitative aspect of the physical system under investigation. An event's weight might be either independent or dependent on its objective likelihood. To distinguish between the events$\omega_1, \omega_2, \dots, \omega_n$, f in a goal-directed experiment based on their significance concerning a particular qualitative feature of the physical system under consideration or their importance concerning the experimenter's goal, we will assign to each event ω_k a nonnegative number $\omega_k \geq 0$ that is directly proportional to the significance or importance indicated above. We shall call $\omega_k$the weight of the elementary eventω_k. We define the weighted entropy by the expression.

$$I_n = I_n(\omega_1, \omega_2, \dots, \omega_n; \, p_1, p_2, \dots, p_n) = -\sum_{k=1}^{n} w_k \, p_k \log p_k$$

Let us notice briefly some obvious properties of the weighted entropy.

1. $I_n(\omega_1, \omega_2, \dots, \omega_n; \, p_1, p_2, \dots, p_n) \geq 0$

2. If $\omega_1, \omega_2, \dots, \omega_n = \omega$ then

$$I_n(\omega_1, \omega_2, \dots, \omega_n; \, p_1, p_2, \dots, p_n) = -\omega \sum_{k=1}^{n} p_k \log p_k$$
$$= H_n(p_1, p_2, \dots, p_n)$$

Where H_n is the classical Shannon Entropy.

3. If $p_{k0} = 1, p_k = 0 \; (k = 1,2, \dots, n \, ; k \neq k_0)$

Then $I_n \, (\omega_1, \omega_2, \dots, \omega_n; p_1, p_2, \dots, p_n) = 0$

Whatever are the weights$\omega_1, \omega_2, \dots, \omega_n$

4. If $p_i = 0 \, , \omega_i \neq 0$ for every $i \in I$ and $p_j \neq 0 \, , \omega_j = 0$ for every $j \in J$,where

$I \cup J = \{1,2, \dots, n\}$,$I \cap J = \phi$

then$I_n(\omega_1, \omega_2, \dots, \omega_n; \, p_1, p_2, \dots, p_n) = 0$

5. $I_{n+1}(\omega_1, \omega_2, \dots \omega_n, \omega_{n+1}; \, p_1, p_2, \dots, p_n, 0) = I_n(\omega_1, \omega_2, \dots, \omega_n; \, p_1, p_2, \dots, p_n)$

Whatever are the weights $\omega_1, \omega_2, \dots \omega_n, \omega_{n+1}$ and the complete system of probabilities $p_1, p_2, \dots, p_n$.

6. For every nonnegative,real no. ζwe have

$$I_n(\zeta\omega_1, \zeta\omega_2, \dots, \zeta\omega_n; \, p_1, p_2, \dots, p_n) = \zeta I_n(\omega_1, \omega_2, \dots, \omega_n; \, p_1, p_2, \dots, p_n)$$

7.If the rule (2) for the weights holds, then

$$I_{n+1}(\omega_1, \omega_2, \dots \dots .., \omega_{n+1}, \omega', \omega''; \; p_1, p_2, \dots \dots .., p_{n-1}, p', p'') =$$
$$I_n(\omega_1, \omega_2, \dots \dots .., \omega_n; \; p_1, p_2, \dots \dots .., p_n) + I_2 p_n \left(\omega', \omega'', \frac{p'}{p_n}, \frac{p''}{p_n}\right)$$

2. Weighted Entropy–Type Measures

Definition 2.1. Weighted Entropy

Let us define the weighted entropy(WE) as

$$H_\varphi^W(P) = -\sum_i \varphi(x_i)\, p(x_i) \log p(x_i)$$

Here, we introduce a nonnegative weight function (WF),$x_i \mapsto \varphi(x_i)$, which denotes the value or usefulness of an outcome x_i. The differential entropy of a probability density function (PDF) f can be calculated in a manner akin to this. Give the weighted differential definition.

