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Original Article

SDC Labeling on Path Union and Cycle of Zero Divisor Graphs

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Abstract *- A sum divisor cordial labeling of a graph G with vertex set V(G) is a bijection f from V(G) to {1, 2, 3,…, |V(G)|} such that an edge u v is assigned the label 0 if 2 divides f(u)+f(v) and 1 otherwise; and it Satisfies the condition |e f (0) – e f (1) | ≤ 1, then a graph with a sum divisor cordial labeling is called a sum divisor cordial graph. In this paper, we apply the sum divisor cordial labeling of path union and a cycle of zero divisor graphs. We proved that the path union and cycle of zero divisor graphs are sum divisor cordial.*

Keywords *- Cordial graphs, Cycle of graphs, path union, Sum divisor cordial labeling, Sum divisor cordial graphs, Zero divisor graphs.*

1. Introduction

Let G = (V (G), E(G)) be a simple, finite and undirected graph. Graph labeling is an assignment of labels traditionally represented by integers to edges and vertices. Lourdusamy et al. has introduced the concept of sum divisor cordial labeling in [7]. The concept of the colouring Zero divisor graphs of the commutative ring was introduced by I. Beck [3] and motivated by T. Tamizhchelvam et al. [8]. Let R be a commutative ring with a non-zero identity; Z(R)is the set of all zero devisors in R, and $Z^*(R) = Z(R) \setminus \{0\}$. The Zero-divisor graph of R is the simple undirected graph Γ(R) with vertex set $Z^*(R)$, and two distinct vertices x and y are adjacent if x y =0. [12] S. Sajana and D. Bharathi studied the Intersection graph of zero-divisors of a finite commutative ring. [11, 13] V.J Kaneria studied path union and a cycle of graphs. All graphs considered in this paper are finite, undirected and straightforward. We are interested in the Sum Divisor Cordial (SDC) labeling of path union and a cycle of zero divisor graphs, which are sum divisor cordial graphs.

2. Preliminaries

2.1. Definition 1

For a commutative ring Z_n with unity (1 = 0), the zero-divisor graph of Z_n, denoted by $\Gamma(Z_n)$, is a simple graph with vertices as elements of Z_n and two distinct vertices are adjacent whenever the product of the vertices is zero.

2.2. Definition 2

Let R be a finite ring. An element a ∈ R is called a zero divisor if there exists a non-zero element b ∈ R such that. a. b = 0 or b. a = 0.

2.3. Definition 3

A cycle graph with n vertices is denoted as C_n.

2.4. Definition 4

The path union of graph G is the graph obtained by adding an edge between corresponding vertices of G_i to G_{i+1}, $1 \le i \le n-1$, Where $G_1, G_2, G_3, \ldots, G_n$ ($n \ge 2$) are n copies of G. It is denoted by P (n. G).

2.5. Definition 5

For a cycle C_n, each vertex of C_n is replaced by connected graphs $G_1, G_2, \ldots, G_n$, known as the cycle of graphs. We shall denote it by C ($G_1, G_2, \ldots, G_n$). If we replace each vertex by a graph G, i.e., G_1= G=G_2= …, G_n, such cycle of a graph G is denoted by C (n. G).

2.6. Definition 6

A graph G is considered complete if every pair of distinct vertices are adjacent.

2.7. Definition 7

A bipartite graph is a graph whose vertex set V (G) can be partitioned into two subsets, V1 and V2, such that every edge of G has one end in V1 and the other end in V_2. (V_1, V_2) is called a bipartition of G.

2.8. Definition 8

A complete bipartite graph is a bipartite graph with bipartition (V_1, V_2) such that every vertex of V_1 is joined to all the vertices of V_2. It is denoted by $K_{r, s}$, where $|V_1| = r$ and $|V_2|=s$. A star graph is a complete bipartite graph $K_{1, s}$.

2.9. Definition 9

Let $G = (V(G), E(G))$ be a simple graph and $h: V(G) \rightarrow 1,2, \cdots, |V(G)|\}$ be the bijection. For each edge u v, assign the label 0. If 2/h(u)+h(v) and the label 1 otherwise. The function h is a Sum Divisor Cordial (SDC) labeling if it satisfies the condition $| e_h(1) - e_h(0) | \leq 1$, where $e_h(0)$ is the number of edges labelled with 0 and $e_h(1)$ is the number of edges labelled with 1. A graph that admits Sum Divisor Cordial (SDC) labeling is called a Sum Divisor Cordial (SDC) Graph. A cycle graph is a graph that contains a single cycle or a closed chain of vertices connected by edges.

3. Main Result

3.1. SDC Labeling on Path Union and a Cycle of Zero Divisor Graphs

In this section, SDC labeling is obtained for different path union classes and zero-divisor graph cycles.

3.1.1. Theorem 1

Let p be a prime number with p>2, then $C(3. \Gamma(Z_{2p}))$ is an SDC graph.

Proof

Let $\Gamma(Z_{2p})$ be a zero Divisor graph of Z_{2p}, where p is a prime number and p > 2.
Let the vertex set and edge set of $\Gamma(Z_{2p})$ are
$V(\Gamma(Z_{2p})) = \{2,4, \ldots, 2(p-1), p\} = \{v_1, v_2, \ldots, v_{p-1}, v_p\}$ and
$E(\Gamma(Z_{2p})) = \{v_1 v_j: 2 \leq j \leq p\}$
$| V(\Gamma(Z_{2p})) |= p$ and $| E(\Gamma(Z_{2p})) |= p-1$
Let $C(3. \Gamma(Z_{2p}))$ be a cycle of 3copies
zero divisor Graph $\Gamma(Z_{2p})$, where p be a prime number and p>2.
Let the vertex set and edge set of $C(3. \Gamma(Z_{2p}))$ are
Let G= $C(3. \Gamma(Z_{2p}))$
$V(G) = \{v_{1, 1}, v_{1, 2}, \ldots\ldots v_{1, p}, v_{2, 1}, \ldots v_{2, p}, v_{3, 1}, \ldots, v_{3p}\}$
$E(G) = \{v_{1,1}v_{1, j}, v_{2, 1}v_{2, j}, v_{3, 1}v_{3, j}, v_{1, 1}v_{2, 1}, v_{2, 1}v_{3, 1}, v_{3, 1}v_{1, 1} : 2 \leq j \leq p\}$.

Therefore, |V(G)|=3p and |E(G) |=3p
We define h: V(G) → {1,2, 3…3p} by
$h(v_{1, j}) = j$ for $2 \leq j \leq p$,
$h(v_{2, j}) = p + j$ for $2 \leq j \leq p$,
$h(v_{3, j}) = 2p + j$ for $2 \leq j \leq p$,
$h(v_{1, 1}) = 1$, $h(v_{2, 1}) = p+1$ and $h(v_{1, 1}) = 2p+1$.

