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Original Article

Some Common Fixed Point Theorems on Complete Hilbert Space

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Abstract *- The present research paper aims to establish some generalized Banach fixed point theorems on closed Hilbert spaces using the sequence of mappings and satisfying the Banach Fixed point Property. The Banach fixed point theory plays an important role in modern areas of mathematics, mathematical Science, and different branches of Science. Koparde P. V; Waghmode B. B [5]; Pandhar, M. D and Waghmode, B. B [6] have established some Banach Fixed point theorems on a sequence of mappings on a closed subset S of a Hilbert space H. After that, Veerapandi, T and Kumar, S. A. [9] have introduced some fruitful results in this line. In this research article, some generalized Banach fixed point theorems of Veerapandi, T and Kumar, S. A. [9] are stated above. The mathematician established a theory on Hilbert spaces in sequences of mappings on closed spaces.*

Keywords *- Cauchy sequence, Closed space, Hilbert space, Fixed Point.*

AMS Subject Classifications (2000): *47H10::54H25*

1. Introduction

The well-known Banach fixed-point theory was introduced by S. Banach in 1922. After that, Brower, Schouder, Fisher, Gahler, S, Dhage, B. C and other mathematicians work in this line to establish different fixed point properties and theorems on different mappings in different metric spaces. In this research paper, some generalized fixed point theorems of Veerapandi, T and Kumar, S. A. [9] have been established on Hilbert space. Koparde and Waghmode proved that, if $\{ T_n \}$ is a sequence of mappings that satisfy the condition,

$$\| T_i\xi - T_j\zeta \|^2 \le \gamma\left(\left(\| \xi - T_i\xi \|^2 + \| \zeta - T_j\zeta \|^2\right) \forall \;\; \xi,\zeta \in S,\;\; \xi \neq \zeta, \text{ also, } 0 \le \gamma < ½,$$

Then $\{T_n\}$ has a common fixed point in S.

Veerapandi, T and kumar, S.A. (1998) proved that, if $\{ T_n \}$ is a sequence of mappings that satisfy the conditions

$$\| T_i\xi - T_j\zeta \|^2 \le \gamma_1 \| \xi - \zeta \|^2 + \gamma_2\left(\| \xi - T_i\xi \|^2 + \| \zeta - T_j\zeta \|^2\right)$$

$$+\frac{\gamma_3}{2}\left(\| \xi - T_j\zeta \|^2 + \zeta - T_i\xi \|^2\right)$$

$\forall \forall \;\; \xi,\zeta \in S,\;\; \xi \neq \zeta,\;\; 0 \le \gamma_1, \gamma_2, \gamma_3 < 1$ and $\gamma_1+2\gamma_2+2\gamma_3<1$ then $\{T_n\}$ has a common fixed point in. Here, the generalized fixed point theorems have been found on Closed Hilbert spaces satisfying the well-known Banach Fixed point condition for the mappings $\{ T_n \}$ defined on a closed subset S of a Hilbert space H.

2. Main Results

Theorem (1.1) Let S be a closed subset of a Hilbert space H and $\{ T_n \} S \to S$ be a sequence of mappings that satisfy the following condition,

$$\| T_i\xi - T_j\zeta \|^2 \le \gamma_1 \| \xi - \zeta \|^2 + \gamma_2\left(\| \xi - T_i\xi \|^2 + \| \zeta - T_j\zeta \|^2\right) +$$

$$+\frac{\gamma_3}{2}\left(\| \xi - T_j\zeta \|^2 + \| \zeta - T_i\xi \|^2\right) + \gamma_4\left(\frac{\| \xi - T_i\xi \|^2 . \| \zeta - T_j\xi \|^2}{\| \xi - \zeta \|^2 + \| \xi - T_j\zeta \|^2}\right)$$

$$+\gamma_5\left(\| \xi - T_j\zeta \| . \| \zeta - T_i\xi \|\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1.1)$$

$\forall\ \xi, \zeta \in S$, $\xi \neq \zeta$, and $0 \leq \gamma_1, \gamma_2, \gamma_3\ \gamma_4, \gamma_5 < 1$, $0 < \gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4 < 1$, $0 < 1 - (2\gamma_2 + \gamma_3) < 1$, $0 < 1 - (\gamma_1 + \gamma_3 + \gamma_5) < 1$.