(WDE) entropy as

$$H_\varphi^W(f) = -\int \varphi(x_i)\, f(x_i) \log f(x_i)$$

Definition 2.2. Weighted Shannon Entropy

Consider a stochastic source that is described by a discrete random variable X of n possible events, with distribution P_X, probability mass function $p = (p_1, p_2, \dots \dots, p_n)^T$ and $(w_1, w_2, \dots \dots, w_n)^T$ as a vector of weights associated with these states, $w \geq 0$, I =1,2.3,…..,n. The weighted Shannon entropy measured is defined by

$$H^w(X) = \sum_{i=1}^{n} w_i p_i \log\left(\frac{1}{p_i}\right)$$

Definition 2.3. Weighted Differential Entropy

The weighted differential entropy defined by Das for random variable $\gamma$with weighted function $w(x) = x$as

$$\psi^w(\gamma) = -\frac{H^w(\gamma) + \int_c x f_\gamma(x) \ln f_\gamma(x) dx - E[\gamma] \ln E[\gamma]}{E[\gamma]}$$

$$= \frac{H^w(\gamma)}{E[\gamma]} + \ln E[\gamma] - \frac{\delta^w}{E[\gamma]}$$

Where

$$\delta^w = \int_c x f_\gamma(x) \ln f_\gamma(x) dx = E(\gamma \ln \gamma)$$

Definition 2.4. Weighted Generalized Entropy

Let X be an absolutely continuous nonnegative random variable having a probability density function $f_X(X)$. Then Shannon's entropy is defined as

$$H(X) = -\int_0^\infty f_X(X) \ln f_X(X)\, dx = -E\, [\ln f_X(X)]$$

3. Our Results

In information theory and other fields, Shannon's concept of entropy, which he established in 1948, is crucial. If X is an absolutely continuous nonnegative random variable with partial differential function f(x), then Shannon's entropy is defined as

$$H(X) = -\int_0^\infty f(x) \log f(x)\, dx$$
$$= -E\, [\log f\, (X)] \; (1)$$

Belis and Guiasu (1968) established the idea of weighted entropy because a shift-dependent measure of uncertainty is preferred in particular real-world scenarios, like reliability or neurobiology.

$$H^W(X) = -\int_0^{\infty} x\, f(x) \log f(x)\, dx (2)$$

The probability density function and absolutely continuous nonnegative random variable are denoted by the partially differential function and random variable X, respectively, throughout this text. Numerous extensions of (1) were put forth in information theory, notably, Consequently, a new three-parameter generalization of this uncertainty measure is created in this article as follows:

$$H_{\alpha,\beta,\gamma}(X) = \frac{1}{1-\alpha} \log \int_0^{\infty} \frac{f^{\alpha+\beta+\gamma-1}(x)}{f^{\beta+\gamma}(x)}\, dx$$

$$= \frac{1}{1-\alpha} \int_0^{\infty} \log \frac{f^{\alpha+\beta+\gamma-1}(x)}{f^{\beta+\gamma}(x)}\, dx \;, \alpha \neq 1, \beta \geq 0, \gamma \geq 0, \beta+\gamma-1 \geq 0 \quad (1)$$

As $\alpha \to 1$the measure reduces to

$$H_{\beta,\gamma}(X) = -\frac{\int_0^{\infty} f^{\beta+\gamma}(x) \log f(x)}{\int_0^{\infty} f^{\beta+\gamma}(x)} \quad (2)$$

If $\beta = 0$ and $\gamma = 1$, then the above measures reduces to Shannon's measures of entropy

Where $H(X) = -\int_0^{\infty} f(x) \log f(x)$

The measure (1) reduces to Renyi's measures when $\beta + \gamma = 1$

As stated by Ebrahimi (1996), The metric is not appropriate for determining the uncertainty over a system's remaining life $X_t = |X - t|, X > t$ if the system with lifespan X is still alive at time t.Consequently, the residual lifetime $X_t = |X - t|, X > t$ concept, which is provided by

$$H(X;t) = -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)}\, dx \quad (3)$$

The weighted residual entropy version of (3) by Di Crescenzo and Longobardi (2006) is given by

$$H^w(X;t) = -\int_t^{\infty} x \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)}\, dx \quad (4)$$

