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Review Article

Some Conditions that Turn a Certain Semigroup into a Group

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Abstract *- Semigroups are an important class of algebraic structures. One of the most important subclasses of semigroups are groups. There are some subclasses of semigroups that are "close" to groups, such as regular semigroups, orthodox semigroups and inverse semigroups. Finding conditions for semigroups that turn them into groups is interesting and important. In this paper, we give our original proofs of some propositions, which show how certain semigroups turn into groups, satisfying some conditions. Among other things, we give a counterexample to show that there is a cancellative and infinite semigroup, which is not a group.*

Keywords *- Semigroup, Regular semigroup, Inverse semigroup, Cancellative semigroup, Group.*

1. Introduction

Semigroup theory plays an important role in the development of abstract algebra. Semigroups have many applications in science and technology. There are some special semigroups, such as regular semigroups [1], orthodox semigroups [3], [4] and inverse semigroups [5], [6], which are not so far from groups. It is of interest to find conditions which turn certain semigroups into groups. In this paper, we give our original versions for some statements that show when a semigroup is a group, and one can find different proof ways.

2. Materials and Methods

Let S be a set and "$\cdot$" a binary operation on it. The structure $(S,\cdot)$ is called *semigroup* if this operation has the *associative* property, i.e., for each of the three elements $a, b, c \in S,$ it holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. An element a S is called an *idempotent* element [2] if $a \cdot a = a$. If a is an element for which at least one element exists in S, then the element is called a *regular* element. If each element of the semigroup S is regular, then S it is called a *regular* semigroup. Let E be the set of idempotent elements of the semigroup S. If E it is a *subsemigroup* of S, then S it is called an *orthodox* semigroup.

Furthermore, if E is a subsemigroup of S and each two elements of E *commute* with each other, then S is called an *inverse* semigroup. From the above definitions, it is evident that the class of the inverse semigroups is a subclass of the class of the orthodox semigroups, and the last one is a subclass of the class of the regular semigroups. If a semigroup S has an element e such that for each element a of $S,$ we have $a \cdot e = e \cdot a = a,$ then the element e of S is called *identity* element of the semigroup S. If there is an element b of S such that $a \cdot b = b \cdot a = e$ where e the identity element of the semigroup is, then it is called the inverse element a. A semigroup G is called *a group if it has the identity element and each element has a unique inverse element. This unique inverse of the element* is usually denoted by a^{-1}. The elements a a^{-1} are inverses of each other. If a semigroup S has the property that for each of three elements $a, b, c \in S,$ it holds the implication $ca = cb \Rightarrow a = b,$ it is called *left cancellative* [2], while if $ac = bc \Rightarrow a = b,$ it is called *right* cancellative [2].

Furthermore, if it is *both* left and right cancellative [2], it is called *cancellative* semigroup. If $\mathcal{L}$ and $\mathcal{R}$ are the known Green's relations [1], the semigroup S is called *left simple* [1],[2] if it has the property $\mathcal{L} = S \times S$, *right simple* if $\mathcal{R} = S \times S$ and *simple* if it is both left and right simple, i.e. $\mathcal{L} = \mathcal{R} = S \times S$, where $S \times S$ is the cartesian product of the set S. All unexplained concepts and statements were taken from [2] and [4].

3. Results and Discussion

In this section, we present our approach to the proof of some propositions, which show when a certain semigroup turns into a group that satisfies some conditions.

Proposition 3.1. *If a semigroup S is left and right simple, then it is a group.*

Proof. Let *S* be a left and right simple semigroup. Hence, we get $\mathcal{L} = S \times S$ and $\mathcal{R} = S \times S$, i.e. $a\mathcal{L}b$ and $a\mathcal{R}b$, for each of two elements a, b of *S*. Hence, it follows that:

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} = (S \times S) \cap (S \times S) = S \times S.$$

From Green's Theorem [4] (Theorem 2.2.5), we deduce that for each $\mathcal{H}$-class *H* on *S*, we have either $H^2 \cap H = \emptyset$ or $H^2 = H$, and if the last equality holds, then *H* is a subsemigroup of *S*. But, in our situation, where $\mathcal{H} = \mathcal{L} \cap \mathcal{R} = S \times S$, we have only one *H*-class of equivalence that is $H = S$, furthermore $H^2 \subset S$ and obviously $H^2 \cap H = \emptyset$. So, *H* is a group, and consequently, *S* is also a group. ■

Theorem 3.2 *Let S be as finite semigroup. The following propositions are equivalent*:

a. *S is a group*

b. *S is left and right cancellative semigroup.*

Proof. *Clearly*, if *S* is a finite semigroup that is a group, then it is a left and right cancellative semigroup.

Conversely, suppose that *S* is a finite, left and right cancellative semigroup. Let *n* be the number of elements of *S*. In these conditions, we must prove that *S* is a group. Let a be any element of *S* and consider the powers of this element: $a, a^2, a^3, \dots, a^r$, ... Since *S* is a finite semigroup, there exists a number $k \in N$, such that the elements $a, a^2, a^3, \dots\dots, a^k$ are all different from each other and a^{k+1} will be one of these elements, i.e. $a^{k+1} = a^p$ for some $p \in N$ and $1 \leq p \leq k$. It is evident that $1 \leq k \leq n$. Let now see that $G = \{a, a^2, a^3, \dots\dots, a^k\}$ is a subgroup of *S*. *Indeed*, by the property of the number *k*, we have $a^{k+1} = a^p$ Where $1 \leq p \leq k$, and we can obtain:

$$a^{k+1} = a^p \Rightarrow a^{p-1}a^{k-p+2} = a^{p-1}a$$

And from the last equality, since *S* is a left cancellative semigroup, we get $a^{k-p+2} = a$. Again, from the property of *k*, we have $k - p + 2 = k + 1$ or $p = 1$ and hence. $a^{k+1} = a$. Now, assume that. $a^r \in G$ where $1 \leq r \leq k$. We have:

$$a^r a^k = a^{r-1}a^{k+1} = a^{r-1}a = a^r \text{ and } a^k a^r = a^{k+1}a^{r-1} = aa^{r-1} = a^r.$$

