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Original Article

Subjacent Manifold whose Lie Algebra is of Analytic and Invariant Vector Fields under Left Translations

Francisco Bulnes

IINAMEI, Chalco, State of Mexico, Mexico.
Research Department in Mathematics and Engineering, TESCHA, Chalco, State of Mexico, Mexico.
Postgraduate Department, Selinus University of Science and Literature, Bologna, Italy.

Corresponding Author : francisco.bulnes@tesch.edu.mx

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Abstract *- The study of the Lie algebra of invariant analytic fields under left translations leads us to establish that the dimension of the subjacent manifold of the analytic group whose Lie algebra has immersed a Lie algebra of invariant analytic fields under left translations is finite and is equal to the dimension of the analytic group. This will be demonstrated punctually and with precision.*

Keywords *- Analytic Group, Analytic and Invariant Vector Fields, Left Translations, Lie Algebra, Lie Group.*

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1. Introduction

The following results define questions of analytic and invariant vector fields under left translations. Many authors on research in Lie groups and the representation theory of these groups have mentioned only some aspects in this regard [1-9]. However, there is not a precise demonstration detailed on the Lie structure of an analytic group G and their linear representation through an analytic vector fields algebra $\mathfrak{g}$, under left translations where the subjacent manifold V(G), of the analytic group, also accept said representation through the corresponding automorphisms of the Lie algebra of vector fields under left translations, which we denote as $\mathfrak{g}_{\mathrm{L}}$.

Likewise, invariant on an analytic field can be described as: If is defined an operation in the function of translations and is applied such transformation to an analytic vector field, then the image is fixed, not necessarily analytic. Then is deduced that it is invariant when it is itself, the said given transformation for the left translations induces in whole manifold an automorphism Aut($\mathfrak{g}$), of the analytic fields algebra $\mathfrak{g}$, (huge lie algebra). Then, those analytic fields are immersed in the Lie algebra of the Lie group *G*. The subjacent manifold is of finite dimension and equal to the analytic group *G* dimension.

1.1. Remark 1

The algebra of invariant analytic fields under left translations form a Lie algebra of finite dimension that is isomorphic to the tangent space $\mathfrak{g}_e$. Likewise, with this research, we are strengthening the consideration of the linear structure of an analytic group *G*, whose audomorphisms are induced in all the analytic manifolds of the group *G*. Finally, these permits establish categorically that any operators or transformations in the analytic manifold of *G*, has a linear representation in an analytic fields algebra $\mathfrak{g}$, under left translations. Then, elements of the analytic manifold of *G* can be represented by elements of the $\mathfrak{g}$, under left translations. Then, their dimension is equal.

2. Conjectures and Backgrounds

A study of the Lie algebra of invariant analytic fields under left translations leads us to conjecture the following [1, 2]:

2.1. Conjecture 1

Exists an immersion of the Lie algebra of invariant analytic fields under left translations (that we denote $\mathfrak{g}_L$) in the Lie algebra of the analytic group *G*.

Also is important to give the following conjecture.

2.2. Conjecture 2

The left translations on analytic fields $\mathfrak{g}$, induce a space of automorphisms whose transformations are induced in all of the analytic manifolds of group *G*. Also is important to give the following conjecture.

2.3. Conjecture 3

The immersion space of the subjacent manifold of the Lie algebra of invariant fields under translations $\mathfrak{g}_L$, in the Lie algebra $\mathfrak{g}$, is of finite dimension. Thus, the Lie algebra of invariant analytic fields is of finite dimension and its dimension is equal to the dimension of the analytic group. Then demonstrating these conjectures can be announced:

2.4. Theorem 1 (F. Bulnes and F. Recillas)

The subjacent manifold of the analytic group whose Lie algebra has immersed a Lie algebra of invariant analytic fields under left translations is of finite dimension, and its dimension is equal to the analytic group dimension. Previously, the following exposition can be considered.

2.4.1. Proposition 1

Let *G* be an analytic group. Let ρ_0, be an element of the analytic group *G*. Then the application

$$\Phi_0: G \to G, \tag{1}$$

Defined for the correspondence

$$\sigma \mapsto \Phi_0(\sigma) = \rho_0\sigma^{-1}, \tag{2}$$

Is an analytic field everywhere.