Then induced edge labeling as follows g: E(G) → {0, 1} is given by

$$g(v_{1, 1} v_{1, j}) = \begin{cases} 0 & \text{if } j \text{ is odd and } 2 \leq j \leq p \\ 1 & \text{if } j \text{ is even and } 2 \leq j \leq p; \end{cases}$$

$$g(v_{2, 1} v_{2, j}) = \begin{cases} 0 & \text{if } j \text{ is odd and } 2 \leq j \leq p \\ 1 & \text{if } j \text{ is even and } 2 \leq j \leq p; \end{cases}$$

$$g(v_{3, 1} v_{3, j}) = \begin{cases} 0 & \text{if } j \text{ is odd and } 2 \leq j \leq p \\ 1 & \text{if } j \text{ is even and } 2 \leq j \leq p; \end{cases}$$

g ($v_{1,1}v_{2,1}$) =1, g ($v_{2,1}v_{3,1}$) =1, g ($v_{3,1}v_{1,1}$) =0.

Also, we have that $|e_g(0)| = \frac{3p-1}{2}$ and $|e_g(1)| = \frac{3p+1}{2}$
Hence $|e_g(0) - e_g(1)| \leq 1$, and so C (3. Γ (Z_{2p})) is an SDC graph.

Example 1

A SDC labeling of the cycle of 3 copies of Γ(Z_{10}) graph C (3. Γ(Z_{10})) is given in Figure 1.

Let Γ(Z_{10}) be a zero Divisor graph of Z_{10}
Let the vertex set and edge set of Γ(Z_{10}) are
V(Γ(Z_{10})) = {2,4, 6,8,5} = {v_1, v_2, v_3, v_4, v_5} and
E(Γ(Z_{10})) = {v_1v_j: $2 \leq j \leq 5$}
| V(Γ(Z_{10})) |= 5 and | E(Γ(Z_{10})) |= 4
Let G = C (3. Γ(Z_{10}))
V(G) = {$v_{1,1}$, $v_{1,2}$, $v_{1,3}$, $v_{1,4}$, $v_{1,5}$, $v_{2,1}$, $v_{2,2}$, $v_{2,3}$, $v_{2,4}$, $v_{2,5}$, $v_{3,1}$, $v_{3,2}$, $v_{3,3}$, $v_{2,4}$, $v_{3,5}$}
E(G) = {$v_{1,1}v_{1,j}$, $v_{2,1}v_{2,j}$, $v_{3,1}v_{3,j}$, $v_{1,1}v_{2,1}$, $v_{2,1}v_{3,1}$, $v_{3,1}v_{1,1}$:$2 \leq j \leq 5$}

Therefore, |V(G) |=15 and |E(G) |=15
Here $|e_g(0)|$ = 7 and $|e_g(1)|$ =8
Hence $|e_g(0) - e_g(1)| \leq 1$ then C (3. Γ(Z_{10})) is SDC a graph.

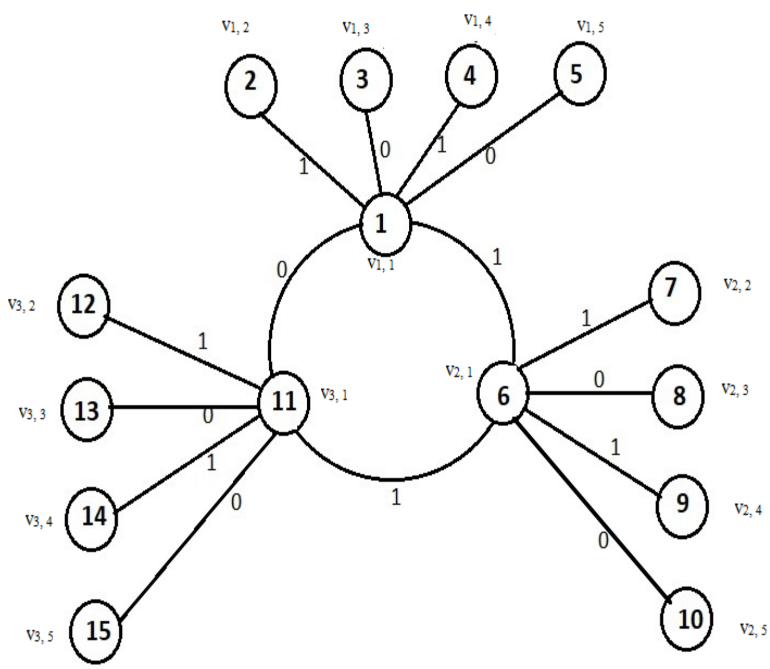

Fig. 1 C (3. Γ(Z_{10}))

3.1.2. Corollary 1

For any prime number p > 2, the path union graph P (2. (Z_{2p})) is a sum divisor cordial graph.

Example 1

A sum divisor cordial labeling of the path union of 2 copies graphs P (2. Γ(Z_{10})) where p= 5 is given in Figure 2.

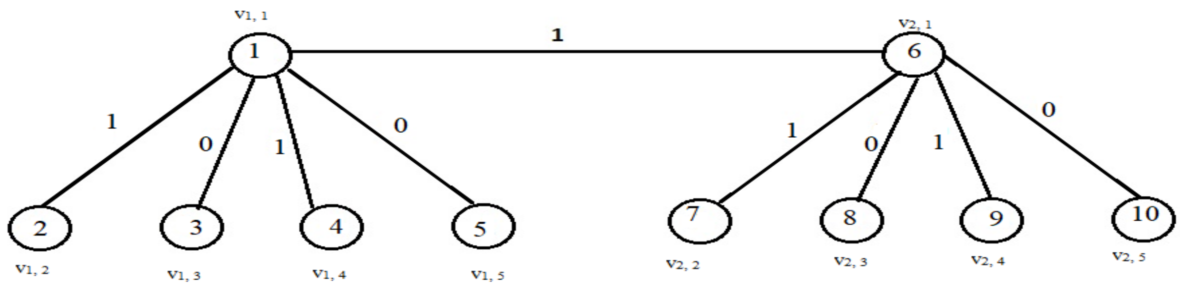

Fig. 2 P (2. Γ(Z_{10}))

Example 2

A sum divisor cordial labeling of the path union of 2 copies graphs P (2. Γ(Z_{14})) where p= 7 is given in Figure 3.

Let Γ(Z_{14}) be a zero Divisor graph of Z_{14}
Let the vertex set and edge set of Γ(Z_{14}) are
V(Γ(Z_{14})) = {2,4, 6,8,10,12, 7} = {$v_1, v_2, \ldots, v_6, v_7$} and
E(Γ(Z_{14})) = {$v_1 v_j$: $2 \leq j \leq 7$}
| V(Γ(Z_{14})) |= 5 and | E(Γ(Z_{14})) |= 6
Let G = P (2. Γ(Z_{14})) then
V (G) = {$u_{1,1}, u_{1,2}, u_{1,3}, u_{1,4}, u_{1,5}, u_{1,6}, u_{1,7}, u_{2,1}, u_{2,2}, u_{2,3}, u_{2,4}, u_{2,5}$ $u_{2,6}, u_{2,7}$}
E (G) = {$v_{1,i} v_{1,7}, v_{1,7}v_{2,7}, v_{2,i}v_{2,7}$ / $1 \leq i \leq 6$}
|V (G) | = 14 and |E (G) | = 13
We define h: V (G) ⟶ {1,2, …, 14}
h ($u_{1,i}$) = i for $1 \leq i \leq 6$,
h ($u_{2,i}$) = 7 + i for $1 \leq i \leq 6$,
h ($u_{1,7}$) = 7,
and h ($v_{2,7}$) = 14.