The two-dimensional version of (1) can be defined as

$$H^W_{\alpha,\beta,\gamma}(X,Y) = \frac{1}{1-\alpha} \log \int_0^{\infty}\int_0^{\infty} xy\, \frac{f^{\alpha+\beta+\gamma-1}(x,y)}{f^{\beta+\gamma}(x,y)} \mathrm{dy\, dx}$$

$$\forall\, \alpha \neq 1, \beta \geq 0, \gamma \geq 0, \beta+\gamma-1 \geq 0$$

$$= \frac{1}{1-\alpha} \left\{ xy \int_0^{\infty}\int_0^{\infty} \left[\log\big(f(x,y)\big)^{\alpha+\beta+\gamma-1} - \log\big(f(x,y)\big)^{\beta+\gamma} \right] dy\, dx \right\} \quad (5)$$

The residual version of (5) is given by

$$H^W_{\alpha,\beta,\gamma}(X,Y;t) = \frac{1}{1-\alpha} \left\{ xy \int_0^{\infty}\int_0^{\infty} \left[\log\left(\frac{f(x,y)}{\bar{F}(t)}\right)^{\alpha+\beta+\gamma-1} - \log\left(\frac{f(x,y)}{\bar{F}(t)}\right)^{\beta+\gamma} \right] dy\, dx \right\}$$

$$\forall\, \alpha \neq 1, \beta \geq 0, \gamma \geq 0, \beta+\gamma-1 \geq 0$$

$$\text{Or } (1-\alpha) H^W_{\alpha,\beta,\gamma}(X,Y;t) = xy \int_0^{\infty}\int_0^{\infty} \log\left(\frac{f(x,y)}{\bar{F}(t)}\right)^{\alpha+\beta+\gamma-1} dy\, dx -$$

$$xy\int_0^\infty\int_0^\infty \log\left(\frac{f(x,y)}{\bar{F}(t)}\right)^{\beta+\gamma} dy\,dx$$

$$= xy\,(\alpha+\beta+\gamma-1)\int_0^\infty\int_0^\infty \log\left(\frac{f(x,y)}{\overline{F\,(t)}}\right) dy\,dx -$$

$$xy\,(\beta+\gamma)\int_0^\infty\int_0^\infty \log\left(\frac{f(x,y)}{\overline{F\,(t)}}\right) dy\,dx$$

$$= (\beta+\gamma)xy\int_0^\infty\int_0^\infty \log\left(\frac{f(x,y)}{\overline{F\,(t)}}\right) dy\,dx + xy(\alpha-1)\int_0^\infty\int_0^\infty \log\left(\frac{f(x,y)}{\overline{F\,(t)}}\right) dy\,dx -$$

$$xy\,(\beta+\gamma)\int_0^\infty\int_0^\infty \log\left(\frac{f(x,y)}{\overline{F\,(t)}}\right) dy\,dx$$

$$= (\alpha-1)xy\int_0^\infty\int_0^\infty \log\left(\frac{f(x,y)}{\overline{F\,(t)}}\right) dy\,dx$$

$$H^W_{\alpha,\beta,\gamma}(X,Y;t) = -xy\int_0^\infty\int_0^\infty \log\left(\frac{f(x,y)}{\overline{F\,(t)}}\right) dy\,dx$$

$$= -xy\int_0^\infty\int_0^\infty \log\big(f(x,y)\big)dy\,dx + xy\int_0^\infty\int_0^\infty \log\bar{F}\,(t)dy\,dx \qquad (6)$$

"Weighted Verma's entropy of order α and β is defined as follows: X is a nonnegative random variable with a probability density function f(x) and an absolutely continuous distribution function F(x).

$$H^W_{\alpha,\beta}(X) = -\frac{1}{\beta-\alpha}\log\int_0^\infty x f^{\alpha+\beta-1}(x)\,dx \quad ,\beta-1<\alpha<\beta\ ,\beta\geq 1$$

Then for $H^W(X)$defined in (2)