Then $\{ T_n \}$ has a unique common fixed point.

Proof. Let $\xi_0 \in S$, be an arbitrary point. The sequence $\{\xi_n\}$ in S is defined as $\xi_{n+1} = T_{n+1}\xi_n$, were $n = 0,1,2,3, \ldots$

At first, here it is to prove that the sequence $\{\xi_n\}$ is a Cauchy sequence.

Therefore, $\|\xi_{n+1} - \xi_n\|^2 = \| T_{n+1}\xi_n - T_n\xi_{n-1} \|^2$

$$\Rightarrow \|\xi_{n+1} - \xi_n\|^2 \leq \gamma_1 \| \xi_n - \xi_{n-1} \|^2 + \gamma_2(\| \xi_n - T_{n+1}\xi_n \|^2 + \| \xi_{n-1} - T_n\xi_{n-1} \|^2)$$

$$+\frac{\gamma_3}{2}\left(\| \xi_n - T_n\xi_{n-1} \|^2 + \| \xi_{n-1} - T_{n+1}\xi_n \|^2\right) + \gamma_4\left(\frac{\| \xi_n - T_{n+1}\xi_n \|^2 . \| \xi_{n-1} - T_n\xi_{n-1} \|^2}{\| \xi_n - \xi_{n-1} \|^2 + \| \xi_n - T_n\xi_{n-1} \|^2}\right)$$

$$+\gamma_5\left(\| \xi_n - T_n\xi_{n-1} \| . \| \xi_{n-1} - T_{n+1}\xi_n \|\right)$$

$$\Rightarrow \|\xi_{n+1} - \xi_n\|^2 \leq a \| \xi_n - \xi_{n-1} \|^2 + b(\| \xi_n - \xi_{n-1} \|^2 + \| \xi_{n-1} - \xi_n \|^2)$$

$$+\frac{\gamma_3}{2}(\| \xi_n - \xi_n \|^2 + \| \xi_{n-1} - \xi_{n+1} \|^2$$

$$+\gamma_4\left(\frac{\| \xi_n - \xi_{n+1} \|^2 . \| \xi_{n-1} - \xi_n \|^2}{\| \xi_n - \xi_{n-1} \|^2 + \| \xi_n - \xi_n \|^2}\right) + \gamma_5(\| \xi_n - \xi_n \| . \| \xi_{n-1} - \xi_{n+1} \|)$$

$$\Rightarrow \|\xi_{n+1} - \xi_n\|^2 \leq \gamma_1 \| \xi_n - \xi_{n-1} \|^2 + \gamma_2(\| \xi_n - \xi_{n+1} \|^2 + \| \xi_{n-1} - \xi_n \|^2)$$

$$+\frac{\gamma_3}{2}(\| \xi_{n-1} - \xi_{n+1} \|^2 + \gamma_4(\| r_n - r_{n+1} \|^2$$

$$\Rightarrow \|\xi_{n+1} - \xi_n\|^2 \leq \gamma_1 \| \xi_n - \xi_{n-1} \|^2 + \gamma_2(\| \xi_n - \xi_{n+1} \|^2 + \| \xi_{n-1} - \xi_n \|^2)$$

$$+\frac{\gamma_3}{2}(\| (\xi_{n-1} - \xi_n) + (\xi_n - \xi_{n+1}) \|^2 + \gamma_4(\| \xi_n - \xi_{n+1} \|^2$$

On simplifying, the above yields,

$$\Rightarrow \|\xi_{n+1} - \xi_n\|^2 \leq (\gamma_1 + \gamma_2 + \gamma_3) \| \xi_n - \xi_{n-1} \|^2 + (\gamma_2 + \gamma_3 + \gamma_4) \| \xi_n - \xi_{n+1} \|^2$$

$$(1 - \gamma_2 - \gamma_3 - \gamma_4)\|\xi_{n+1} - \xi_n\|^2 \leq (\gamma_1 + \gamma_2 + \gamma_3) \| \xi_n - \xi_{n-1} \|^2$$

Which implies that, $\|\xi_{n+1} - \xi_n\|^2 \leq \frac{(\gamma_1 + \gamma_2 + \gamma_3)}{(1 - \gamma_2 - \gamma_3 - \gamma_4)} \| \xi_n - \xi_{n-1} \|^2$