Then, the induced edge labeling function
g: E(G) ⟶ {0, 1} is given by

$$g\,(v_{1,7} v_{1,i}) = \begin{cases} 0 \;\; if\ i\ is\ odd\ 1 \leq i \leq 6 \\ 1 \;\; if\ i\ is\ even\ 1 \leq i \leq 6; \end{cases}$$

g ($v_{1,7}v_{2,7}$) = 1,

$$g\,(v_{2,7} v_{2,i}) = \begin{cases} 1 \;\; if\ i\ is\ odd\ 1 \leq i \leq 6 \\ 0 \;\; if\ i\ is\ even\ 1 \leq i \leq 6; \end{cases}$$

From this, we have that.

$|e_g\,(0)|$ = 6 and $e_g\,(1) = 7$ and satisfies the condition $|e_g\,(0) - e_g\,(1)| \leq 1$, so (P (2. Γ($Z_{14}$)) is a sum divisor cordial graph.

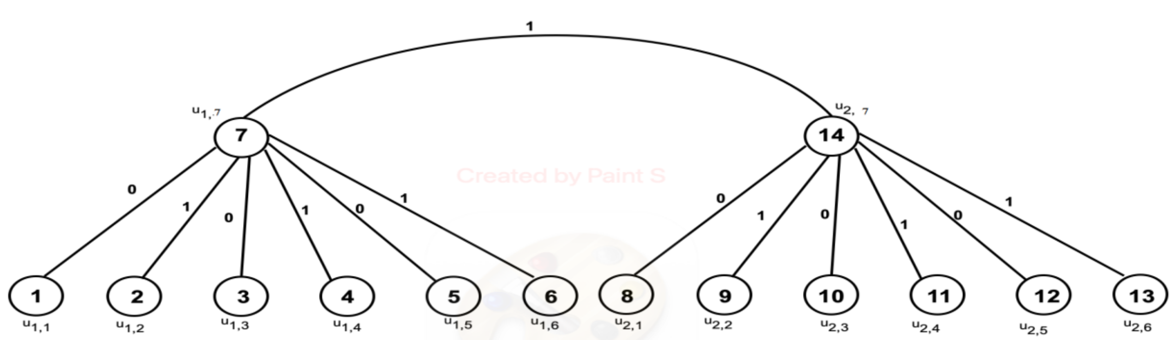

Fig. 3 (P (2. Γ(Z_{14}))

3.1.3. Theorem

For any prime number $p > 3$, C (3. $\Gamma(Z_{3p})$) does not admit an SDC labeling.

Example 1

An SDC labeling of the cycle of 3 copies C (3. $\Gamma(Z_{15})$) is given in Figure 4.

Let $\Gamma(Z_{15})$ be a zero Divisor graph of Z_{15}

Let the vertex set and edge set of $\Gamma(Z_{15})$ are

$V(\Gamma(Z_{3p})) = \{p, 2p\} \cup \{3, 6, \ldots, 3(p-1)\} = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_{p-1}\}$

$E(\Gamma(Z_{3p})) = \{u_1 v_i, u_2 v_i : 1 \leq i \leq p-1\}$.

p=5 then

$V(\Gamma(Z_{15})) = \{5,10\} \cup \{3,6,9,12\} = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_4\}$ and

$E(\Gamma(Z_{15})) = \{u_1 v_i, u_2 v_i : 1 \leq i \leq 4\}$

$|V(\Gamma(Z_{15}))| = 6$ and $|E(\Gamma(Z_{15}))| = 8$

Let G = C (3. $\Gamma(Z_{15})$)

$V(G) = \{u_{1,1}, u_{1,2}, v_{1,1}, v_{1,2}, \ldots, v_{1,4}, u_{2,1}, u_{2,2}, v_{2,1}, v_{2,2}, \ldots, v_{2,4}, u_{3,1}, u_{3,2}, v_{3,1}, v_{3,2}, \ldots, v_{3,4}\}$

$E(G) = \{u_{1,1}v_{1,i}, u_{1,2}v_{1,i}, u_{2,1}v_{2,i}, u_{2,2}v_{2,i}, u_{3,1}v_{3,i}, u_{3,2}v_{3,i}, u_{1,1}u_{2,1}, u_{2,1}u_{3,1}, u_{3,1}u_{1,1} : 1 \leq i \leq 4\}$

$|V(G)| = 18$, $|E(G)| = 27$.

We observe that

Here $|e_g(0)| = 15$ and $|e_g(1)| = 12$

i. e $|e_g(0) - e_g(1)| \nleq 1$, hence C (3. $\Gamma(Z_{15})$) is not a SDC graph.

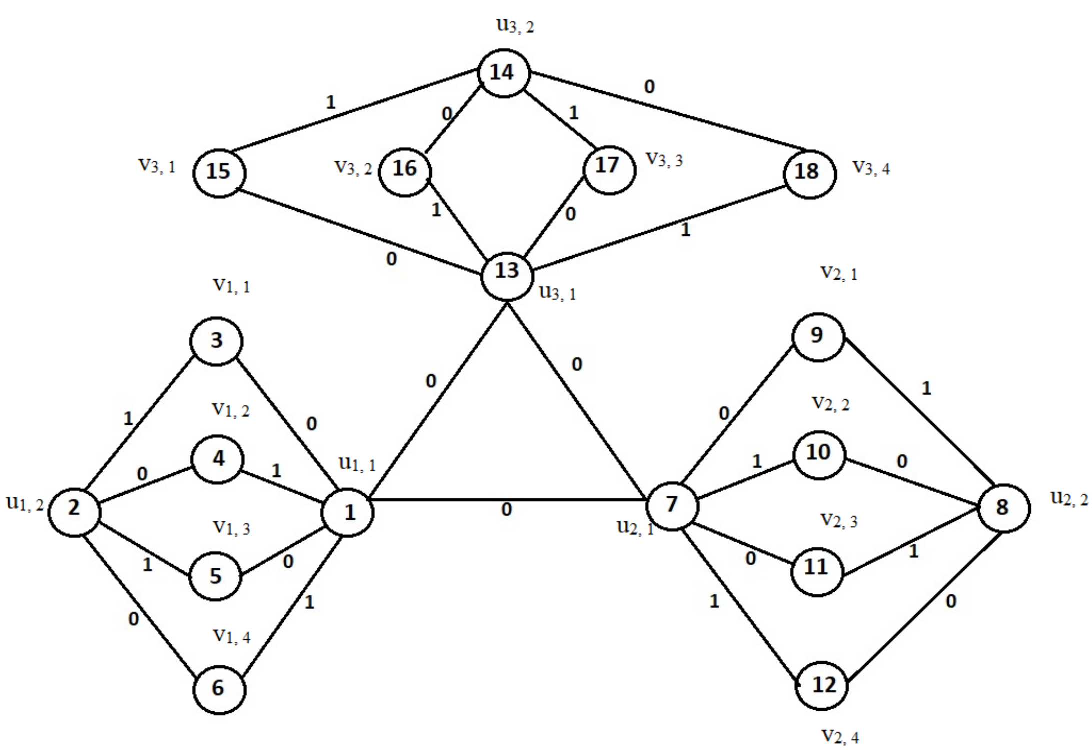

Fig. 4 C (3. Γ(Z_{15}))

3.1.4. Corollary

For any prime number $p > 3$, the path union graph P (2. $\Gamma(Z_{3p})$) is the sum of a divisor cordial graph.