Proof. [3]

2.4.2. Corollary 1

Let *G* be an analytic group. Then, the application

$$\Phi_1(\sigma) = G \to G, \tag{3}$$

With the rule of correspondence

$$\sigma \mapsto \Phi_1(\sigma) = \sigma^{-1}, \tag{4}$$

Proof

The corollary is demonstrated easily to $\rho = \varepsilon$, where ε, in this case, is the identical element of the analytic group.

2.4.3. Remark 2

If *G* is an analytic group, in everything that follows, it will be noted with the symbol $\mathfrak{g}_S$, to the tangent space $T_\sigma G$, on the analytic group *G*, in the element $\sigma \in G$

$$\mathfrak{g}_S \cong T_\sigma G, \tag{5}$$

2.4.4. Proposition 2

Let *G* be an analytic group. $\forall\ \rho \in G$, the left translation Φ_ρ, associated with the element ρ, given by

$$\Phi : G \to G, \tag{6}$$

Defined for the correspondence

$$\sigma \mapsto \Phi_\rho(\sigma) = \rho \bullet \sigma^{-1}, \tag{7}$$

Is an analytic isomorphism of G in itself,

$$\Phi : G \cong G, \tag{8}$$

Whose differential $d\Phi_{\rho}$, in the point $\sigma \in G$, is a linear isomorphism of $\mathfrak{g}_{\sigma}$, on $\mathfrak{g}_{(\sigma)}$,

$$d\Phi_{\rho} = \mathfrak{g}_{\sigma} \cong \mathfrak{g}_{\Phi(\sigma)}, \tag{9}$$

Proof

$\forall \rho \in G$, it is known that the left translation Φ_{ρ}, associated with the element ρ, is a homeomorphism of the subjacent topological group to the analytic group G in itself. Therefore, according to proposition 1 only lacks a demonstration of how much is needed that Φ_{ρ}, and $_{\rho}$-1$= 0$, are analytic mappings of G in themselves. This is the immediate consequence of proposition 1 and the corollary 1. Indeed, if we calculate $\forall \sigma \in G$,

$$\Phi_0 \circ \Phi_1(\sigma) = \Phi_1(\Phi_0 (\sigma)) = \Phi_0(\sigma^{-1}) = \rho(\sigma^{-1}) = \rho \bullet \sigma = \Phi_{\rho}(\sigma),$$

we obtain

$$\Phi_0 \circ \Phi_1(\sigma) = \Phi_{\rho}(\sigma), \quad \forall\ \sigma \in G, \tag{10}$$

And as the composition of analytic mappings is an analytic mapping, we have demonstrated that the application Φ_{ρ}, is an analytic mapping everywhere $\forall\ \sigma \in G$. Proposition 1 implies that Φ_{ρ}, is an analytic isomorphism. Then, proposition 2 is demonstrated.

This shows the existence of an immersion of $\mathfrak{g}$, in $\mathfrak{g}$. With it is demonstrated the conjecture 1.

2.4.5. Definition 1

Let be V, a manifold. A vector field X, we understand an application

$$X : G \rightarrow \cup_{\sigma \in G} \mathfrak{g}_{\sigma}, \tag{11}$$

With the correspondence rule

$$\sigma \mapsto X(\sigma), \tag{12}$$

Let be f, a function defined and analytic in each point of an open set $U \in \mathbb{V}(\mathrm{V})$. Let be X, a vector field defined on the manifold V. We suppose that the function

$$_{X}F : U \rightarrow \mathbb{R}, \tag{13}$$

With the rule of correspondence

$$q \mapsto Xf(q) = X_q f, \tag{14}$$

Is a real analytic function. If this happens for all analytic functions of the manifold V, we say that the vector field X is an analytic vector field.

