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Original Article

Degree of Approximation of Functions in Besov Space by Eular Means of Fourier Series

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Abstract *- In this paper, we established a result on the degree of approximation of functions in the Besov Space by Eular means of trigonometric Fourier series.*

Keywords *- Besov space, Degree of Approximation, Eular Summability, Periodic functions, Trigonometric Fourier Series.*

Subject Classification Number : *30H25, 42A10, 41A10, 41A25, 42A24.*

1. Introduction

Several researchers have determined the level of approximation of function f, belonging to various summability methods such as Riesz mean, Nörlund mean, Matrix mean, and generalized Matrix mean, etc., through different spaces. The Besov spaces extend the scope of basic functional spaces such as Lipschitz, Hölder, and generalized Hölder spaces. The Besov spaces are significant in assessing the regularity characteristics of functions. Therefore, in the present work, we obtain the degree of approximation of a function in a Besov space using Eular means.

We will discuss the degree of approximation of functions in the H_α Class of Fourier series in the supremum norm. This topic has been extensively studied by P. Chandra [5], P. Chandra and R.N. Mahapatra [7], T. Singh [9], and Z. Krasniqi and B. Szal [11] in relation to the approximation of functions using Hölder metrics.

2. Definitions and Notations

2.1. Modulus of Continuity (DeVore et al [3])

Let $A = R,\ R_+, [a, b]$*, or T (which is usually taken to be R with the identification of points modulo* 2π*).*

The modulus of continuity $w(f, t) = w(t)$ *of a function* f *on A can be defined as*

$$w(t) = w(f, t) = \sup_{|x-y|\le t, x,y\in A} |f(x) - f(y)|\ , t \ge 0. \tag{2.1}$$

2.2. Modulus of Smoothness (DeVore et al [3])

The k^{th} *The order modulus of smoothness of a function* $f: A \to R$ *is defined by*

$$w_k(f, t) = \sup_{0<h\le t} \{|\Delta_h^k(f, x)|: x, x + kh \in A\}, t \ge 0. \tag{2.2}$$

$$\Delta_h^k(f, x) = \sum_{i=0}^{k} \binom{k}{i} f(x + ih),\ k \in N. \tag{2.3}$$

For $k = 1$*,* $w_1(f, t)It$ *is called the modulus of continuity of* f*. The function* w *is continuous at* $t = 0$ *if and only if* f *is uniformly continuous on A, that is* $f \in C(A)$*. The* k^{th} *order modulus of smoothness of* $f \in L_p(A)$*,*

$0 < p < \infty$ *or of* $f \in C(A)$, *if* $p = \infty$, *is defined by*

$$w_k(f,t)_p = \sup_{0<h\leq t} ||\Delta_h^k(f,x)||_p \quad ,t \geq 0. \tag{2.4}$$

if $p \geq 1, k = 1$, *then* $w_1(f,t)_p = w(f,t)_p$ *It is a modulus of continuity (or integral modulus of continuity). If* $p = \infty$, $k = 1$ *and* f *is continuous, the* $w_k(f,t)_p$ *reduces to the modulus of continuity* $w_1(f,t)$ *(or* $w(f,t)$*).*

2.3. Lipschitz Space (DeVore et al [3])

Let $f \in C(A)$ *and*

$$w(f,t) = O(t^\alpha), \ 0 < \alpha \leq 1. \tag{2.5}$$

then we write $f \in Lip\ \ \alpha$, *If* $w(f,t) = O(t)$ *as* $t \to 0+$ *(in particular (3.5) holds for* $\alpha > 1$*) then* f *reduces to a constant.*

If $f \in L_p(A), 0 < p < \infty$ *and*

$$w(f,t)_p = O(t^\alpha), \ 0 < \alpha \leq 1. \tag{2.6}$$

then we write $f \in Lip\ \ (\alpha,p),\ 0 < p < \infty, 0 < \alpha \leq 1.$

The case $\alpha > 1$ *is of no interest as the function reduces to a constant whenever*

$$w(f,t)_p = O(t) \ \text{ as } \ t \to 0+ \tag{2.7}$$

We note that if $p = \infty$ *and* $f \in C(A)$, *then* $Lip(\alpha,p)$ *class reduces to* $Lip\ \ \alpha$ *class.*