Example 1

P (2. $\Gamma(Z_{15})$) is shown in Figure 5.

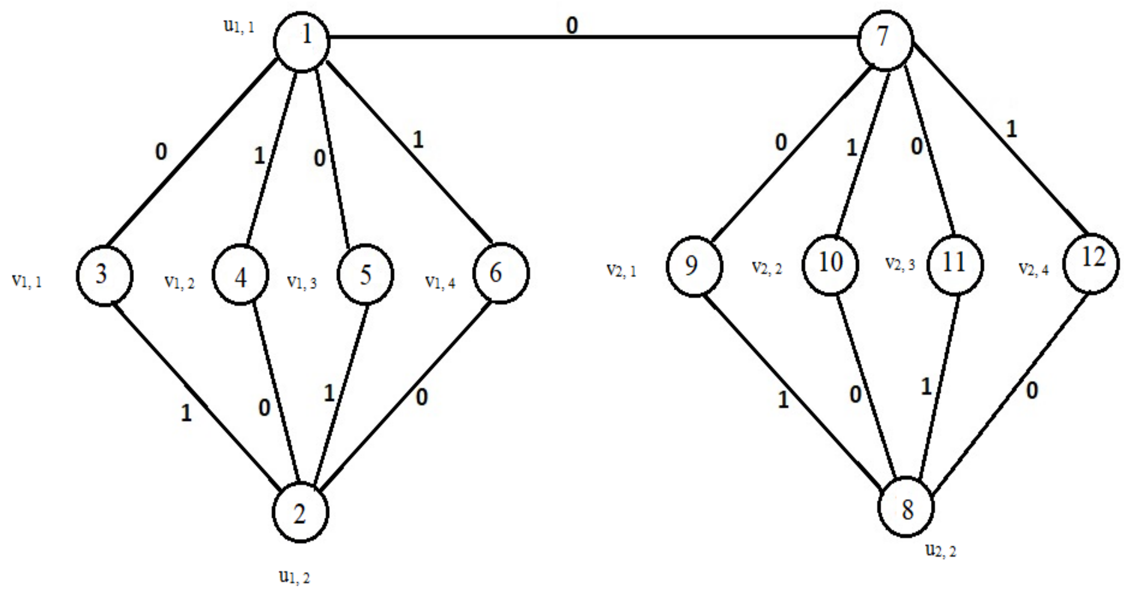

Fig. 5 P (2. Γ(Z15))

Example 2

Let G = P (2. $\Gamma(Z_{3p})$ where p=7 then G = P (2. $\Gamma(Z_{21})$ is shown in Figure 6.
Let p >3 be a prime number
Let P_2 be a path of length 1 and $\Gamma(Z_{21})$ be the Zero-Divisor graph of Z_{21}.
Let G = P (2. $\Gamma(Z_{21})$)
V (G) = $\{u_{1,1}, u_{1,2}, v_{1,1}, v_{1,2}, \ldots, v_{1,6}, u_{2,1}, u_{2,2}, v_{2,1}, v_{2,2}, \ldots, v_{2,6}\}$,
E(G) = $\{u_{1,1}v_{1,j}, u_{1,2}v_{1,1}, u_{2,1}v_{2,i}, u_{2,2}v_{2,i}, 1 \leq i \leq 6\}$.

Note that
$|V(G)| = 16$, $|E(G)| = 25$.
Define h: V(G) → {1, 2, …,16} by h ($u_{1,1}$) = 1, h ($u_{1,2}$) = 2, h ($u_{2,1}$) = 9, h ($u_{2,2}$) = 10 and
h ($v_{1,i}$) = i + 2; $1 \leq i \leq 6$,
h ($v_{2,i}$) = i + p+3; $1 \leq i \leq 6$.

Then the induced edge labeling function g: E (G) → {0, 1}.

$$g(u_{1,1}v_{1,j}) = \begin{cases} 0 & if\ j\ is\ odd \\ 1 & if\ j\ is\ even; \end{cases}$$

$$g(u_{1,2}v_{1,j}) = \begin{cases} 1 & if\ j\ is\ odd \\ 0 & if\ j\ is\ even; \end{cases}$$

$$g(u_{2,1}v_{2,j}) = \begin{cases} 0, & if\ j\ is\ odd\ ;\ 1 \leq j \leq 6 \\ 1, & if\ j\ is\ even; \end{cases}$$

g ($u_{11}u_{21}$) =0;

$$g(u_{2,2}v_{2,j}) = \begin{cases} 0, & if\ j\ is\ even\ ; 1 \leq j \leq 6 \\ 1, & if\ j\ is\ odd; \end{cases}$$

Also, we have that
$|e_g(0)| = 13$ and $|e_g(1)| = 12$ then also satisfies the condition $|e_g(0) - e_g(1)| \leq 1$,
So, P (2. $\Gamma(Z_{21})$) is a sum divisor cordial graph shown in Figure 4.

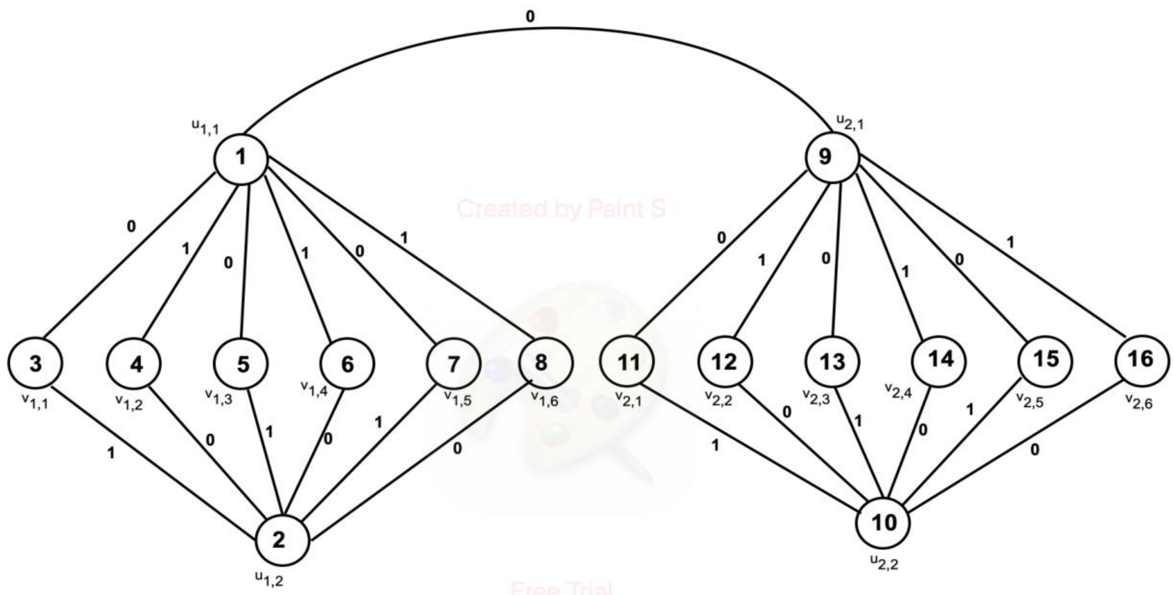

Fig. 6 P (2. $\Gamma(Z_{21})$)