2.5. Theorem 2

Let be G, an analytic group. Then $\forall\ \rho \in G$, the left translation Φ_{ρ}, associated with the element ρ, induces an automorphism $\underline{\Phi}_{\rho}$, of Lie algebras of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, of all vector fields defined and analytic on the analytic group G, to know

$$\underline{\Phi}_{\rho} = \mathfrak{g} \cong \mathfrak{g},\ \forall\ \rho \in G, \tag{15}$$

Proof

The application can be considered

$$\underline{\Phi} : \mathfrak{g} \rightarrow \mathfrak{g}, \tag{16}$$

Defined by the correspondence

$$X \mapsto \underline{\Phi}_{\rho}(X) = Y, \tag{17}$$

Where

$$Y : G \rightarrow \cup_{\tau \in G} \mathfrak{g}_{\tau}, \tag{18}$$

Defined by the correspondence, too

$$\tau \mapsto Y(\tau) = Y_{\tau} = d\Phi_{\rho} \bullet X_{\rho}\text{-}1_{\tau}, \tag{19}$$

It can be said that this vector field Y, like that defined, is an analytic vector field. Indeed, according to proposition 2, left translation Φ_{ρ}, is an analytic isomorphism whose differential $d\,\Phi_{\rho}$, in the point $\tau \in G$, satisfies the relation

$$d\,\Phi_{\rho}\text{-}1\mathfrak{g}_{\tau} = \mathfrak{g}_{\Phi\rho}\text{-}1_{\tau}(\tau), \tag{20}$$

Which implies $X_{\Phi_{\rho}\text{-}1_{\tau}} \in d\,\Phi_{\rho}\text{-}1\mathfrak{g}_{\tau}$, $\forall\ \tau \in G$. Due to that X, is a vector field defined and analytic on G, and the left translation Φ_{ρ}, is a regular mapping, proposition 2, assures the existence of a unique vector field Y', defined and analytic on the analytic group G, such that

$$d\,\Phi_{\rho}\text{-}1Y'_{\tau} = X_{\Phi\rho}\text{-}1_{\tau}(\tau), \tag{21}$$

Relation that can be written

$$Y'_{\tau} = d\,\Phi_{\rho}\,X_{\Phi_{\rho}\text{-}1_{\tau}}(\tau), \tag{22}$$

Therefore, by the existence unicity, we have $Y = Y'$, which means $\underline{\Phi}_{\rho}(X) = Y \in \mathfrak{g}$. In other words, the application $\underline{\Phi}_{\rho}$, of $\mathfrak{g}$, is well-defined. The same argument permits us to demonstrate that the application $\underline{\Phi}_{\rho}$, is a bijection. Indeed, suppose that it had a vector field Y, defined and analytic on the analytic group G, $\forall\ \tau \in G$, always exists an unique element $\sigma \in G$, ($\sigma = \rho^{-1}\tau$) such that the left translation Φ_{ρ}, maps to τ, as

$$\Phi_{\rho}(\sigma) = \tau, \tag{23}$$

And due to that is an analytic isomorphism of G itself, its differential satisfies the relation

$$d\,\Phi_{\rho}\mathfrak{g}_{\sigma} = \mathfrak{g}_{\Phi_{\rho}(\sigma)}, \tag{24}$$

Therefore, $Y_{\Phi_{\rho}(\sigma)} \in d\,\Phi_{\rho}\mathfrak{g}_{\sigma}$, $\forall\ \sigma \in G$. Hypothesis Y is a vector field defined and analytic on the analytic group G, and due to that, the left translation Φ_{ρ}, is a regular mapping, proposition 2, assures the existence of a unique vector field X, defined and analytic on the analytic group G, such that

$$Y = d\,\Phi_{\rho}\,X_{\Phi_{\rho}\text{-}1_{\tau}}, \forall\ \tau \in G, \tag{25}$$

According to the definition of the application $\underline{\Phi}_{\rho}$, this says us only that

$$\underline{\Phi}_{\rho}(X) = Y,$$

That is to say, $\underline{\Phi}_{\rho}$, is a suprajective application. Further, as was mentioned, X is a unique vector field defined and analytic on G, with the property (25), which means that $\underline{\Phi}_{\rho}$, is bijective. It is verified without any difficulty that $\underline{\Phi}_{\rho}$, is linear

$$\underline{\Phi}(\alpha X + \beta Y) = \alpha\,\underline{\Phi}_{\rho}(X) + \beta\,\underline{\Phi}_{\rho}(Y), \forall\ X, Y \in \mathcal{L}(G), \tag{26}$$

In other words, $\underline{\Phi}_\rho$, is a linear isomorphism.