As $\alpha\to 1,\beta\to 1$

$$H^W_{\alpha,\beta}(X) = \frac{H^W(X)}{E(X)} \qquad (7)$$

Therefore, from (6) using equation (7)

$$H^W_{\alpha,\beta,\gamma}(X,Y;t) = \frac{H^W(X,Y)}{E(X,Y)} + xy\int_0^\infty\int_0^\infty \log\bar{F}\,(t)dy\,dx$$

$$= \frac{H^W(X,Y)}{E(X,Y)} + xy\int_0^\infty\int_0^\infty \log\bar{F}\,(t)dy\,dx - xy\int_0^\infty\int_0^\infty \log f\,(x)dy\,dx +$$

$$xy\int_0^\infty\int_0^\infty \log f\,(x)dy\,dx$$

$$= \frac{H^W(X,Y)}{E(X,Y)} - xy\int_0^\infty\int_0^\infty \log\frac{f(x)}{\bar{F}(t)}\,dy\,dx + xy\int_0^\infty\int_0^\infty \log f\,(x)dy\,dx$$

$$= \frac{H^W(X,Y)}{E(X,Y)} - \frac{H^W(X\,;t)}{E(X;t)} + y\int_0^\infty\left[\int_0^\infty x\log f(x)\,dx\right]dy$$

$$= \frac{H^W(X,Y)}{E(X,Y)} - \frac{H^W(X\,;t)}{E(X;t)} - y\int_0^\infty H^W\,(X)dy$$

$$= \frac{H^W(X,Y)}{E(X,Y)} - \frac{H^W(X\,;t)}{E(X;t)} - H^W(X)\int_0^t\int_t^\infty y\,dy$$

$$= \frac{H^W(X,Y)}{E(X,Y)} - \frac{H^W(X\,;t)}{E(X;t)} - \frac{t^2}{2}H^W(X)$$

Or $\quad H^W_{\alpha,\beta,\gamma}(X,Y;t) = \frac{H^W(X,Y)}{E(X,Y)} - \frac{H^W(X\,,t)}{E(X,t)} - \frac{t^2}{2}H^W(X)$

Consequently, we note that the formula presented in equation (5) is a generalized probabilistic entropy measure. Next, we examine a few significant characteristics of this overall measurement

The measure (5) satisfies the following properties:

The Measure $H_{\alpha,\beta,\gamma}(P)$, where $P = p_1 + p_2 + \cdots \ldots. + p_n, \sum_{i=1}^n p_i = 1$ is a probability distribution, as characterized in the preceding section, that satisfies certain properties, which are given in the following.

3.1. Non-negative

The measure $H^w_{\alpha,\beta,\gamma}(P)$ is non-negative for $\alpha \neq \beta \neq \gamma, \quad \alpha,\beta,\gamma > 0$

I) $\boldsymbol{\alpha} > \gamma\,;\ \beta < \gamma$

$$\Rightarrow \frac{\alpha}{\gamma} > 1\ , \qquad \frac{\beta}{\gamma} < 1$$

$$\Rightarrow \frac{1}{1-{}^{\alpha}/_{\gamma}}\log\sum_{i=1}^n (p_i)^{\frac{\alpha}{\gamma}} < 1\ , \frac{1}{1-{}^{\beta}/_{\gamma}}\log\sum_{i=1}^n (p_i)^{\frac{\beta}{\gamma}} > 1$$

$$\Rightarrow \left\{\left[\frac{1}{1-{}^{\alpha}/_{\gamma}} - \frac{1}{1-{}^{\beta}/_{\gamma}}\right]\log\sum_{i=1}^n\left[(p_i)^{\frac{\alpha}{\gamma}} - (p_i)^{\frac{\beta}{\gamma}}\right]\right\} < 0$$

$$\Rightarrow \left\{\left[\frac{\frac{\alpha}{\gamma} - \frac{\beta}{\gamma}}{\frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma^2}}\right]\log\sum_{i=1}^n\left[(p_i)^{\frac{\alpha}{\gamma}} - (p_i)^{\frac{\beta}{\gamma}}\right]\right\} < 0$$