3.1.5. Theorem 3

For any prime number $p \geq 2$, the cycle of graph C ($3.\Gamma(Z_{4p})$) is an SDC graph.

Proof

Let C ($3.\Gamma(Z_{4p})$) be a cycle of 3 copies of zero divisor graph $\Gamma(Z_{4p})$ of $Z_{4p.}$

If p=2 then C ($3.\Gamma(Z_8)$)

$\Gamma(Z_8)$, being a path on three vertices and $c_{3,}$ a cycle of three vertices, is an SDC graph.

Let $\Gamma(Z_{4p})$ be a zero-divisor graph of Z_{4p} where p is a prime number and $p \geq 3$.

The vertex set of $\Gamma(Z_{4p})$ is partitioned into two sets V_1 and V2, are

$V_1 = \{p, 2p, 3p\} = \{u_1, u_2, u_3\}$ and

$V_2 = \{2,4, \ldots, 2(p-1), 2(p+1), \ldots, 2(2p-1)\} = \{v_1, v_2, \ldots v_{p-1}, v_{p+1}, \ldots, v_{2p-1}\}$

Here, the vertex set of C ($3.\Gamma(Z_{4p})$) is partitioned into 2 sets, V_1 and $V_{2,}$ where

$V_1 = \{u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, u_{2,3}, u_{3,1}, u_{3,2}, u_{3,3}\}$ and

$V_2 = \{v_{1,1}, v_{1,2}, \ldots v_{1,p-1}, v_{1,p+1}, \ldots, v_{1,2p-1}, v_{2,1}, v_{2,2}, \ldots, v_{2,p-1}, v_{2,p+1}, \ldots, v_{2,2p-1}, v_{3,1}, v_{3,2}, \ldots, v_{3,p-1}, v_{3,p+1}, \ldots v_{3,2p-1}\}$

Therefore,

$|V(C(3.\Gamma(Z_{4p}))| = 6p+3$ and

$|E(C(3.\Gamma(Z_{4p}))| = 12p-9$

Let G= C ($3.\Gamma(Z_{4p})$)

We define $h: V(G) \rightarrow \{1,2,3, \ldots 6p+3\}$

by $h(u_{1,1}) = 1$, $h(u_{1,2}) = 2$, $h(u_{1,3}) = 3$,

$h(u_{2,1}) = 2p+2$, $h(u_{2,2}) = 2p+3$, $h(u_{2,3}) = 2p+4$,

$h(u_{3,1}) = 4p+3$, $h(u_{3,2}) = 4p+4$, $h(u_{3,3}) = 4p+5$,

$h(v_{1,j}) = j+3$ for $1 \leq j \leq p-1$ and

$h(v_{1,j}) = j+2$ for $p+1 \leq j \leq 2p-1$

$h(v_{2,j}) = 2p+j+4$ for $1 \leq j \leq p-1$ and

$h(v_{2,j}) = 2p+j+3$ for $p+1 \leq j \leq 2p-1$

$h(v_{3,j}) = 4p+j+3$ for $1 \leq j \leq p-1$ and

$h(v_{3,j}) = 4p+j+2$ for $p+1 \leq j \leq 2p-1$.

The induced edge labeling is given by g: E (G) → {0, 1}.

$$g(u_{1,1}v_{1,j}) = \begin{cases} 0 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 1 \;\; if\ j\ is\ even\ and\ p+1 \le j \le 2p-1; \end{cases}$$

$$g(u_{1,2}v_{1,j}) = \begin{cases} 0 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 1 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 0 \;\; if\ j\ is\ even\ and\ p+1 \le j \le 2p-1 \\ 1 \;\; if\ j\ is\ odd\ and\ p+1 \le j \le 2p-1 \end{cases}$$

$$g(u_{1,3}v_{1,j}) = \begin{cases} 1 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 0 \;\; if\ j\ is\ even\ and\ p+1 \le j \le 2p-1 \end{cases}$$

$$g(u_{2,1}v_{2,j}) = \begin{cases} 0 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 1 \;\; if\ j\ is\ even\ and\ p+1 \le j \le 2p-1 \end{cases}$$

$$g(u_{2,2}v_{2,j}) = \begin{cases} 0 \;\; if\ j\ is\ odd\ and\ 1 \le j \le p-1 \\ 1 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 0 \;\; if\ j\ is\ even\ and\ p+1 \le j \le 2p-1 \\ 1 \;\; if\ j\ is\ odd\ and\ p+1 \le j \le 2p-1 \end{cases}$$

$$g(u_{2,3}v_{2,j}) = \begin{cases} 1 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 0 \;\; if\ j\ is\ even\ and\ p+1 \le j \le 2p-1 \end{cases}$$

$$g(u_{3,1}v_{3,j}) = \begin{cases} 0 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 1 \;\; if\ j\ is\ even\ and\ p+1 \le j \le 2p-1 \end{cases}$$

$$g(u_{3,2}v_{3,j}) = \begin{cases} 0 \;\; if\ j\ is\ odd\ and\ 1 \le j \le p-1 \\ 1 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 0 \;\; if\ j\ is\ even\ and\ p+1 \le j \le 2p-1 \\ 1 \;\; if\ j\ is\ odd\ and\ p+1 \le j \le 2p-1 \end{cases}$$

$$g(u_{3,3}v_{3,j}) = \begin{cases} 0 \;\; if\ j\ is\ even\ and\ 1 \le j \le p-1 \\ 1 \;\; if\ j\ is\ even\ and\ p+1 \le j \le 2p-1 \end{cases}$$

$g(u_{1,1}u_{2,1}) = 1$, $g(u_{2,1}u_{3,1}) = 1$, $g(u_{3,1}u_{1,1}) = 0$.

Therefore,
$|e_g(0)| = 6p-5$ and $|e_g(1)| = 6p-4$
Hence $|e_g(0) - e_g(1)| \le 1$ and so $C(3.\ \Gamma(Z_{4p}))$ is SDC graph.

Example 1

An SDC labeling of $C(3.\ \Gamma(Z_8))$ is given in the below Figure 7.