We consider now the vector fields X_1, and X_2, defined and analytics on the analytic group $G : X_1, X_2 \in L(G)$, and let be

$$\underline{\Phi}_\rho X_i = Y_i \ (1 \leq i \leq 2), \tag{27}$$

According to the definition of the application $\underline{\Phi}_\rho$,

$$Y_{i,\tau} = d\,\underline{\Phi}_\rho X_{i,\rho}\text{-}1_\tau, \ \forall\ \tau \in G, \tag{28}$$

If it is written $\sigma = \rho^{-1}\tau$, then

$$Y_{i,\Phi\rho(\sigma)} = d\Phi_\rho X_{i,\sigma}, \ \forall\ \sigma \in G, \tag{29}$$

Which means that X_i and Y_i are Φ_ρ -related to $(1 \leq i \leq 2)$. When this happens, the fields defined and analytic on the group G, the Lie brackets $[X_1, X_2]$, and $[Y_1, Y_2]$, are Φ_ρ -related, that is to say,

$$[Y_1, Y_2]_{\Phi(\sigma)} = d\Phi_\rho[X_1, X_2], \ \forall\ \sigma \in G, \tag{30}$$

And as $\sigma = \rho^{-1}\tau$, is had that

$$[Y_1, Y_2]_\tau = d\Phi[X_1, X_2]_\rho\text{-}1, \ \forall\ \sigma \in G, \tag{31}$$

Which, according to the definition of $\underline{\Phi}_\rho$, says us only that:

$$d\underline{\Phi}[X_1, X_2] = [Y_1, Y_2] = [\underline{\Phi}_\rho X_1, \underline{\Phi}_\rho X_2], \tag{32}$$

This demonstrates that the linear isomorphism $\underline{\Phi}_\rho$, is an isomorphism of Lie algebras. Then, the theorem is demonstrated.

2.5.1. Definition 2

Let be G, an analytic group. We will say that a vector field X, defined on the group G (even non-necessary analytic), is invariant for left if it satisfies the following condition:

$$\underline{\Phi}_\rho, X = X, \ \forall\ \sigma, \tau \in G, \tag{33}$$

2.5.2. Proposition 3

Let be G, an analytic group, and let be X, a vector field defined on the analytic group G. Then the vector field X is an invariant vector field if and only if,

$$X_\tau = d\Phi_{\tau\sigma}\text{-}1X_\sigma, \ \forall\ \sigma, \tau \in G, \tag{34}$$

That is to say, $\underline{\Phi}_\rho$, $X = X$, $\forall\ \rho \in G$, equivalent to (34).

Proof

The condition is necessary. Suppose that $\underline{\Phi}_\rho$, $X = X$, $\forall\ \rho \in G$. Let be σ, and τ, fix elements but arbitrarily of the group G, which implies the existence of an element $\rho \in G$, such that $\rho\sigma = \tau$, equivalent to $\sigma = \rho^{-1}\tau$. If the now is taken into consideration the hypothesis, it can be written

$$X_\tau = d\Phi_\rho X_\rho\text{-}1_\tau, \tag{35}$$

Such that if it is written as

$$X_\tau = d\Phi_{\tau\sigma}\text{-}1\, X_\sigma, \tag{36}$$

And remembering that σ, and τ, were chosen arbitrarily in group G, then it had (36) $\forall\ \sigma, \tau \in G$; in other words, the condition is necessary. Reciprocally, suppose now that the condition is satisfied. Let be ρ, a fixed element but arbitrarily of the group G. For all element $\tau \in G$, exists a unique element $\sigma \in G$, such that $\rho\sigma = \tau$, which implies that $\rho = \tau\sigma^{-1}$, such that it is taken the hypothesis, can be written (35) $\forall\ \tau \in G$. This implies that $\underline{\Phi}_\rho$, $X = X$, and if it is remembered that ρ, was chosen arbitrarily in G, then $\underline{\Phi}_\rho$, $X = X$, $\forall\ \rho \in G$; in other words, the condition is sufficient. Then, proposition 3 is demonstrated.