$$\Rightarrow \left\{\left[\frac{\gamma(\alpha-\beta)}{(\gamma-\alpha)(\gamma-\beta)}\right]\log\sum_{i=1}^n\left[(p_i)^{\frac{\alpha}{\gamma}} - (p_i)^{\frac{\beta}{\gamma}}\right]\right\} < 0$$

$$\Rightarrow [\gamma(\alpha-\beta)][(\gamma-\alpha)(\gamma-\beta)]^{-1}\log\sum_{i=1}^n\left[(p_i)^{\frac{\alpha}{\gamma}} - (p_i)^{\frac{\beta}{\gamma}}\right] < 0$$

$$\Rightarrow [\gamma(\alpha-\beta)]\log\sum_{i=1}^n\left[(p_i)^{\frac{\alpha}{\gamma}} - (p_i)^{\frac{\beta}{\gamma}}\right] > 0$$

II) Similarly for $\boldsymbol{\alpha} < \gamma\,;\ \beta > \gamma$ **,** we get

$$[\gamma(\alpha-\beta)]\log\sum_{i=1}^n\left[(p_i)^{\frac{\alpha}{\gamma}} - (p_i)^{\frac{\beta}{\gamma}}\right] > 0$$

Therefore, from case I ,case II ,we get

$$H^w_{\alpha,\beta,\gamma}(P) \geq 0$$

3.2. Additive Property

The nature of measure (2.1) is additive. The following joint entropy is taken into consideration in order to demonstrate this additivity feature.

$$\mathrm{H}_{\mathrm{mn}}^{\mathrm{w}}(P * Q;\alpha,\beta,\gamma) = G_n(P;\alpha,\beta,\gamma)H_m(Q;\alpha,\beta,\gamma) + G_m(Q;\alpha,\beta,\gamma)H_n(P;\alpha,\beta,\gamma)$$
$$\alpha,\beta,\gamma > 0$$

where, $G_n(P;\alpha,\beta,\gamma) = \frac{1}{2}\sum_{i=1}^{n}\left[(p_i)^{\frac{\alpha}{\gamma}} + (p_i)^{\frac{\beta}{\gamma}}\right];\ \alpha,\beta,\gamma > 0$

from equation (2.5) we get

$$= \frac{1}{1-\alpha}\left\{xy\int_0^\infty\int_0^\infty\left[\log\big(f(x,y)\big)^{\alpha+\beta+\gamma-1} - \log\big(f(x,y)\big)^{\beta+\gamma}\right] dy\,dx\right\} = \frac{1}{1-\alpha}\left\{p\,q\int_0^\infty\int_0^\infty\left[\log\big(f(p,q)\big)^{\alpha+\beta+\gamma-1} - \log\big(f(p,q)\big)^{\beta+\gamma}\right] dq\,dp\right\}.\frac{1}{2}\sum_{i=1}^{n}\left[(p_i)^{\frac{\alpha}{\gamma}} + (p_i)^{\frac{\beta}{\gamma}}\right]$$

$$= \frac{1}{1-\alpha}\left\{q\int_0^\infty \log\frac{\big(f(p,q)\big)^{\alpha+\beta+\gamma-1}}{\big(f(p,q)\big)^{\beta+\gamma}}\right\} + \frac{1}{1-\alpha}\left\{p\int_0^\infty \log\frac{\big(f(p,q)\big)^{\alpha+\beta+\gamma-1}}{\big(f(p,q)\big)^{\beta+\gamma}}\right\}$$

Where, $\sum_{i=1}^{n} p_i = 1$

$$= H_m^w(Q;\alpha,\beta,\gamma) + H_n^w(P;\alpha,\beta,\gamma)$$

Thus, the entropy is given by

$$\mathrm{H}_{\mathrm{mn}}^{\mathrm{w}}(P * Q;\alpha,\beta,\gamma) = H_m^w(Q;\alpha,\beta,\gamma) + H_n^w(P;\alpha,\beta,\gamma)$$