Let $\frac{(\gamma_1 + \gamma_2 + \gamma_3)}{(1 - \gamma_2 - \gamma_3 - \gamma_4)} = \lambda^2 < 1$, then the above gives,

$$\|\xi_{n+1} - \xi_n\|^2 \leq \lambda^2 \| \xi_n - \xi_{n-1} \|^2 \qquad (1.1.2)$$

$$\| \xi_{n+1} - \xi_n \| \leq \lambda \| \xi_n - \xi_{n-1} \|, \forall \text{ n}$$

$$\| \xi_{n+1} - \xi_n \| \leq \lambda^2 \| \xi_{n-1} - \xi_{n-2} \|, \text{ for every n} \quad (1.1.3)$$

. . . .

Thus, in general, it yields,

$$\| \xi_{n+1} - \xi_n \| \leq \lambda^n \| \xi_1 - \xi_0 \|, \text{ for every n} \quad (1.1.4)$$

Now, for any positive integer $m \geq n \geq 1$,

$$\| \xi_m - \xi_n \| \leq \| \xi_n - \xi_{n+1} \| + \| \xi_{n+1} - \xi_{n+2} \| + \cdots + \| \xi_{m-1} - \xi_m \|$$

$$\| \xi_m - \xi_n \| \leq \lambda^n \| \xi_1 - \xi_0 \| + \lambda^{n+1} \| \xi_1 - \xi_0 \| + \cdots + \lambda^{m-1} \| \xi_1 - \xi_0 \|$$

$$\| \xi_m - \xi_n \| \leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) \| \xi_1 - \xi_0 \|$$

$$\| \xi_m - \xi_n \| \leq \left(\frac{\lambda^n}{1-\lambda}\right) \| \xi_1 - \xi_0 \| \to 0 \ as\ n \to \infty.$$

Therefore, $\{\xi_n\}$ is a Cauchy in S. As S is a closed subset of H, so $\{\xi_n\}$ convergent sequence and converges to a point μ in S.

Now we see that μ is a common fixed point of $\{T_n\}$ on S.

Let, $T_n\mu \neq \mu$ for all n. For any positive integer, inequation (1.1.1) gives,

$$\| \mu - T_n\mu \|^2 = \| (\mu - \xi_n) + (\xi_n - T_n\mu) \|^2$$

$$\leq 2 \| \mu - \xi_n \|^2 + 2 \| \xi_n - T_n\mu \|^2$$

$$\leq 2 \| -r_n \|^2 + 2 \| T_n r_{n-1} - T_n u \|^2$$

$$\leq 2 \| \mu - \xi_n \|^2 + 2\{\gamma_1 \| \xi_{n-1} - \mu \|^2 + \gamma_2 (\| \xi_{n-1} - T_n\xi_{n-1} \|^2 + \| \mu - T_n\mu \|^2)$$

$$+ \frac{\gamma_3}{2} (\| \xi_{n-1} - T_n\mu \|^2 + \| \mu - T_n\xi_{n-1} \|^2) + \gamma_4 \left(\frac{\| \xi_{n-1} - T_n\xi_{n-1} \|^2 . \| \mu - T_n\mu \|^2}{\| \xi_{n-1} - \mu \|^2 + \| \xi_n - T_n\mu \|^2} \right)$$

$$+ \gamma_5 (\| \xi_{n-1} - T_n\mu \| . \| \mu - T_n\xi_{n-1} \|)\}$$

Which implies that,

$$\| \mu - T_n\mu \|^2 \leq 2 \| \mu - \xi_n \|^2 + 2\{\gamma_1 \| \xi_{n-1} - \mu \|^2 + \gamma_2 (\| \xi_{n-1} - \xi_n \|^2 + \| \mu - T_n\mu \|^2)$$

$$+ \frac{\gamma_3}{2} (\| \xi_{n-1} - T_n\mu \|^2 + \| \mu - \xi_n \|^2) + \gamma_4 \left(\frac{\| \xi_{n-1} - \xi_n \|^2 . \| \mu - T_n\mu \|^2}{\| \xi_{n-1} - \mu \|^2 + \| \xi_{n-1} - T_n\mu \|^2} \right)$$