2.5.3. Corollary 2

Let be G, an analytic group; let be ε, the unitary element of the analytic group G, and let be X, a vector field defined on the analytic group. Then X is an invariant vector field if the following affirmations are equivalents:

1. $\underline{\Phi}_\rho$, $X = X$, $\forall\ \rho \in G$,

2. $X_\tau = d\Phi_\tau X_\varepsilon$, $\forall\ \tau \in G$.

Proof

Suppose that the condition is satisfied. Let be σ, and τ, fix elements but arbitrary of the group G. If it is written

$$X_\tau = d\Phi_\tau X_\tau = d\Phi(d\Phi_\rho\text{-}1 d\Phi_\sigma) = d\Phi_{\tau\rho}\text{-}1 d\Phi_\sigma X_\varepsilon = (d\Phi_{\tau\rho}\text{-}1 X_\sigma), \tag{37}$$

Where we obtain that

$$X_\tau = d\Phi_{\tau\rho}\text{-}1 X_\sigma, \tag{38}$$

Due that the elements σ, and τ, were chosen arbitrarily in group G, then is obtained (38) $\forall\ \sigma, \tau \in G$, which implies, according to proposition 3, that in particular for the elements $\sigma, \tau \in G$, then is obtained $X_\tau = d\Phi_\tau X_\varepsilon$, $\forall\ \tau \in G$, which demonstrates that the condition is necessary. Then, the implication of the corollary 2 is demonstrated.

With the symbol Ж(G), denoted the space of all the vector fields defined on the analytic group G, and with the symbol $\mathfrak{g}_{\mathcal{L}}$, the space of all the vector fields defined and invariants on analytic group G, that is to say,

$$\mathfrak{g}_{\mathcal{L}} = \{X \in Ж(G) \mid \underline{\Phi}_\rho, X = X, \forall\ \rho \in G\}, \tag{39}$$

2.6. Theorem 3 (C. Chevalley)

[1] Let be G, an analytic group. Then all vector field X, defined and invariant on the analytic group G, such that

$$\underline{\Phi}_\rho, X = X, \forall\ \rho \in G,$$

Is a vector field defined and analytic on the analytic group G (that is to say, if $X \in \mathfrak{g}_{\mathcal{L}}$, then $X \in \mathfrak{g}$, the Lie algebra of all the vector fields defined and analytic on the analytic group G).

Proof

Let be f, a fixed function but arbitrarily defined and analytic in each point of an open set U, of the analytic group G: $U \subset$ V(V), and let be σ_0, a fixed point but arbitrarily of the open set U: $\sigma_0 \in U$, and let

$$: (\{x_1, x_2, x_3, ..., x_n\}, V_1, a), \tag{40}$$

a coordinates system on G, in the point $\sigma_0 \in G$. As $\sigma_0^{-1} V_1$, is a neighborhood of the unitary element ε, of the group G, to know, $\sigma_0^{-1} V_1 \in$ V(ε), then exists a neighborhood V'_2, of the element $\varepsilon \in G$, to know, $V'_2 \in$ V(ε), such that

$$V'_2\, V'_2 \subset \sigma_0^{-1} V_1, \tag{41}$$

If it is written as

$$\sigma_0 V'_2\, \sigma_0^{-1} \sigma_0 V'_2 \subset V_1, \tag{42}$$

And if is defined $V_2 = \sigma_0 V'_2$, then obtain

$$V_2\, \sigma_0^{-1} V'_2 \subset V_1, \tag{43}$$

It is observed that the application

$$: G \times G \to G, \tag{44}$$

Defined by the correspondence

$$(\sigma, \tau) \mapsto (\sigma, \tau) = \sigma\sigma_0^{-1}\tau, \tag{45}$$

Is an analytic mapping everywhere. Indeed, it is calculated for all $(\sigma, \tau)\in G \times G$,

$$\Phi \circ ((\Phi_1 \circ \Phi_0) \times \Phi_1)(\sigma, \tau) = \Phi \circ ((\Phi_1 \circ \Phi_0)(\sigma), \Phi_1(\tau))$$

$$= \Phi \circ ((\Phi_1)(\sigma_0\sigma^{-1}), \tau^{-1}) = \Phi(\sigma_0\sigma^{-1}, \tau^{-1}) = \sigma_0\sigma^{-1}\tau = \Psi(\sigma, \tau),$$