From the above equalities, we deduce that. $a^k = e$ is the identity element of *G*. Furthermore, for each $a^r \in G$ where $1 \leq r \leq k - 1$, we see that $a^r a^{k-r} = a^k = e$ and also $a^{k-r}a^r = a^k = e$. So, *G* is a subgroup of the semigroup *S* with respect to the operation of this semigroup. Now, if $G = S$, we have proved that *S* is a group. If $G \neq S$, then there exists an element $b \in S$ such that $b \notin G$. Similarly to the element *a*, we can prove that $G' = \{b, b^2, b^3, \dots\dots, b^q\}$ is also a subgroup of the semigroup *S*, where $b^q = e'$ will be the identity element of *G'*, and for each element $b^r \in G', b^{q-r}$ For $1 \leq r \leq q$ will be its inverse element. Now, we prove that the groups *G* and *G'* have the same identity element, i.e. $e = e'$. Indeed:

$$a^{k+1}b = a^k(ab) = e(ab) = ab \Rightarrow b(a^{k+1}b) = b(ab) \Rightarrow (ba^k)(ab) = b(ab) \Rightarrow ba^k = b \quad (1)$$

since *S* is a right cancellative semigroup. But, on the other hand, we have

$$b^{q+1} = b \quad (2)$$

Hence, from (1) and (2) we obtain that $b^{q+1} = ba^k$ And since *S* is a left cancellative semigroup, it follows that. $b^q = a^k$ or $e' = e$, which means the groups G and G' have the same identity element e. So, we deduce that all the subgroups of type G have the same identity element; *e* is the identity element of *S*. On the other hand, each element $s \in S$ will belong to one of the subgroups of the type *G* of *S*, therefore *s* has a unique inverse element as we showed above. Finally, we conclude that *S* is a group. ■

Corollary 3.3 *The property "finite" of the semigroup S in proposition 2 is necessary and might not be removed from this proposition.* ■

We can show this with a counterexample as follows:

Counter example Let $\mathbb{Z}^*$ Be the set of nonzero integers and "$\cdot$" the usual multiplication on it. Take into consideration the semigroup. $(\mathbb{Z}^*, \cdot)$. This semigroup is both left and right-cancellative but not finite. Obviously, $\mathbb{Z}^*$ is infinite. Let $a, b, c \in \mathbb{Z}^*$. We have to show that $ac = bc \Rightarrow a = b$. We use contrapositive proof to prove the implication. There are four cases, as follows:

$$a > b \wedge c > 0 \Rightarrow ac > bc$$
$$a > b \wedge c < 0 \Rightarrow ac < bc$$
$$a < b \wedge c > 0 \Rightarrow ac < bc$$
$$a < b \wedge c < 0 \Rightarrow ac > bc.$$

In all cases, we have a contradiction with the fact that $ac = bc$. Hence, it remains that $a = b$ and the semigroup $(\mathbb{Z}^*,\cdot)$ is right cancellative. One can analogously show that $ca = cb \Rightarrow a = b$, so $\mathbb{Z}^*$ is also left cancellative. On the other hand, this semigroup has the identity element, which is number 1, but is not a group because not every element of $\mathbb{Z}^*$ has the inverse element on it.

Indeed, if $a, b \in \mathbb{Z}^*$ and different from 1, such that $a \cdot b = 1$, we will have that $a = b = 1 \vee a = b = -1$. Hence, we can deduce that every $a \in \mathbb{Z}^*\backslash\{1, -1\}$ has not inverse element in $\mathbb{Z}^*$ and consequently $(\mathbb{Z}^*,\cdot)$ is a left and right cancellative semigroup but not a group.

Proposition 3.4 *Let S be a semigroup satisfying the property that for every element a of S, there exists only one element b of S, for which it holds that* $aba = a$*, then S is a group.*

Proof: Let a be an element of the semigroup S, for which there exists only one element b of S such that $aba = a$. Then, we have:

$$a(bab)a = (aba)(ba) = aba = a$$

Now, since b is the only element of S such that $aba = a$, it follows that $bab = b$; furthermore, a is also the only element of S such that $bab = b$, so S is an inverse semigroup. Denote by E be the semilattice of idempotents of S and let u, v be two elements of E. Then, since E is a commutative subsemigroup of S, we have:

$$(uv)u(uv) = (uv)(uuv) = (uv)(uv) = (uv)^2 = uv \qquad (7)$$

$$(uv)v(uv) = (uvv)(uv) = (uv)(uv) = (uv)^2 = uv \qquad (8)$$

Now, from (7) and (8) and from the property of the semigroup S, we obtain $u = v$. Hence, we deduce that $E = \{u\}$, i.e. S has a unique idempotent. On the other hand, since S is an inverse semigroup, for each element $a \in S$, there exists a unique element $b \in S$ such that $aba = a$ and $bab = b$; furthermore ab and ba are idempotents. Hence, $\forall a \in S$ we have $ab = ba = e$ where b is unique. Now, putting $b=a^{-1}$ we have that S is a group. ■

This paper aims to give some original proofs of propositions that help us show that, in certain circumstances, a semigroup turns into a group. In the future, examining the same issue for groupoids is interesting.

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