$$+ \gamma_5 (\| \xi_{n-1} - T_n\mu \| . \| \mu - \xi_n \|)\}$$

After simplifying the above gives,

$\| \mu - T_n\mu \|^2 \leq 2\gamma_2 \| u - T_n u \|^2 + \gamma_3 \| u - T_n u \|^2$

i.e, $\| \mu - T_n\mu \|^2 \leq (2\gamma_2 + \gamma_3) \| \mu - T_n\mu \|^2$

i.e, $\{1-(2\gamma_2+\gamma_3)\}\ \| \mu - T_n\mu \|^2 \leq 0$, which is a contradiction as $0<1-(2\gamma_2+\gamma_3)<1$, the only possibility is that $\| \mu - T_n \mu \| = 0$

$\Rightarrow T_n \mu = \mu$

Therefore $T_n\mu = \mu$.

Hence, μ is a common fixed point of $\{T_n\}$ on S.

Lastly, to prove μ is unique, if possible, let ω be another fixed point in S.

i.e, $T_n\mu = \mu$, also $T_m\omega = \omega$.

Therefore, the inequality (1.1.1) gives,

$$\| \mu - \omega \|^2 = \| T_n\mu - T_m\omega \|^2$$

$$\Rightarrow \| \mu - \omega \|^2 \leq \gamma_1 \| \mu - \omega \|^2 + \gamma_2(\| \mu - T_n\mu \|^2 + \| \omega - T_m\omega \|^2) + \frac{\gamma_3}{2}(\| \mu - T_m\omega \|^2 + \| \omega - T_n\mu \|^2)$$

$$+\gamma_4\left(\frac{\| \mu - T_n\mu \|^2 \cdot \| \omega - T_m\omega \|^2}{\| \mu - \omega \|^2 + \| \mu - T_m\omega \|^2}\right) + \gamma_5((\| \mu - T_n\omega \| . \| \omega - T_m\mu \|)$$

Which implies that $\| \mu - \omega \|^2 \leq \gamma_1 \| \mu - \omega \|^2 + \gamma_3 \| \mu - \omega \|^2 + \gamma_5 \| \mu - \omega \|^2$

$$\| \mu - \omega \|^2 \leq (\gamma_1 + \gamma_3 + \gamma_5) \| \mu - \omega \|^2$$

$$(1 - \gamma_1 - \gamma_3 - \gamma_5) \| \mu - \omega \|^2 \leq 0 \ [\because \gamma_1 + \gamma_3 + \gamma_5 < 1].$$

Again, a contradiction as, $0<1-(\gamma_1+\gamma_3+\gamma_5)<1$, the only possibility is that $\| \mu - \omega \| = 0$,

thus μ = ω

Which completes the proof of the theorem.

Theorem (1.2) Let S be a closed subset of a Hilbert space H and $\{T_n\} S \to S$ be a sequence of mappings that satisfy the following condition,

$$\| T_i\xi - T_j\zeta \|^2 \leq \gamma\ max\ \{ \| \xi - \zeta \|^2, \frac{1}{2}(\| \xi - T_i\xi \|^2 + \| \zeta - T_j\zeta \|^2), \frac{1}{4}(\| \xi - T_j\zeta \|^2 + \| \zeta - T_i\xi \|^2,)$$

$$\| \xi - T_i\xi \| . \| \zeta - T_j\zeta \|,\ \ \| \xi - T_j\zeta \| . \| \zeta - T_i\xi \| \} \ \ldots\ldots\ldots\ldots (1.2.1)$$

$\forall\ \xi, \zeta \in S$, $\xi \neq \zeta$, and $0 < \gamma < 1$.

Proof. Let $\xi_0 \in S$ be an arbitrary point. Let the sequence $\{\xi_n\}$ on S is defined as $\xi_{n+1} = T_{n+1}\xi_n$,

where $n = 0,1,2,3, \ldots$

Firstly, it will be proved that $\{\xi_n\}$ is Cauchy.

