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Original Article

A Non-Existence Result for Positive Periodic Solutions to the Relativistic Singular Liénard Equation

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Abstract - *In this paper, we establish the non-existence of positive periodic solutions to the attractive singular Liénard equation with relativistic acceleration*

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{v^2}}}\right)' + f(x)x' + a(t)x^{\mu} + \frac{b(t)}{x^{\rho}} = s$$

Where v is the speed of light in a vacuum? Distinct from the classical Liénard equation, the inertial term arises from the relativistic momentum, which introduces strong nonlinearity and imposes a natural velocity bound.

Keywords - *Attractive singularity, Liénard equation, Non-existence, Relativistic acceleration.*

1. Introduction

Singular differential equations have significant applications in various scientific fields, such as cell cycle models in biomathematics [1] and the study of closed convex hypersurfaces in geometry [8]. The study of periodic solutions plays an important role in the theory of singular differential equations, due to its close connection with a wide range of periodic phenomena observed in nature and society.

The study of periodic solutions in singular differential equations dates back to 1944, when Nagumo first addressed this topic in the literature [9]. Later, Lazer and Solimini (1987) [7] made a major breakthrough by introducing topological degree theory into the analysis of periodic solutions for singular equations. Their work revealed the essential distinction between repulsive and attractive singularities and greatly advanced the development of the field. Inspired by their results, many researchers have carried out extensive studies on the existence of periodic solutions to singular differential equations [2, 3, 6].

Among various types of singular equations, the Liénard equation has attracted considerable attention. In recent years, several important results have been obtained. In 2021, Xin and Cheng [12] investigated the non-existence of positive periodic solutions for the following ϕLaplacian generalized Liénard equation with a singularity

$$(\phi(x'))' + f(t,x)x' + \frac{b(t)}{x^{\rho}} = h(t)x^{m},$$

Where ρ [?] is a positive constant and [?] is a constant. More recently, in 2024, Yu et al. [13] considered the Liénard equation with an attractive singularity

$$x'' + f(x)x' + a(t)x^{\mu} + \frac{b(t)}{x^{\rho}} = s, \tag{1.1}$$

Where $f(x) \in C((0,+\infty);\mathbb{R}), a(t), b(t) \in C(\mathbb{R}/T\mathbb{Z},\mathbb{R}), \mu, \rho$ are positive constants, and $s \in \mathbb{R}$ is a parameter. Using the upper and lower solutions method, they established the multiplicity of periodic solutions. The Liénard equations are not only of significant theoretical value, but also find wide applications in physical models, such as the classical bubble dynamics model discussed in the monograph by Professor Torres [11], and the Micro-electro-mechanical systems (MEMS) mass-spring model.

Motivated by the above works, in this paper, we investigate the non-existence of periodic solutions to the relativistic Liénard equation with an attractive singularity

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{v^2}}}\right)' + f(x)x' + a(t)x^{\mu} + \frac{b(t)}{x^{\rho}} = s, \tag{1.2}$$

Where $f(x) \in C((0,+\infty);\mathbb{R}), a(t) \in C(\mathbb{R}/T\mathbb{Z},\mathbb{R})$ [?] and [?] are positive constants and [?] is the speed of light in a vacuum. Compared with the classical Liénard equation, the equation's main difference lies in the inertial term, which no longer takes the standard Newtonian form. Instead, it involves the relativistic modification $\left(\frac{x'}{\sqrt{1-\frac{x'^2}{v^2}}}\right)'$, which originates from the expression for relativistic momentum. From a mathematical viewpoint, this term introduces a strong nonlinearity as $x' \to v$ the denominator tends to zero, and thus the entire term becomes unbounded. This behavior reflects a nonlinear growth in inertia and imposes a natural upper bound on the admissible velocity. This phenomenon is a manifestation of relativistic effects. Such mechanisms are useful in modeling the dynamics of high-energy particles under strong-field conditions [10]. By using the styles defined in this document.

