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Original Article

On the Diophantine Equations $(x_1x_2x_3 \dots 6)^x + (y_1y_2y_3 \dots 1)^y = z^2$ and $(x_1x_2x_3 \dots 4)^{2x} + (y_1y_2y_3 \dots 2)^{5y} = z^2$

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Abstract *- In this paper, the Diophantine equations $(x_1x_2x_3 \dots 6)^x + (y_1y_2y_3 \dots 1)^y = z^2$ and $(x_1x_2x_3 \dots 4)^{2x} + (y_1y_2y_3 \dots 2)^{5y} = z^2$ have been discussed for positive integer solutions. Here $x_1, x_2, x_3 \dots$ and $y_1, y_2, y_3 \dots$ are digits 0, 1, 2,…9.*

***Keywords* -** *Exponential, Diophantine equation and integral solution.*

1. Introduction

Several authors discussed the exponential Diophantine equations. Poonen, B. (1998) studied some Diophantine equations of the form. $x^n + y^n = z^m$. Sroysang, B. (2013) discussed the Diophantine equation $5^x + 11^y = z^2$. It was shown that this Diophantine equation has the solutions $(x, y, z) = (1, 1, 4)$ and $(2, 1, 6)$. Burshtein, N. (2018) discussed the Diophantine equations $p^3 + q^3 = z^2$ and $p^3 - q^3 = z^2$ where p q are primes for possible integer solutions. Burshtein, N. (2019) obtained the solutions to the Diophantine equations $5^x + 103^y = z^2$ and $5^x + 11^y = z^2$ with positive integers *x*, *y*, *z*. Burshtein, N. (2019) discussed the Diophantine equations $6^x + 11^y = z^2$ and $6^x - 11^y = z^2$ in positive integers *x*, *y*, *z*.

Here the Diophantine equations $(x_1x_2x_3 \dots 6)^x + (y_1y_2y_3 \dots 1)^y = z^2$ and $(x_1x_2x_3 \dots 6)^{2x} + (y_1y_2y_3 \dots 2)^{5y} = z^2$ have been discussed for positive integer solutions where $x_1, x_2, x_3 \dots$ and $y_1, y_2, y_3 \dots$ are digits 0, 1, 2,…9. This is some generalization of Burshtein, N. [3] and further extension.

2. Preliminaries

2.1. Lemma 1(Sroysang, B. (2013)): The Diophantine equation $5^x + 11^y = z^2$ has the solutions $(x, y, z) = (1, 1, 4)$ and $(2, 1, 6)$.

2.2. Lemma 2(Burshtein (2019)): The exponential Diophantine equation $6^x + 11^y = z^2$ has no positive integer solution.

2.3. Lemma 3(Burshtein (2019)): The exponential Diophantine equation $6^x - 11^y = z^2$ has the solution $(x, y, z) = (2, 1, 5)$.

3. Analysis

3.1. Theorem 1: The Diophantine equation $(x_1x_2x_3 \dots 6)^x + (y_1y_2y_3 \dots 1)^y = z^2$ has no positive integer solution in *x*, *y*, *z* where $x_1, x_2, x_3 \dots$ and $y_1, y_2, y_3 \dots$ are digits 0, 1, 2,…9.

Proof: It is obvious that the term $(x_1x_2x_3 \dots 6)^x$ Has the last digit equal to 6 for each positive integer value of x. Similarly the term $(y_1y_2y_3 \dots 1)^y$ Has the last digit equal to 1 for each positive integer value of y? Therefore, the sum of these two terms

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$(x_1x_2x_3 \dots 6)^x + (y_1y_2y_3 \dots 1)^y$ It is an odd integer whose last digit is equal to $6 + 1 = 7$. However, there is no odd square. z^2 which has seven as the last digit. Therefore, the given Diophantine equation $(x_1x_2x_3 \dots 6)^x + (y_1y_2y_3 \dots 1)^y = z^2$ has no solution in *x, y, z.*

3.2. Particular Cases

3.2.1. Case 1: The Diophantine equation $16^x + 11^y = z^2$ has no positive integer solution in x, y, z.

Proof: It is evident that the term 16^x Has the last digit equal to 6 for each positive integer value of x. Similarly the term 11^y Has the last digit equal to 1 for each positive integer value of y? Thus the term $16^x + 11^y$It is an odd integer, and its last digit equals $6 + 1 = 7$. But no odd square z^2 has the last digit equal to 7. Therefore, the Diophantine equation $16^x + 11^y = z^2$ has no positive integer solution in *x, y, z.*