2.4. Generalized Lipschitz Space (DeVore et al [3])

Let $\alpha > 0$ *and suppose that* $k = [\alpha] + 1$. *For* $f \in L_p(A),\ 0 < p < \infty$, *if*

$$w_k(f,t) = O(t^\alpha), t > 0, \tag{2.8}$$

Then we write

$$f \in Lip^*(\alpha,p), \alpha > 0, \ 0 < p \leq \infty. \tag{2.9}$$

and say that f *belongs to generalized Lipschitz space. The seminorm is then.*

$$|f|_{Lip^*(\alpha,L_p)} = \sup_{t>0}(t^{-\alpha}w_k(f,t)_p)$$

It is known ([3], p-52) that the space $Lip^*(\alpha,L_p)$ *contains* $Lip(\alpha,L_p)$. *For* $0 < \alpha < 1$, *the spaces coincide. (for* $p = \infty$ *, it is necessary to replace* L_∞ *By the C of uniformly continuous functions on A. For* $0 < \alpha < 1$ *and* $p = \infty$ *the space* $Lip^*(\alpha,L_p)$ *coincide with* $Lip\ \ \alpha$.

For $\alpha = 1, p = \infty$, *we have*

$$Lip(1,C) = Lip\ \ 1 \tag{2.10}$$

but

$$Lip^*(1,C) = Z \tag{2.11}$$

is the Zygmund space [1] which is characterized by (2.8) with $k = 2$.

2.5. Hölder (H_α) *Space (M. Mohanty et al. [4])*

For $0 < \alpha \leq 1$, *let*

$$H_\alpha = \{f \in C_{2\pi} : w(f,t) = O(t^\alpha)\}. \tag{2.12}$$

It is known [8] that H_α *is a Banach Space with the norm* $||.||_\alpha$ *defined by*

$$\| f \|_\alpha = \| f \|_C + \sup_{t>0} t^{-\alpha} w(t), 0 < \alpha \leq 1, \tag{2.13}$$

$$\| f \|_0 = \| f \|_C$$

and

$$H_\alpha \subseteq H_\beta \subseteq C_{2\pi}, \quad 0 < \beta \leq \alpha \leq 1. \tag{2.14}$$

2.6. Besov Space (DeVore et al [3])

Let $\alpha > 0$ *be given and let* $k = [\alpha] + 1$. *For* $0 < p, q \leq \infty$, *the Besov space([3],p-54)* $B_q^\alpha(L_p)$ *is defined as follows:*

$$B_q^\alpha(L_p) = \{f \in L_p : |f|_{B_q^\alpha(L_p)} = ||w_k(f,.)||_{(\alpha,q)} is \ finite\}$$

where

$$||w_k(f,.)||_{(\alpha,q)} = \{ \begin{array}{ll} [\int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t}]^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{t>0} (t^{-\alpha} w_k(f,t)_p), & q = \infty \end{array}$$

It is known ([3], p-55) that $||w_k(f,.)||_{(\alpha,q)}$ *It is a seminorm if* $1 \leq p, q \leq \infty$ *and a quasi-seminorm in other cases.*

The Besov norm for $B_q^\alpha(L_p)$ *is*

$$||f||_{B_q^\alpha(L_p)} = ||f||_p + ||w_k(f,.)||_{(\alpha,q)} \tag{2.15}$$

It is known ([10], p-237) that for 2π *-periodic function* f *, the integral* $[\int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t}]^{\frac{1}{q}}$ *is replaced by* $[\int_0^\pi (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t}]^{\frac{1}{q}}$.

We know ([3],[10]) the following inclusion relations.

For fixed α *and p*

$$B_q^\alpha(L_p) \subset B_{q_1}^\alpha(L_p), \quad q < q_1.$$

For fixed p and q

$$B_q^\alpha(L_p) \subset B_q^\beta(L_p \beta < \alpha.$$

For fixed α *and q*

$$B_q^\alpha(L_p) \subset B_q^\alpha(L_{p_1}), \quad p_1 < p.$$

2.7 Through out the paper we will use the following notations:

Let f *be a* 2π *periodic function and* $f \in L_p[0,2\pi]$, *where* $p \geq 1$. *Then the Fourier series is given by*

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx). \tag{2.16}$$

Let $S_n(x) = S_n(f,x)$ *denote the* n^{th} *Partial sum of the Fourier series. Then as we know ([2]) that*

$$S_n(x) - f(x) = \frac{1}{\pi} \int_0^\pi \phi_x(u) D_n(u) \ du. \tag{2.17}$$

where

$$\phi_x(u) = \phi(x,u) = f(x+u) + f(x-u) - 2f(x) \tag{2.18}$$

and the Dirichlet kernel

$$D_n(u) = \frac{1}{2} + \sum_{k=0}^n cos \ \ ku = \frac{sin(k+\frac{1}{2})u}{2sin\frac{u}{2}} \quad . \tag{2.19}$$