2. Main Theorem and Proof

We introduce some notations that will be used throughout the paper. Let C_T denote the space of continuous T-periodic functions, i.e. $C_T = \{x \in C(\mathbb{R},\mathbb{R}) | x(t) = x(t+T), \forall t \in \mathbb{R}\}$. C_T^1 Is the space of T periodic functions with continuous derivatives. For each $x \in C_T$, we define $\bar{x} := \frac{1}{T}\int_0^T x(t)dt. \|x\|_{\infty} := \max_{t\in[0,T]}|x(t)|, x_{min} := \min_{t\in[0,T]} x(t)$. The main result is as follows.

Theorem 2.1. If equation (1.1) has no positive T-periodic solution, then there exists $v_* > 0$ such that equation (1.2) (under the same choice of $f(x), a(t), b(t)$ and μ, ρ, s) has no positive T-periodic solution for any $v > v_*$.

Remark 2.1. The non-existence of positive T-periodic solutions for equation (1.1) can be found in the literature [13, Theorem 3.1].

We mainly prove Theorem 2.1 by establishing a priori bounds of solutions (which are independent of v) and passing to the limit using the Ascoli-Arzelà Theorem. We first present one lemma essential for the proof of a priori bounds.

Lemma 2.1. *[5]* For any $x \in C_T^1$, it yields:

$$\left(\max_{t\in[0,T]} x(t) - \min_{t\in[0,T]} x(t)\right)^2 \leq \frac{T}{4}\int_0^t |x'(t)|^2 dt.$$

In what follows, we estimate *a priori bounds* of positive periodic solutions x(t) of equation (1.2). Specifically, any $x(t)$ has a natural bound $\|x'\|_{\infty} < v$.

Lemma 2.2. For all positive T-periodic solutions $x(t)$ of equation (1.2), there exists a positive constant M_0 such that

$$x(t) > M_0, \quad \text{for } t \in [0,T].$$

Proof. Let be $t_0 \in (0,T)$ such that $x(t_0) = \min_{t\in[0,T]} x(t)$. Then $x''(t_0) \geq 0, x'(t_0) = 0$ and

$$\left.\left(\frac{x'(t)}{\sqrt{1-\frac{x'^2(t)}{v^2}}}\right)'\right|_{t=t_0} + f(x(t_0))x'(t_0) + a(t_0)x^{\mu}(t_0) + \frac{b(t_0)}{x^{\rho}(t_0)} = s,$$

where $\left.\left(\frac{x'(t)}{\sqrt{1-\frac{x'^2(t)}{v^2}}}\right)'\right|_{t=t_0} = \left.\frac{x''(t)}{\left(1-\frac{x'^2(t)}{v^2}\right)^{\frac{3}{2}}}\right|_{t=t_0} = \frac{x''(t_0)}{\left(1-\frac{x'^2(t_0)}{v^2}\right)^{\frac{3}{2}}} \geq 0, f(x(t_0))x'(t_0) = 0$, then

$$0 \leq s - a(t_0)x^{\mu}(t_0) - \frac{b(t_0)}{x^{\rho}(t_0)} < s + |a_{min}|x^{\mu}(t_0) - \frac{b_{min}}{x^{\rho}(t_0)}. \tag{2.1}$$

Suppose $x(t_0) < \left(\frac{b_{min}}{2|a_{min}|+1}\right)^{\frac{1}{\mu+\rho}}$, then, it follows that

$$|a_{min}|x^{\mu}(t_0) < \frac{b_{min}}{2x^{\rho}(t_0)}.$$

Substituting into (2.1), we obtain

$$0 < s + \frac{b_{min}}{2x^{\rho}(t_0)} - \frac{b_{min}}{x^{\rho}(t_0)} = s - \frac{b_{min}}{2x^{\rho}(t_0)},$$

which implies

$$x(t_0) > \left(\frac{b_{min}}{2s}\right)^{\frac{1}{\rho}}.$$

Therefore, we conclude that.

$$x(t_0) \geq min\left\{\left(\frac{b_{min}}{2s}\right)^{\frac{1}{\rho}}, \left(\frac{b_{min}}{2|a_{min}|+1}\right)^{\frac{1}{\mu+\rho}}\right\}.$$