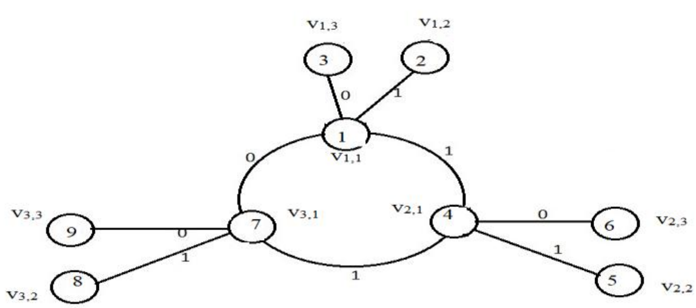

Fig. 7 C (3. Γ(Z₈))

Example 2

An SDC labeling of C (3. Γ(Z_{12})) is given in the below Figure 8.

$V_1 = \{u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}, u_{2,2}, u_{2,3}, u_{3,1}, u_{3,2}, u_{3,3}\}$ and

$V_2 = \{v_{1,1}, v_{1,2}, v_{1,4}, v_{1,5}, v_{2,1}, v_{2,2}, v_{2,4}, v_{2,5}, v_{3,1}, v_{3,2}, v_{3,4}, v_{3,5}\}$

Therefore,

| V (C (3. Γ(Z_{12}))) |= 21 and | E (C (3. Γ(Z_{12}))) |= 27

Let G= C (3. Γ(Z_{12}))

We define h: V(G) → {1,2,3, …21}

by h ($u_{1,1}$) =1, h ($u_{1,2}$) =2, h ($u_{1,3}$) =3,

h ($u_{2,1}$) =8, h ($u_{2,2}$) =9, h ($u_{2,3}$) =10,

h ($u_{3,1}$) =15, h ($u_{3,2}$) =16, h ($u_{3,3}$) =17,

h ($v_{1,j}$) =j+3 for $1 \le j \le 2$and

h ($v_{1,j}$) =j+2 for $3 \le j \le 5$

h ($v_{2,j}$) =2p+j+4 for $1 \le j \le 2$ and

h ($v_{2,j}$) =2p+j+3 for $3 \le j \le 5$

h ($v_{3,j}$) =4p+j+3 for $1 \le j \le 2$ and

h ($v_{3,j}$) =4p+j+2 for $3 \le j \le 5$.

The induced edge labeling is given by g: E (G) → {0, 1}.

$|e_g(0)|$ = 13 and $|e_g(1)|$ = 14

Hence $|e_g(0) - e_g(1)| \le 1$ and so C (3. (Γ(Z_{12})) is a SDC graph.

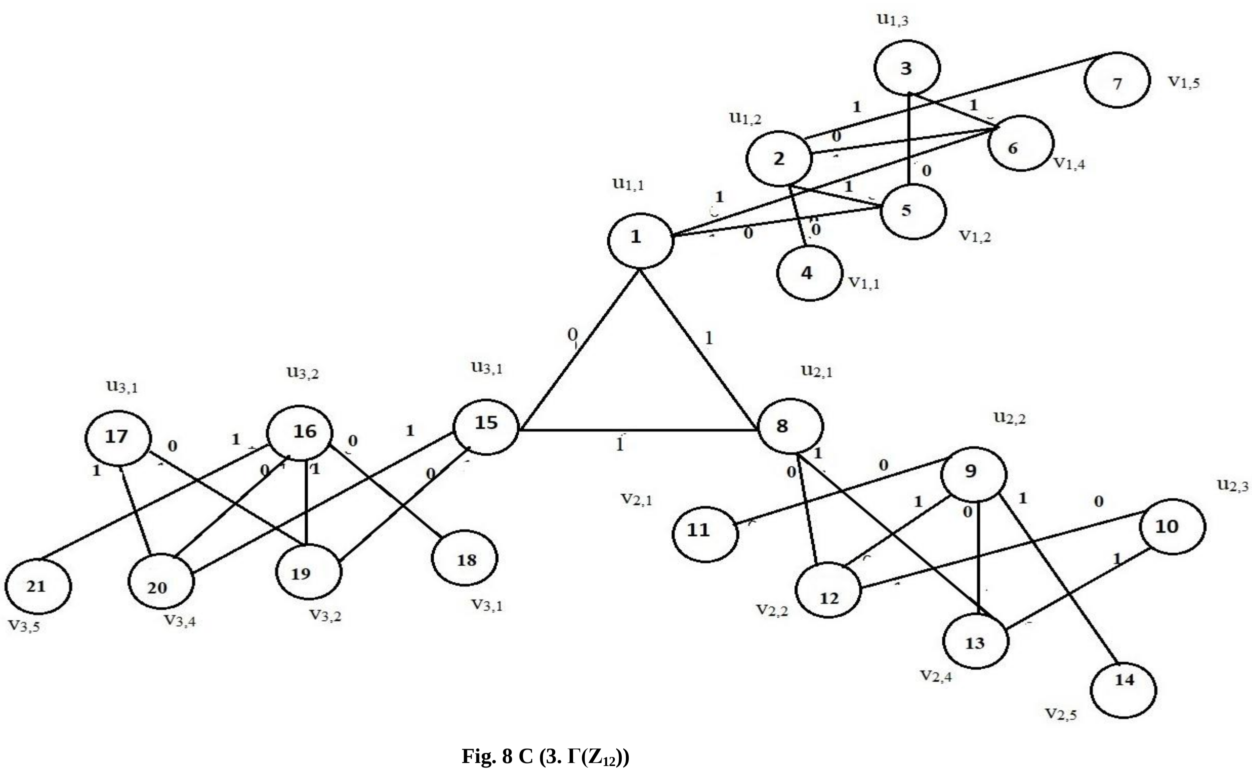

Fig. 8 C (3. Γ(Z_{12}))

3.1.6. Corollary 3

For any prime number p ≥ 3, the path union graph (P (2. Γ(Z_{4p}))) is a sum divisor cordial graph.

Example 1

An SDC labeling of (P (2. Γ(Z_{12}))) is given in below Figure 9.

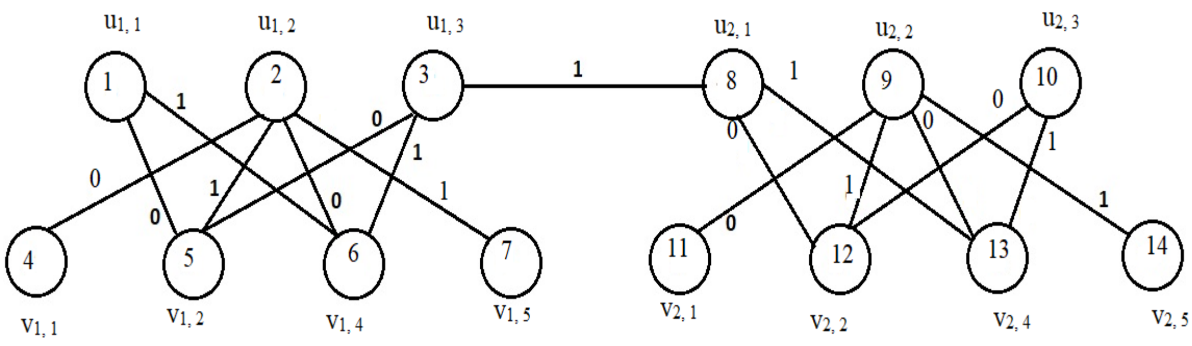

Fig. 9 P (2. Γ(Z_{12}))

Example 2

An SDC labeling of (P (2. Γ(Z_{20}))) is given in below Figure 10.