Now introducing, $\xi = \xi_n$ and $\zeta = \xi_{n-1}$ on (1.2.1) and it gives,

$$\|\xi_{n+1} - \xi_n\|^2 = \| T_{n+1}\xi_n - T_n\xi_{n-1} \|^2$$

$$\|\xi_{n+1} - \xi_n\|^2 \leq \gamma\ max\ \{ \| \xi_n - \xi_{n-1} \|^2, \frac{1}{2}(\| \xi_n - T_{n+1}\xi_n \|^2 + \| \xi_{n-1} - T_n\xi_{n-1} \|^2),$$

$$\frac{1}{4}(\| \xi_n - T_n\xi_{n-1} \|^2 + \| \xi_{n-1} - T_{n+1}\xi_n \|^2), \| \xi_n - T_{n+1}\xi_n \|.\| \xi_{n-1} - T_n\xi_{n-1} \|,$$

$$\| \xi_n - T_n\xi_{n-1} \|.\| \xi_{n-1} - T_{n+1}\xi_n \|\}$$

Which implies that,

$$\|\xi_{n+1} - \xi_n\|^2 \leq \gamma\, max\{ \| \xi_n - \xi_{n-1} \|^2, \frac{1}{2}(\| \xi_n - \xi_{n+1} \|^2 + \| \xi_{n-1} - \xi_n \|^2), \frac{1}{4}(\| \xi_n - \xi_n \|^2 + \| \xi_{n-1} - \xi_{n+1} \|^2), (\| \xi_n - \xi_{n+1} \|.\| \xi_{n-1} - \xi_n \|), (\| \xi_n - \xi_n \|.\| \xi_{n-1} - \xi_{n+1} \|)\}$$

$$\Rightarrow \|\xi_{n+1} - \xi_n\|^2 \leq \gamma\, max\{ \| \xi_n - \xi_{n-1} \|^2, \frac{1}{2}(\| \xi_n - \xi_{n+1} \|^2 + \| \xi_{n-1} - \xi_n \|^2),$$

$$\frac{1}{4}(\| \xi_{n-1} - \xi_{n+1} \|^2), \| \xi_n - \xi_{n+1} \|.\| \xi_{n-1} - \xi_n \|\}$$

After simplifying the above yields,

$$\|\xi_{n+1} - \xi_n\|^2 \leq \gamma \max\{ \| \xi_n - \xi_{n-1} \|^2, \frac{1}{2}(\| \xi_n - \xi_{n+1} \|^2 + \| \xi_{n-1} - \xi_n \|^2),$$

$$\| \xi_n - \xi_{n+1} \|.\| \xi_{n-1} - \xi_n \|\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.2.1)$$

Case (i).

If $max\left\{ \| \xi_n - \xi_{n-1} \|^2, \frac{1}{2}(\| \xi_n - \xi_{n+1} \|^2 + \| \xi_{n-1} - \xi_n \|^2), \| \xi_n - \xi_{n+1} \|.\| \xi_{n-1} - \xi_n \|\right\}$

$= \| \xi_n - \xi_{n-1} \|^2$ Then it yields,

$$\|\xi_{n+1} - \xi_n\|^2 \leq \gamma \| \xi_{n-1} - \xi_n \|^2$$

Let $\gamma = \lambda_1^2$ then

$$\|\xi_{n+1} - \xi_n\|^2 \leq \lambda_1^2 \| \xi_{n-1} - \xi_n \|^2$$

This implies that,

$$\|\xi_{n+1} - \xi_n\| \leq \lambda_1 \|\xi_n - \xi_{n-1}\| \quad \ldots\ldots\ldots \quad (1.2.2)$$

Case(ii).

If $max\left\{ \| \xi_n - \xi_{n-1} \|^2, \frac{1}{2}(\| \xi_n - \xi_{n+1} \|^2 + \| \xi_{n-1} - \xi_n \|^2), \| \xi_n - \xi_{n+1} \|.\| \xi_{n-1} - \xi_n \|\right\}$

$= \frac{1}{2}(\| \xi_n - \xi_{n+1} \|^2 + \| \xi_{n-1} - \xi_n \|^2)$ Then it produces,

$$\|\xi_{n+1} - \xi_n\|^2 \leq \frac{\gamma}{2}(\| \xi_n - \xi_{n+1} \|^2 + \| \xi_{n-1} - \xi_n \|^2)$$

$$\left(1 - \frac{\gamma}{2}\right)\|\xi_{n+1} - \xi_n\|^2 \leq \frac{\gamma}{2} \| \xi_{n-1} - \xi_n \|^2$$

$$\|\xi_{n+1}-\xi_n\|^2 \le \left(\frac{\frac{\gamma}{2}}{\left(1-\frac{\gamma}{2}\right)}\right) \| \xi_{n-1}-\xi_n \|^2$$

Let $\left(\frac{(\frac{\gamma}{2})}{\left(1-\frac{\gamma}{2}\right)}\right) = \lambda_2^2$ Then it gives,

$$\|\xi_{n+1}-\xi_n\|^2 \le \lambda_2^2 \| \xi_{n-1}-\xi_n \|^2$$

Which implies that,

$$\|\xi_{n+1}-\xi_n\| \le \lambda_2\|\xi_n-\xi_{n-1}\| \quad (1.2.3)$$

Case (iii).