Consequently, a sufficiently large constant exists.

$$x(t_0) > \left(\frac{b_{min}}{2k}\right)^{\frac{1}{\rho}} := M_0 > 0\cdot$$

Lemma 2.3. For all positive Tperiodic solutions $x(t)$ of equation (1.2), there exists a positive constant M_1 such that

$$x(t) < M_1, \text{for } t \in [0,T], \tag{2.2}$$

where $M_1 = \left(A_\kappa + \frac{s}{(1-\kappa)\bar{a}}\right)^{\frac{1}{\mu}} + M_0, \kappa \in (0,1)$.

Proof. In fact, to establish (2.2), it suffices to prove that $M_1 = \left(A_\kappa + \frac{s}{(1-\kappa)\bar{a}}\right)^{\frac{1}{\mu}}, \kappa \in (0,1)$. For the sake of contradiction, suppose this inequality does not hold. Then there exists $\kappa \in (0,1)$ a sequence $\{x_n\}_{n=1}^{+\infty}$ such that

$$M_n > \left(n + \frac{s}{(1-\kappa)\bar{a}}\right)^{\frac{1}{\mu}}, \text{for } n \in \mathbb{N} \tag{2.3}$$

where $M_n := max\{x_n(t): t \in [0,T]\}$, x_n denote the positive T -periodic solutions of equation (1.2). Then, from equation (2.3), we deduce

$$\lim_{n\to+\infty} M_n = +\infty. \tag{2.4}$$

Moreover, it follows from Lemma 2.1 and $\|x'\|_\infty < v$ that

$$M_n - m_n \le \frac{Tv}{2},$$

Where $m_n := min\{ x_n(t): t \in [0,T]\}$. Combining with equation (2.4), we obtain

$$\lim_{n\to+\infty} \frac{m_n}{M_n} \ge \lim_{n\to+\infty} \frac{M_n - \frac{Tv}{2}}{M_n} \ge 1.$$

On the other hand, by the definitions of the M_n, m_n, it is evident that

$$\lim_{n\to+\infty} \frac{m_n}{M_n} \le 1.$$

Combining the above estimates, we conclude that.

$$\lim_{n\to+\infty} \frac{m_n}{M_n} = 1.$$

Therefore, based on the above equation and the condition $\bar{a} > 0$, it can be deduced that there exists a constant $N > 0$ such that

$$m_n > \left(\frac{\kappa\overline{[a]_-} + (1-\kappa)\overline{[a]_+}}{\overline{[a]_+}}\right)^{\frac{1}{\mu}} M_n, \text{ for } n > N. \tag{2.5}$$

Given that $x_n(t) \in C_T^1$, it satisfies

$$\left(\frac{x_n'(t)}{\sqrt{1-\frac{x_n'^2(t)}{v^2}}}\right)' + f(x_n(t))x_n'(t) + a(t)x_n^\mu(t) + \frac{b(t)}{x_n^\rho(t)} = s, \tag{2.6}$$

Integrating (2.6) over $[0,T]$ yields

$$\int_0^T \left(\frac{x_n'(t)}{\sqrt{1-\frac{x_n'^2(t)}{v^2}}}\right)' dt + \int_0^T f(x_n(t))x_n'(t)dt + \int_0^T a(t)x_n^\mu(t)dt + \int_0^T \frac{b(t)}{x_n^\rho(t)} dt = s\int_0^T dt.$$

Therefore

$$\int_0^T a(t)x_n^\mu(t)dt + \int_0^T \frac{b(t)}{x_n^\rho(t)} dt = Ts, \text{ for } n \in \mathbb{N}. \tag{2.7}$$

Since $b(t) > 0$ then $t \in [0,T]$, it follows that

$$Ts \ge \int_0^T [a]_+(t)x_n^\mu(t)dt - \int_0^T [a]_-(t)x_n^\mu(t)dt$$
$$\ge T\overline{[a]}_+ m_n^\mu - T\overline{[a]}_- M_n^\mu \quad \text{for } n \in \mathbb{N}. \tag{2.8}$$