Let G = (P (2. Γ(Z_{20})))

Define h: V(G) → {1,2, …, 22} by

| V (G) | =22 and | E (G) | = 33

$h(u_{1,1}) = 1$, $h(u_{1,2}) = 2$, $h(u_{1,3}) = 3$,

$h(v_{1,j}) = 3+j;\ 1 \le j \le 4$ and

$h(v_{1,j}) = j+2;\ 6 \le j \le 9$.

$h(u_{2,1}) = 13$, $h(u_{2,2}) = 14$, $h(u_{2,3}) = 15$,

$h(v_{2,j}) = 15+j;\ 1 \le j \le 4$ and

$h(v_{2,j}) = 14+j;\ 6 \le j \le 9$.

The induced edge labeling function g: E → {0, 1}, is given by

$$g(u_{1,1}v_{1,j}) = \begin{cases} 0 & if\ j\ is\ even\ and\ 1 \le j \le 4 \\ 1 & if\ j\ is\ even\ and\ 6 \le j \le 9; \end{cases}$$

$$g(u_{1,2}v_{1,j}) = \begin{cases} 0 & if\ j\ is\ odd\ and\ 1 \le j \le 4 \\ 1 & if\ j\ is\ even\ and\ 1 \le j \le 4 \\ 0 & if\ j\ is\ even\ and\ 6 \le j \le 9; \\ 1 & if\ j\ is\ odd\ and\ 6 \le j \le 9; \end{cases}$$

$$g(u_{1,3}v_{1,j}) = \begin{cases} 0 & if\ j\ is\ even\ and\ 1 \le j \le 4 \\ 1 & if\ j\ is\ even\ and\ 6 \le j \le 9; \end{cases}$$

$$g(u_{2,1}v_{2,j}) = \begin{cases} 0 & if\ j\ is\ even\ and\ 1 \le j \le 4 \\ 1 & if\ j\ is\ even\ and\ 6 \le j \le 9; \end{cases}$$

$$g(u_{2,2}v_{2,j}) = \begin{cases} 0 & if\ j\ is\ odd\ and\ 1 \le j \le 4 \\ 1 & if\ j\ is\ even\ and\ 1 \le j \le 4 \\ 0 & if\ j\ is\ even\ and\ 6 \le j \le 9; \\ 1 & if\ j\ is\ odd\ and\ 6 \le j \le 9; \end{cases}$$

$$g(u_{2,3}v_{2,j}) = \begin{cases} 0 & if\ j\ is\ even\ and\ 1 \le j \le 4 \\ 1 & if\ j\ is\ even\ and\ 6 \le j \le 9 \end{cases}$$

$g(u_{2,1}u_{1,1}) = 0$.

$|e_g(0)| = 17$ and $|e_g(1)| = 16$.

Hence $|e_g(0) - e_g(1)| \le 1$ and so (P (2. Γ(Z_{20}))) is a SDC graph.

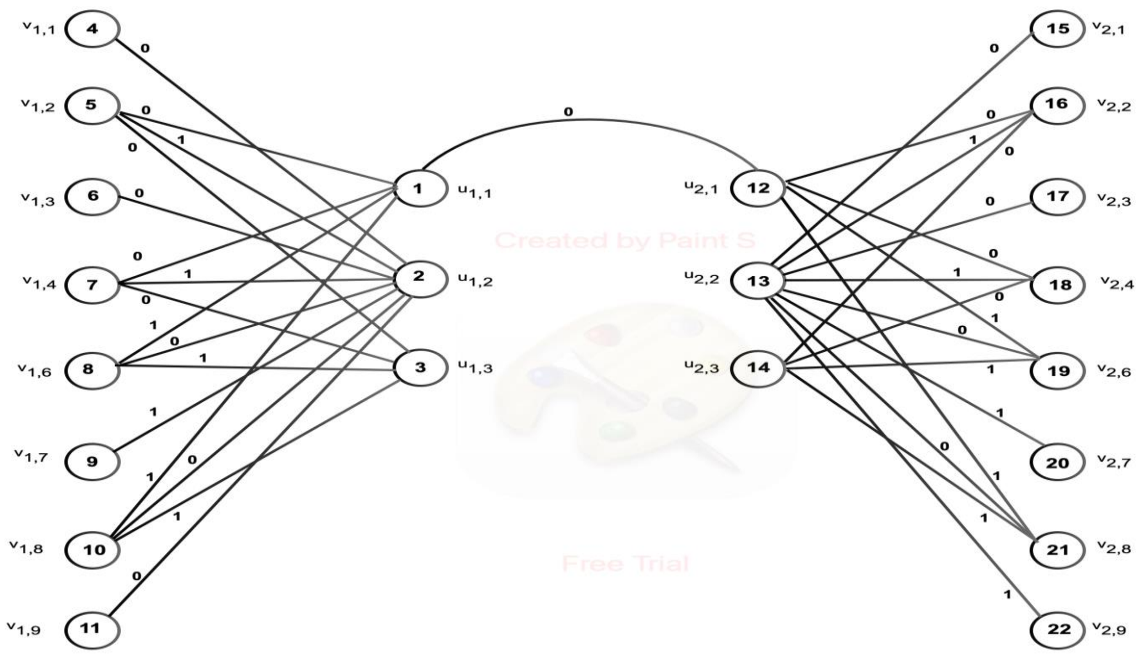

Fig. 10 P (2. Γ(Z_{20}))

3.1.7. Theorem 4

For any prime number $p > 3$, the Zero-Divisor graph Γ (Z_{np}) where n is an odd number, n > 3 and n ≠ p is an SDC graph.

3.1.8. Theorem 5

For any prime number $p > 3$, the cycle of graph C (3. Γ (Z_{np})) where n is an odd number, n > 3 and n ≠ p does not admit the SDC labeling.

Example 1

ASDC labeling of C (3. Γ (Z_{35})) where n=5, p=7 is given in the below Figure 11.

Here $|e_g\ (0)|$ = 39 and $|e_g\ (1)\ |$=36, hence C (3. Γ(Z_{35})) does not admit the SDC labeling.