If $max\left\{ \| \xi_n-\xi_{n-1} \|^2, \frac{1}{2}(\| \xi_n-\xi_{n+1} \|^2 + \| \xi_{n-1}-\xi_n \|^2), \| \xi_n-\xi_{n+1} \|. \| \xi_{n-1}-\xi_n \|\right\}$

$=\| \xi_n-\xi_{n+1} \|. \| \xi_{n-1}-\xi_n \|$, then

$$\|\xi_{n+1}-\xi_n\|^2 \le \gamma \| \xi_n-\xi_{n+1} \|. \| \xi_{n-1}-\xi_n \|$$

$$\|\xi_{n+1}-\xi_n\| \le \gamma\|\xi_n-\xi_{n-1}\|$$

Let, $\gamma = \lambda_3$Then above yields,

$$\|\xi_{n+1}-\xi_n\| \le \lambda_3\|\xi_n-\xi_{n-1}\|$$

Which implies, $\|\xi_{n+1}-\xi_n\| \le \lambda_3\|\xi_n-\xi_{n-1}\| \ \forall n$ …………….. (1.2.4)

Let $\lambda = \max\{ \lambda_1,\lambda_2,\lambda_3$Then the above gives,

$$\|\xi_{n+1}-\xi_n\| \le \lambda \|\xi_n-\xi_{n-1}\| \ \forall n.$$

$$\le \lambda^2 \|\xi_{n-1}-\xi_{n-2}\|$$

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$$\le \lambda^n \|\xi_1-\xi_0\| \to 0 \ as \ n\to\infty$$

Now, for any positive integer $m \ge n \ge 1$, the above produces as follows,

$$\| \xi_m-\xi_n \| \le \| \xi_n-\xi_{n+1} \| + \| \xi_{n+1}-\xi_{n+2} \| + \cdots + \| \xi_{m-1}-\xi_m \|$$

$$\| \xi_m-\xi_n \| \le \lambda^n \| \xi_1-\xi_0 \| + \lambda^{n+1} \| \xi_1-\xi_0 \| + \cdots + \lambda^{m-1} \| \xi_1-\xi_0 \|$$

$$\| \xi_m-\xi_n \| \le (\lambda^n+\lambda^{n+1}+\cdots+\lambda^{m-1}) \| \xi_1-\xi_0 \|$$

$$\| \xi_m-\xi_n \| \le \left(\frac{\lambda^n}{1-\lambda}\right) \| \xi_1-\xi_0 \| \to 0 \ as\, n\to\infty.$$

Hence, $\{\xi_n\}$ is Cauchy on S. As S is a closed subset of H, so $\{\xi_n\}$ converges to a point μ in S.

Here it will be proved that μ is a common fixed point of $\{T_n\}$ of mappings on S.

Let$T_n\mu \neq \mu$.

Now, for positive integer n, (1.2.1) gives,

$$\| \mu - T_n\mu \|^2 = \| (\mu - \xi_n) + (\xi_n - T_n\mu) \|^2$$

$$\| \mu - T_n\mu \|^2 \leq 2 \| \mu - \xi_n \|^2 + 2 \| \xi_n - T_n\mu \|^2$$

$$\| \mu - T_n\mu \|^2 \leq 2 \| \mu - \xi_n \|^2 + 2 \| T_n\xi_{n-1} - T_n\mu \|^2$$

$$\Rightarrow \| \mu - T_n\mu \|^2 \leq 2 \| \mu - \xi_n \|^2 + 2\gamma\, max \{ \| \xi_{n-1} - \mu \|^2, \tfrac{1}{2}(\| \xi_{n-1} - T_n\xi_{n-1} \|^2 + \| \mu - T_n\mu \|^2),$$