Substituting (2.3) and (2.5) into the inequality above and noting that $\bar{a} > 0$ with $\kappa \in (0,1)$, we conclude

$$\begin{aligned} s &\ge \overline{[a]}_+ m_n^\mu - \overline{[a]}_- M_n^\mu \\ &\ge \left(\kappa\overline{[a]_-} + (1-\kappa)\overline{[a]_+}\right)M_n^\mu - \overline{[a]_-}M_n^\mu \\ &= (1-\kappa)\bar{a}M_n^\mu \\ &\ge n(1-\kappa)\bar{a} + s > s \quad \text{for } n > N. \end{aligned}$$

This leads to a contradiction, and hence the desired conclusion is established.

Lemma 2.4. For all positive T -periodic solutions $x(t)$ of equation (1.2), there exists a positive constant M_2 such that

$$|x'(t)| < M_2.$$

Proof. Since $x(0) = x(T)$ there is a point $t_2 \in (0,T)$ such that $x'(t_2) = 0$. Integrating (1.2) from t_2 to t yields

$$\int_{t_2}^{t} \left(\frac{x'(\sigma)}{\sqrt{1-\frac{x'^2(\sigma)}{v^2}}} \right)' d\sigma + \int_{t_2}^{t} f(x(\sigma))x'(\sigma)d\sigma + \int_{t_2}^{t} a(\sigma)x^{\mu}(\sigma)d\sigma + \int_{t_2}^{t} \frac{b(\sigma)}{x^{\rho}(\sigma)} d\sigma = s\int_{t_2}^{t} d\sigma,$$

Where $t \in [t_2, t_2 + T]$. It follows from (2.7), Lemmas 2.2 and 2.3 that.

$$\begin{aligned}
|x'(t)| &\le \left| \frac{x'(t)}{\sqrt{1-\frac{x'^2(t)}{v^2}}} \right| \\
&= \left| s(t-t_2) - F(x(t)) + F(x(t_2)) - \int_{t_2}^{t} a(\sigma)x^{\mu}(\sigma)d\sigma - \int_{t_2}^{t} \frac{b(\sigma)}{x^{\rho}(\sigma)} d\sigma \right| \\
&\le Ts + 2\max_{x\in[M_0,M_1]} |F(x)| + \left| \int_{t_2}^{t} a(\sigma)x^{\mu}(\sigma)d\sigma + \int_{t_2}^{t} \frac{b(\sigma)}{x^{\rho}(\sigma)} d\sigma \right| \\
&\le Ts + 2\max_{x\in[M_0,M_1]} |F(x)| + \int_{t_2}^{t_2+T} |a(\sigma)|x^{\mu}(\sigma)d\sigma + \int_{t_2}^{t_2+T} \frac{b(\sigma)}{x^{\rho}(\sigma)} d\sigma \\
&= Ts + 2\max_{x\in[M_0,M_1]} |F(x)| + Ts + \int_{t_2}^{t_2+T} (-a(\sigma) + |a(\sigma)|)|x^{\mu}(\sigma)d\sigma \\
&\le Ts + 2\max_{x\in[M_0,M_1]} |F(x)| + Ts + 2T\overline{[a]_-}M_1 \\
&< 2\max_{x\in[M_0,M_1]} |F(x)| + 2Ts + 2T\overline{[a]_-}M_1 + 1 := M_2
\end{aligned}$$

Where F is the primitive function of f?

We now proceed to prove Theorem 2.1 by contradiction. Suppose a sequence and corresponding -periodic solutions of (1.2) exist $v = v_n$. According to Lemmas 2.3 and 2.4, the sequences $x_n(t)$ and $x'_n(t)$ are uniformly bounded. Moreover, equation (1.2) can be written as

$$\frac{x''}{(1-\frac{x'^2}{v^2})^{\frac{3}{2}}} + f(x)x' + a(t)x^{\mu} + \frac{b(t)}{x^{\rho}} = s, \tag{2.9}$$

By Lemmas 2.2, 2.3 and 2.4, the sequence $x''_n(t)$ is also uniformly bounded. Then by the Ascoli-Arzelà Theorem, subsequences exist such that both are uniformly convergent and convergent to a certain $x_{\infty}(t)$ in C_T^1. It is important to note that all the bounds above are independent of v_n each other.