3.2.2. Case 2: The Diophantine equation $116^x + 111^y = z^2$ has no positive integer solution in x, y, z.

Proof: It is evident that the term 116^x Has the last digit equal to 6 for each positive integer value of x. Similarly the term 111^y Has the last digit equal to 1 for each positive integer value of y? Thus the term $116^x + 111^y$It is an odd integer, and its last digit equals $6 + 1 = 7$. But no odd square z^2 has the last digit equal to 7. Therefore, the Diophantine equation $116^x + 111^y = z^2$ has no positive integer solution in *x, y, z.*

3.2.3. Case 3: The Diophantine equation $126^x + 211^y = z^2$ has no positive integer solution in x, y, z.

Proof: It is evident that the term 126^x Has the last digit equal to 6 for each positive integer value of x. Similarly the term 211^y Has the last digit equal to 1 for each positive integer value of y? Thus the term $126^x + 211^y$It is an odd integer, and its last digit equals $6 + 1 = 7$. But no odd square z^2 has the last digit equal to 7. Therefore, the Diophantine equation $126^x + 211^y = z^2$ has no positive integer solution in *x, y, z.*

3.3. Theorem 2: The Diophantine equation $(x_1x_2x_3 \dots 4)^{2x} + (y_1y_2y_3 \dots 2)^{5y} = z^2$ has no positive integer solution in *x, y, z* where x_1, x_2, x_3 ... and y_1, y_2, y_3 ... are digits 0, 1, 2,…9.

Proof: It is obvious that the term $(x_1x_2x_3 \dots 6)^{2x}$ has the last digit equal to 6 for each positive integer value of x. Similarly the term $(y_1y_2y_3 \dots 1)^{5y}$ has the last digit equal to 2 for each positive integer value of y. Therefore, the sum of these two terms $(x_1x_2x_3 \dots 6)^{2x} + (y_1y_2y_3 \dots 1)^{5y}$ is an even integer whose last digit is equal to $6 + 2 = 8$. But there is no even square z^2 which has eight as the last digit. Therefore, the given Diophantine equation $(x_1x_2x_3 \dots 6)^{2x} + (y_1y_2y_3 \dots 1)^{5y} = z^2$ has no positive integer solution in *x, y, z.*

3.4. Particular Cases

3.4.1 Case 1: The Diophantine equation $14^{2x} + 12^{5y} = z^2$ has no positive integer solution in x, y, z.

Proof: It is obvious that the term 14^{2x} has the last digit equal to 6 for each positive integer value of x. Similarly the term 12^{5y} has the last digit equal to 2 for each positive integer value of y. Thus the term $14^{2x} + 12^{5y}$is an even integer, and its last digit is equal to $6 + 2 = 8$. But no even square z^2 has the last digit equal to 8. Therefore, the Diophantine equation $14^{2x} + 12^{5y} = z^2$ has no positive integer solution in *x, y, z.*

3.4.2. Case 2: The Diophantine equation $114^{2x} + 112^{5y} = z^2$ has no positive integer solution in x, y, z.

Proof: It is obvious that the term 114^{2x} has the last digit equal to 6 for each positive integer value of x. Similarly the term 112^{5y} has the last digit equal to 2 for each positive integer value of y. Thus the term $114^{2x} + 112^{5y}$ is an even integer, and its last digit is equal to $6 + 2 = 8$. But no odd square z^2 has the last digit equal to 8. Therefore, the Diophantine equation $114^{2x} + 112^{5y} = z^2$ has no positive integer solution in *x, y, z.*

3.4.3. Case 3: The Diophantine equation $124^{2x} + 212^{5y} = z^2$ has no positive integer solution in x, y, z.

Proof: It is obvious that the term 124^{2x} has the last digit equal to 6 for each positive integer value of x. Similarly the term 212^{5y} has the last digit equal to 2 for each positive integer value of y. Thus the term $124^{2x} + 212^{5y}$ is an even integer, and its last digit is equal to $6 + 2 = 8$. But no odd square z^2 has the last digit equal to 8. Therefore, the Diophantine equation $124^{2x} + 212^{5y} = z^2$ has no positive integer solution in *x*, *y*, *z*.

4. Conclusion

Here, it has been shown that the Diophantine equations $(x_1 x_2 x_3 \ldots 6)^x + (y_1 y_2 y_3 \ldots 1)^y = z^2$ and $(x_1 x_2 x_3 \ldots 6)^{2x} + (y_1 y_2 y_3 \ldots 2)^{5y} = z^2$ have no positive integer solution in *x, y, z*. Some particular solutions have also been discussed.

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