Is to say,

$$\Psi = \Phi \circ ((\Phi_1 \circ \Phi_0) \times \Phi_1), \tag{46}$$

Demonstrating that Ψ, is an analytic mapping everywhere. As $V_2\, \sigma_0^{-1} V'_2 \subset V_1$, if only if $\Psi(V_2 \times V_2) \subset V_1$, and as $x_i \in \boldsymbol{\mathcal{F}}(p)$, $\forall\, p\in V_1$ $(1\leq i \leq n)$, in particular for all pair $(\sigma, \tau)\in V_2 \times V_2$, $x_i\in \boldsymbol{\mathcal{F}}(\Psi(\sigma, \tau))$, then $\Psi * x_i\in \boldsymbol{\mathcal{F}}(\Psi(\sigma, \tau))$, and whose expression in the coordinates system $\{z_1, z_2, z_3, ..., z_n, z_{n+1}, ..., z_{2n}\}$, where $z_i = x_i \circ \theta_1$, $z_{n+i} = x_i \circ \theta_2$, for $(1\leq i \leq n)$, is given by

$$\Psi * x_i = \mathrm{F}_i(z_1, z_2, z_3, ..., z_n, z_{n+1}, ..., z_{2n}), \tag{47}$$

Where $\mathrm{F}_i(z_1, z_2, z_3, ..., z_n, z_{n+1}, ..., z_{2n})$, is a real function of $2n$, real variables, defined and analytic in each point of an open neighborhood of the point $(\varphi(\sigma_0), \varphi(\sigma_0))\in\mathbb{R}^{2n}$, as by hypothesis X, is a left-invariant vector field then $\underline{\Phi}_\rho, X = X$, $\forall\, \rho\in G$, and can be written and calculated

$$X_\sigma x_i = d\Phi_{\sigma\sigma_0\text{-}1} X_{\sigma_0} \bullet x_i$$

$$= X_{\sigma_0}(\Phi_{\sigma\sigma_0\text{-}1}) \bullet x_i$$

$$= X_{\sigma_0}\, x_i \circ \Phi_{\sigma\sigma_0\text{-}1},$$

To obtain

$$X_\sigma x_i = X_{\sigma 0}\, x_i \circ \Phi_{\sigma\sigma 0}\text{-}1, \tag{48}$$

On another side, it is considered the function

$$x_i \circ \Phi_{\sigma\sigma 0}\text{-}1 : V_2 \to \mathbb{R}, \tag{49}$$

With the rule of correspondence

$$\tau \mapsto x_i \circ \Phi_{\sigma\sigma_0\text{-}1}(\tau), \tag{50}$$

If it is calculated

$$x_i \circ \Phi_{\sigma\sigma 0}\text{-}1(\tau) = x_i(\sigma\sigma_0^{-1}\tau)$$

$$= (\sigma, \tau) = \boldsymbol{\mathcal{F}}_i(x_1(\sigma), x_2(\sigma), x_3(\sigma), ..., x_n(\sigma), x_1(\tau), x_2(\tau), x_3(\tau), ..., x_n(\tau)), \tag{51}$$

For all $\tau \in V_2$. It is written said function in the form

$$x_i \circ \Phi_{\sigma\sigma 0}\text{-}1(\tau) = \boldsymbol{\mathcal{F}}_i(x_1(\sigma), x_2(\sigma), x_3(\sigma), ..., x_n(\sigma), x_1(\tau), x_2(\tau), x_3(\tau), ..., x_n(\tau)),$$

On V_2, and is calculated $X_\sigma x_i$, can be obtained

$$X_\sigma x_i = X_{\sigma 0}\, x_i \circ \Phi_{\sigma\sigma 0}\text{-}1 = X_{\sigma 0} \circ \boldsymbol{\mathcal{F}}_i(x_1(\sigma), x_2(\sigma), x_3(\sigma), ..., x_n(\sigma), x_1(\tau), x_2(\tau), x_3(\tau), ..., x_n(\tau))$$

$$= \boldsymbol{\Sigma}_j^n\, X_{\sigma 0} x_j (\partial \boldsymbol{\mathcal{F}}_i / \partial x_j)|_{(\sigma,\, \sigma 0)},$$