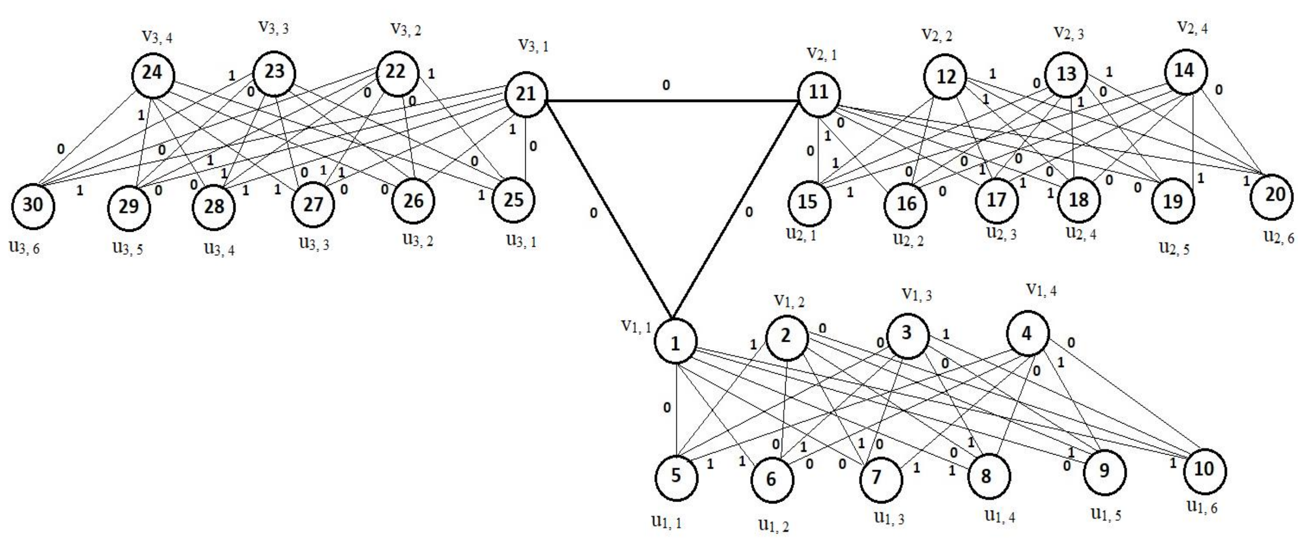

Fig. 11 C (3. Γ (Z_{35}))

3.1.9. Theorem 6

For any prime number $p > 2$, the cycle of graph C (3. Γ (Z_{np})) where n is an even number, n > 4 is not an SDC graph.

3.1.10. Theorem 7

For two distinct primes p and q with p< q, the path union graph (P (2. Γ (Z_{pq}))) is a sum divisor cordial graph.

Proof

The vertex set of Γ (Z_{pq}) is partitioned into two sets, V_1 and V2, which are
V_1= {p, 2p, 3p, …, (q-1) p} = {$u_1, u_2, u_3, …, u_{q-1}$} and
V_2 = {q, 2q, 3q, …, (p-1) q} = {$u_1, u_2, u_3, …, u_{p-1}$}
The edge set of Γ (Z_{pq}) is given by
E (Γ (Z_{pq})) = $\{v_j u_i: u_i \in V_1 \; and \; v_j \in V_2 , 1 \le j \le p-1, 1 \le i \le q-1\}$
| V (Γ (Z_{pq})) |= p+q-2 and | E (Γ (Z_{pq})) |= (p-1) (q-1)

Here, the vertex set of P (2. Γ (Z_{pq})) is partitioned into 2 sets,
V_1 and V_2, where
V_1 = {$u_{1,1}, u_{1,2}, u_{1,3}, … u_{1,(q-1)}$ $u_{2,1}, u_{2,2}, u_{2,3}, …, u_{2,(q-1)}$} and
V_2 = {$v_{1,1}, v_{1,2}, … v_{1,(p-1)}, v_{2,1}, v_{2,2}, …, v_{2,(p-1)}$}

Therefore,
|V (P (2. Γ (Z_{pq}))) | = 2p+2q-4 and
|E (P (2. Γ (Z_{pq}))) | =2(p-1) (q-1) +1

Let G= P (2. Γ (Z_{pq}))
We define h: V(G) → {1,2,3, …2p+2q-4} by
h ($v_{1,j}$) =i for $1 \le i \le p-1$and
h ($u_{1,j}$) = p-1+i for $1 \le i \le q-1$
h ($v_{2,j}$) = p+q-2+i for $1 \le i \le p-1$ and
h ($u_{2,j}$) = 2p+q-3+i for $1 \le j \le q-1$ and

The induced edge labeling is given by g: E (G) → {0, 1}.
For $1 \le j \le \frac{p-1}{2}$ and $1 \le i \le q-1$

$$g(v_{1,2j-1}u_{1,i}) = \begin{cases} 0 & if\ i\ is\ odd \\ 1 & if\ i\ is\ even \end{cases};$$

$$g(v_{1,2j}u_{1,i}) = \begin{cases} 1 & if\ i\ is\ odd \\ 0 & if\ i\ is\ even \end{cases};$$

$$g(v_{2,2j-1}u_{2,i}) = \begin{cases} 0 & if\ i\ is\ odd \\ 1 & if\ i\ is\ even \end{cases};$$

$$g(v_{2,2j}u_{2,i}) = \begin{cases} 1 & if\ i\ is\ odd \\ 0 & if\ i\ is\ even \end{cases};$$

g ($v_{1,1}u_{2,1}$) = 0

$|e_g(0)| = \frac{(p-1)(q-1)}{2} + 1$ and $|e_g(1)| = \frac{(p-1)(q-1)}{2}$.

Hence $|e_g(0) - e_g(1)| \le 1$ and so (P (2. Γ (Z_{pq}))) is a Group difference cordial graph.

3.1.11. Remark

The existence of labeling depends on the allotment of vertices.

4. Conclusion

The path union and cycle of graphs

- P (n. $\Gamma(Z_{2p})$), n = 2 and p> 2, then it is a SDC graph. If n > 2 is not an SDC graph.
- P (n. $\Gamma(Z_{3p})$), n = 2 and p> 2, then it is a SDC graph. If n > 2 is not an SDC graph.
- P (n. $\Gamma(Z_{4p})$), n = 2 and p> 2, then it is a SDC graph. If n > 2, not an SDC graph.
- P (n. Γ ($Z_{p\,q}$)), n = 2 and p< q, then it is an SDC graph. If n>2 is not an SDC graph.
- P (n. Γ ($Z_{k\,p}$)), n = 2, $k \geq 2$ *be a positive integer* and p> 2 then it is a SDC graph. If n > 2 is not an SDC graph.
- C (n. $\Gamma(Z_{2p})$), n = 3 and p> 2, then it is an SDC graph if n > 3 is not an SDC graph.
- C (n. $\Gamma(Z_{3p})$), $n \geq 3$ and p> 3, then it is not an SDC graph.
- C (n. $\Gamma(Z_{4p})$), n = 3 and $p \geq 4$, then it is a SDC graph. If n > 3 is not an SDC graph.
- C (3. Γ (Z_{np})), n is even and $n \geq 2$, $p > 2$, then it is an SDC graph.
- C (3. Γ (Z_{np})), n is odd and $n \geq 3$, $p > 2$, then it is not an SDC graph.

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