$$\frac{1}{4}(\| \xi_{n-1} - T_n\mu \|^2 + \| \mu - T_n\xi_{n-1} \|^2), \| \xi_{n-1} - T_n\xi_{n-1} \|. \| \mu - T_n\mu \|,$$

$$\| \xi_{n-1} - T_n\mu \|. \| \mu - T_n\xi_{n-1} \| \}$$

$$\Rightarrow \| \mu - T_n\mu \|^2 \leq 2 \| \mu - \xi_n \|^2 + 2\gamma\, max \{ \| \xi_{n-1} - \mu \|^2, \tfrac{1}{2}(\| \xi_{n-1} - \xi_n \|^2 + \| \mu - T_n\mu \|^2)$$

$$\frac{1}{4}(\| \xi_{n-1} - T_n\mu \|^2 + \| \mu - \xi_n \|^2), \| \xi_{n-1} - \xi_n \|. \| \mu - T_n\mu \|,$$

$$\| \xi_{n-1} - T_n\mu \|. \| \mu - \xi_n \| \}$$

$$\Rightarrow \| \mu - T_n\mu \|^2 \leq 2\gamma\, max \left\{ \tfrac{1}{2} \| \mu - T_m\mu \|^2, \tfrac{1}{4} \| \mu - T_n\mu \|^2 \right\}$$

$$\| \mu - T_n\mu \|^2 = \gamma \| \mu - T_n\mu \|^2$$

$$\| \mu - T_n\mu \|^2 < \| \mu - T_n\mu \|^2$$

Which is a contradiction, as 0 <γ<1. The only possibility is that T_n μ = μ. To prove uniqueness, suppose that $\mu \neq \omega, \forall$ n, $T_m\omega = \omega$.

$$\| \mu - \omega \|^2 = \| T_n\mu - T_m\omega \|^2$$

$$\Rightarrow \| \mu - \omega \|^2 \leq \gamma\, max \{ \| \mu - \omega \|^2, \tfrac{1}{2}(\| \mu - T_n\mu \|^2 + \| \omega - T_m\omega \|^2),$$

$$\frac{1}{4}(\| \mu - T_m\omega \|^2 + \| \omega - T_n\mu \|^2), \| \mu - T_n\mu \|. \| \omega - T_m\omega \|, \| \mu - T_m\omega \|. \| \omega - T_n u \| \}$$

$$\Rightarrow \| \mu - \omega \|^2 \leq \gamma\, max \{ \| \mu - \omega \|^2, 0, \tfrac{1}{4}(\| \mu - \omega \|^2 + \| \omega - \mu \|^2), 0, \| \mu - \omega \|. \| \omega - \mu \| \}$$

$$\Rightarrow \| \mu - \omega \|^2 \leq \gamma\, max \{ \| \mu - \omega \|^2, \tfrac{1}{4} \| \mu - \omega \|^2, \| \mu - \omega \|^2$$

$$\Rightarrow \| \mu - \omega \|^2 \leq \gamma \| \mu - \omega \|^2$$

Again, a contradiction as 0 < γ <1. Hence $\mu = \omega$. This is complete proof of our theorem.

Theorem (1.3) Let S be a closed subset of a Hilbert space H and $\{T_n\} S \rightarrow S$ be a sequence of mappings that satisfy the following condition,

$$\| T_i\xi - T_j\zeta \|^2 \leq \gamma \, max \, \{ \| \xi - \zeta \|^2, (\| \xi - T_i\xi \|^2 + \| \zeta - T_j\zeta \|^2), \frac{1}{2}(\| \xi - T_j\zeta \|^2 + \| \zeta - T_i\xi \|^2,)$$

$$\| \xi - T_i\xi \| . \| \zeta - T_j\zeta \|, \left(\frac{\| \xi - T_i\xi \|^2 . \| \zeta - T_j\xi \|^2}{\| \xi - \zeta \|^2 + \| \xi - T_j\zeta \|^2} \right), \| \xi - T_j\zeta \| . \| \zeta - T_i\xi \| \}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1.3.1)$$

$\forall \ \xi, \zeta \ \in S, \ \xi \neq \zeta,$ and $0 < \gamma < 1$.

Proof of the above is the same as the theorem (1.2).

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