Then can be obtained

$$X_\sigma x_i = \boldsymbol{\Sigma}_j^n\, X_{\sigma 0}\, x_j (\partial \boldsymbol{\mathcal{F}}_i / \partial x_j)|_{(\sigma,\, \sigma 0)}, \tag{52}$$

Where the symbol $(\partial \boldsymbol{\mathcal{F}}_i/\partial x_j)|_{(\sigma,\, \sigma 0)}$ means that the partial derivatives are evaluated in the point $(x_1(\sigma), x_2(\sigma), x_3(\sigma), ..., x_n(\sigma), x_1(\tau), x_2(\tau), x_3(\tau), ..., x_n(\tau)) \in \mathbb{R}^{2n}$.

Further $X_{\sigma_0} x_i$, is constant, then it is obtained that $X_\sigma x_i$, $(1 \leq i \leq n)$, is an analytic function in the point $\sigma_0 \in U$. As σ_0, was chosen arbitrarily in U., then it has been demonstrated that $X_\sigma x_i$, is an analytic function in each point of an open set U. This implies that.

$$X_\sigma f = \boldsymbol{\Sigma}_j^n\, X_{\sigma 0}\, x_j (\partial \boldsymbol{\mathcal{F}}_i / \partial x_j), \tag{53}$$

Let be an analytic function on the open set $U \in \mathrm{V}(G)$, and as the analytic function f was chosen arbitrarily in the function class $\mathbb{F}(p)$, it has been demonstrated that $X \in \mathfrak{g}$, (the Lie algebra of the defined and analytic fields of the analytic group G) then the theorem 3, is demonstrated.

2.6.1. Corollary 3

If G is an analytic group, then the space $\mathfrak{g}_L$, of the vector fields defined and invariant by left on the analytic group G, is a Lie subalgebra of the Lie algebra $\mathfrak{g}$, of the vector fields defined and analytic on the group G.

Proof

If $X \in \mathfrak{g}_L$, Chevalley's theorem (theorem 3) implies that $X \in \mathfrak{g}$, and therefore $\mathfrak{g}_L$, is a subset of the Lie algebra $\mathfrak{g}$. Due to that $\mathfrak{g}_L$, is identified with the tangent space of $\mathfrak{g}$ (by immersion in $\mathfrak{g}$), $\mathfrak{g}_L$, without difficulty, is a vector subspace of $\mathfrak{g}$. To demonstrate that $\mathfrak{g}_L$, is a Lie subalgebra, is considered two elements X_1 and $X_2 \in \mathfrak{g}_L$, and is considered that of $\mathfrak{g}$, is a Lie subalgebra then $[X_1, X_2] \in \mathfrak{g}_L$.

As the theorem 2, affirms that $\underline{\Phi}_\rho$, is an automorphism of Lie algebras is has that

$$\underline{\Phi}_\rho [X_1, X_2] = [\underline{\Phi}_\rho\, X_1, \underline{\Phi}_\rho\, X_2], \tag{54}$$

And as by hypothesis $\underline{\Phi}_\rho X_i = X_i$ $(i = 1, 2)$, $\forall\, \rho \in G$, is obtained

$$\underline{\Phi}_\rho [X_1, X_2] = [X_1, X_2],\ \forall\, \rho \in G, \tag{55}$$

Therefore, $[X_1, X_2] \in \mathfrak{g}$, and all this demonstrates that $\mathfrak{g}_L$, is a Lie subalgebra of the Lie algebra $\mathfrak{g}$. Then, stays demonstrates the corollary.

As this subalgebra $\mathfrak{g}_L$, of the corollary 3 is a Lie algebra, by proper right, it has the following definition.

2.6.2. Definition 3

Let be G, an analytic group. The Lie algebra $\mathfrak{g}_L$, of the vector fields defined on the analytic group G and invariant by the left is called the Lie algebra of the analytic group.