The final step is to write the equation as an integral and take the limit. Starting from (2.9), we write.

$$x'' - x = (1 - \frac{x'^2}{v^2})^{\frac{3}{2}} \left(s - f(x)x' - a(t)x^{\mu} - \frac{b(t)}{x^{\rho}} \right) - x.$$

Then the problem of finding a Tperiodic solution of (1.2) is reduced to finding a Tperiodic solution of the integral equation.

$$x = \int_0^T G(t,s) \left[(1 - \frac{x'^2(s)}{v^2})^{\frac{3}{2}} \left(s - f(x(s))x'(s) - a(s)x^{\mu}(s) - \frac{b(s)}{x^{\rho}(s)} \right) - x(s) \right] ds,$$

Where $G(t,s)$ is the Green function of the linear operator $x'' - x$ with periodic conditions uniformly bounded on the square $[0,T] \times [0,T]$? Consequently, x_{n_j} satisfies

$$x_{n_j} = \int_0^T G(t,s)\left[(1 - \frac{x'^2_{n_j}(s)}{v^2_{n_j}})^{\frac{3}{2}}\left(s - f(x_{n_j}(s))x'_{n_j}(s) - a(s)x^{\mu}_{n_j}(s) - \frac{b(s)}{x^{\rho}_{n_j}(s)}\right) - x_{n_j}(s)\right] ds.$$

Taking the limit and using the uniform boundedness $G(t,s)$, we can pass the limit inside the integral. Hence

$$x_{\infty} = \int_0^T G(t,s)\left[(s - f(x_{\infty}(s))x'_{\infty}(s) - a(s)x^{\mu}_{\infty}(s) - \frac{b(s)}{x^{\rho}_{\infty}(s)} - x_{\infty}(s)\right] ds.$$

Or equivalently, $x_{\infty}(t)$ it is a Tperiodic solution of (1.1). This completes the proof.

3. Application to MEMS

In this section, we consider the application of equation (1.2) to the MEMS mass-spring model with relativistic acceleration.

$$m\left(\frac{y'}{\sqrt{1-\frac{y'^2}{v^2}}}\right)' + cy' + ky = \frac{\varepsilon_0 A}{2}\frac{V^2(t)}{(d-y)^2},$$

By applying time scale transformation and translation transformation $x = d - y$, it is transformed into the following form

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{v^2}}}\right)' + \frac{c}{m}x' + \frac{k}{m}x + \frac{\varepsilon_0 A}{2m}\frac{V^2(t)}{x^2} = \frac{kd}{m}, \tag{3.1}$$

Where $k > 0$ is the stiffness coefficient of a linear spring? $V(t) = v_{dc} + v_{ac}\cos\omega t$ The applied voltage $V(t) > 0$ $d > 0$ is the initial distance between two parallel capacitor plates, y represents the vertical displacement of the movable plate (y always less than d), $m > 0$ is its mass, $c > 0$is a viscous damping coefficient, $\varepsilon_0 > 0$ is the absolute dielectric constant of vacuum and$A > 0$ is the area of the capacitor plates.

The equation (3.1) is a special case of (1.2) where $f(x) = \frac{c}{m}$, $a(t) = \frac{k}{m}$, $b(t) = \frac{\varepsilon_0 A V^2(t)}{2m}$, $\mu = 1, \rho = 2, s = \frac{kd}{m}$. By applying Theorem 2.1, we obtain that when the equation.

$$x'' + \frac{c}{m}x' + \frac{k}{m}x + \frac{\varepsilon_0 A}{2m}\frac{V^2(t)}{x^2} = \frac{kd}{m},$$

Admits no positive T-periodic solution [4], there exists $v_* > 0$ such that equation (3.1) also admits no positive T-periodic solution for any $v > v_*$.

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