3.3. Sub Additive

We have

$$\mathrm{H}_{\mathrm{mn}}^{\mathrm{w}}(P * Q;\alpha,\beta,\gamma) = G_n(P;\alpha,\beta,\gamma)H_m(Q;\alpha,\beta,\gamma) + G_m(Q;\alpha,\beta,\gamma)H_n(P;\alpha,\beta,\gamma)$$
$$\alpha,\beta,\gamma > 0$$

where $G_n(P;\alpha,\beta,\gamma) = \frac{1}{2}\sum_{i=1}^{n}\left[(p_i)^{\frac{\alpha}{\gamma}} + (p_i)^{\frac{\beta}{\gamma}}\right] \leq 1\ for\ \alpha,\beta \geq \gamma$

Therefore

$$\mathrm{H}_{\mathrm{mn}}^{\mathrm{w}}(P * Q;\alpha,\beta,\gamma) \leq H_m^w(Q;\alpha,\beta,\gamma) + H_n^w(P;\alpha,\beta,\gamma)$$

3.4. Maximum Value

To find the maximum value of (2.5), we apply Lagrange's method of maximum multipliers and differentiating (2.5) with respect to p_i and taking $\frac{\partial H_{\alpha,\beta,\gamma}^{w}}{\partial p_i} = 0$, we get

Let $f(x,y) = \frac{1}{1-\alpha}\left\{xy\int_0^\infty\int_0^\infty\left[\log\big(f(x,y)\big)^{\alpha+\beta+\gamma-1} - \log\big(f(x,y)\big)^{\beta+\gamma}\right] dy\,dx\right\}$

So, $f(p) = \frac{1}{1-\alpha}\left[\sum_{i=1}^{n} p_i \log(p_i)^{\alpha+\beta+\gamma-1}\right] - \frac{1}{1-\alpha}\left[\sum_{i=1}^{n} p_i \log(p_i)^{\beta+\gamma}\right]$

$$-\lambda\sum_{i=1}^{n}(p_i - 1)$$

$$= \frac{(\alpha+\beta+\gamma-1)}{1-\alpha}\left[\sum_{i=1}^{n} p_i \log(p_i)\right] - \frac{\beta+\gamma}{1-\alpha}\left[\sum_{i=1}^{n} p_i \log(p_i)\right] - \lambda\sum_{i=1}^{n} p_i - \lambda$$

$$= \frac{(\alpha+\beta+\gamma-\beta-\gamma-1)}{1-\alpha}\left[\sum_{i=1}^{n} p_i \ \log(p_i)\right] - \lambda\sum_{i=1}^{n} p_i - \lambda$$

$$= \frac{\alpha-1}{1-\alpha}\sum_{i=1}^{n} p_i \ \sum_{i=1}^{n}\log(p_i) - \lambda\sum_{i=1}^{n} p_i - \lambda$$

$$= -\sum_{i=1}^{n}\log(p_i) - \lambda - \lambda$$

Taking $\frac{\partial f(p)}{\partial p_i} = 0, \forall \ i = 1,2,3, \dots\dots, n$, we get

$p_1 = p_2 = \cdots \dots\dots\dots = p_n = \frac{1}{n}$and $\sum_{i=1}^{n} p_i = 1$.

Therefore $\frac{\partial f(p)}{\partial p_i} =$ -log $\left(\frac{1}{n}\right)$ -2λ

$$= \log n - 2\lambda$$

$$= \frac{-1}{n^2} < 0$$

which shows that the maximum value is a concave function of n.

4. Conclusion

A weighted generalized entropy of order $\alpha, \beta,$ and type γ its residual version were developed. Several important features and inequalities of the weighted measure were investigated, and the suggested measure characterizes the distribution function uniquely. The expressions of these measures were taken into consideration for a few specific distributions. The present communication introduces a new 'useful weighted generalized information measure, i.e.,$H^{W}_{\alpha,\beta,\gamma}(X,Y)$ of order α, β and γ. The properties of $H^{W}_{\alpha,\beta,\gamma}(X,Y)$were considered Further, the behavior of $H^{W}_{\alpha,\beta,\gamma}(X,Y)$ at different cases of α, β and γwere studied.

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