2.7. Theorem 4

Let be G, an analytic group, and let be $\mathfrak{g}_L$, its Lie algebra. Then the application Ψ, of the Lie algebra $\mathfrak{g}_L$, in the tangent space $\mathfrak{g}_e$:

$$: \mathfrak{g}_L \rightarrow \mathfrak{g}_e, \tag{56}$$

Defined by the correspondence

$$X \mapsto \Psi(X) = X_\varepsilon, \tag{57}$$

Is an isomorphism of the algebra $\mathfrak{g}$, on the tangent space $\mathfrak{g}_e$,

$$: \mathfrak{g}_L \cong \mathfrak{g}_e = T_\varepsilon(G), \tag{58}$$

Proof

The application Ψ, is linear. Indeed, if it is calculated

$$(\alpha X + \beta Y) = (\alpha X + \beta Y)_\varepsilon = \alpha X(\varepsilon) + \beta Y(\varepsilon) = \alpha X_\varepsilon + \beta Y_\varepsilon = \alpha\Psi(X) + \beta\Psi(Y),$$

Is to say,

$$(\alpha X + \beta Y) = \alpha\Psi(X) + \beta\Psi(Y), \tag{59}$$

Then, the application Ψ, is injective. Indeed, let be $X \in \mathfrak{g}_L$, such that $\Psi(X) = 0$. Due that by hypothesis X, is invariant by the left, the corollary 2, implies $X_\sigma = d\Phi_\sigma X_\sigma$, $\forall\ \sigma \in G$. Thus $X_\sigma = d\Phi(\Psi(X)) = 0$, and as this is valid $\forall\ \sigma \in G$, is obtained $X = 0$. We affirm that the application Ψ, is suprajective. Indeed, considering a vector $X_\varepsilon \in \mathfrak{g}_e$, is defined the vector field:

$$X: G \rightarrow \bigcup_{\sigma \in G} \mathfrak{g}_\sigma, \tag{60}$$

Defined by the correspondence

$$\sigma \mapsto X(\sigma) = X = d\Phi_\sigma X_\varepsilon, \tag{61}$$

If now is considered the proposition 3, the vector field X, by its definition, to know; $X_\sigma = d\Phi_\sigma X_\sigma$, $\forall\ \sigma \in G$, is accord with the proposition 3, invariant by the left, is to say; $X \in \mathfrak{g}_L$. Then the suprajectivity is demonstrated. Therefore, the application Ψ, is bijective. This last demonstrates that Ψ, is an isomorphism of the algebra $\mathfrak{g}_L$, in the tangent space $\mathfrak{g}_e$.

3. Results and Discussion

Now will be completed the demonstration of the mean result (theorem 1), considering the following key proposition (equivalent to the conjecture 2. 3) to stablish the equality of the dimensions between the subjacent manifold whose Lie algebra has immersed a Lie algebra of invariant analytic fields under left translations and the analytic group.

3.1. Proposition 4 (F. Bulnes and F. Recillas)

The dimension of the Lie algebra of the invariant and analytic vector fields is finite and equal to the dimension of the analytic group G.

Proof

The immersion of $\mathfrak{g}_L$, in $\mathfrak{g}$, permits the identification between the tangent space of the analytic group G, with the Lie algebra $\mathfrak{g}_L$, (remark 2) of the translations of the Lie algebra $\mathfrak{g}$, (Chevalley's theorem).

Then the isomorphism $\mathfrak{g}_L, \cong \mathfrak{g}_e$, implies that the Lie algebra $\mathfrak{g}_L$, is of finite dimension. Then by the theorem of Lie algebras correspondence, each Lie subalgebra of finite dimension is corresponed a Lie subgroup of finite dimension of the analytic group G, subjacent in the manifold $\mathrm{V}(G)$. Said subgroup is an analytic group.

For another side, due that the invariant and analytic vector fields of the analytic group G, are corresponded in fields defined and analytic in the analytic group G, is induced an automorphism of G, on the points set $\mathrm{V}(G)$. Then

$$\dim \mathfrak{g}_L, = \dim \mathrm{V}(G) = \dim G.$$

4. Conclusion

From this last proposition (proposition 4) and the Chevalley's theorem (theorem 3) is demonstrated the proposed theorem (theorem 1).

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