The Eular mean of the Fourier series is given by:

$$E_n^q = (1+q)^{-n} \sum_{k=0}^n {}^nC_k \ q^{n-k} S_k \tag{2.20}$$

and

$$E_n^q - f(x) = \frac{1}{2\pi}(1+q)^{-n}\int_0^{\pi} \phi_x(u)\sum_{k=0}^{n} {}^nC_k \frac{\sin(k+\frac{1}{2})u}{2\sin\frac{u}{2}} q^{n-k}du \quad .$$

Thus

$$T_n(x) = E_n^q - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi_x(u)K_n(u)du, \tag{2.21}$$

where

$$K_n(u) = (1+q)^{-n}\sum_{k=0}^{n} {}^nC_k q^{n-k}\frac{\sin(k+\frac{1}{2})u}{2\sin\frac{u}{2}} \tag{2.22}$$

We need the following additional notations.

$$\phi(x,t,u) = \begin{cases} \phi_{x+t}(u) - \phi_x(u), & for\ \ 0<\alpha<1, \\ \phi_{x+t}(u) + \phi_{x-t}(u) - 2\phi_x(u) & for\ \ 1 \le \alpha < 2. \end{cases} \tag{2.23}$$

where $\phi_x(u)$ *is as defined in (3.18) and* $f \in B_q^{\alpha}(L_p)$ *for* $k = [\alpha]+1$ *and* $p \ge 1$.
The k^{th} *order modulus of smoothness is :*

$$w_k(f,t)_p = \begin{cases} w_1(f,t)_p & for\ \ 0<\alpha<1, \\ w_2(f,t)_p & for\ \ 1<\alpha<2. \end{cases} \tag{2.24}$$

and

$$T_n(x,t) = \begin{cases} T_n(x+t) - T_n(x) & for\ \ 0<\alpha<1, \\ T_n(x+t) - T_n(x-t) - 2T_n(x) & for\ \ 1 \le \alpha < 2. \end{cases} \tag{2.25}$$

With the definition of $T_n(x,t)$ *in (2.25) and* $w_k(f,t)_p$ *in (2.24),*

$$w_k(T_n,t)_p = ||T_n(x,t)||_p.$$

3. Theorems and Lemmas

Theorem 3.1 (P.Chandra [6])

Let $f \in Lip(\alpha,p), (0<\alpha\le 1, p>1)$. *Then*

$$||f - E_n^q||_p = O(n^{-\alpha}) \qquad 0<\alpha\le 1. \tag{3.1}$$

Theorem 3.2 (P.Chandra [6])

Let $f \in Lip(\alpha,1), (0<\alpha\le 1)$. *Then*

$$||f - E_n^q||_1 = O(n^{-\alpha}\log n). \tag{3.2}$$

Lemma 3.1 (M. Mohanty et al. [4])

Let $1 \le p \le \infty$ *and* $0<\alpha<2$. *If* $f \in L_p[0,2\pi]$, *then for* $0<t,\ u\le\pi$.

1. $||\phi(.,t,u)||_p \le 4w_k(f,t)_p$
2. $||\phi(.,t,u)||_p \le 4w_k(f,u)_p$
3. $||\phi(u)||_p \le 2w_k(f,u)_p$

where $k = [\alpha]+1$.

Lemma 3.2 (M. Mohanty et al.[4])

Let $0<\alpha<2$. *Suppose that* $0\le\beta<\alpha$.*If* $f \in B_q^{\alpha}(L_p)$, $p \ge 1$ *and* $1<q\le\infty$, *then*

1. $\int_0^{\pi} |K_n(u)|(\int_0^{u} \frac{||\phi(.,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t})^{\frac{1}{q}}du = O(1)[\int_0^{\pi} (u^{\alpha-\beta}|K_n(u)|)^{\frac{q}{q-1}}du]^{1-\frac{1}{q}}$

2. $\int_0^\pi |K_n(u)|(\int_u^\pi \frac{||\phi(.,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t})^{\frac{1}{q}}du = O(1)[\int_0^\pi (u^{\alpha-\beta+\frac{1}{q}}|K_n(u)|)^{\frac{q}{q-1}}du]^{1-\frac{1}{q}}$

where $|K_n(u)|$ *is defined as in equation (2.22).*

Lemma 3.3 (M. Mohanty et al.[4])
Let $0 < \alpha < 2$. *Suppose that* $0 \le \beta < \alpha$. *If* $f \in B_q^\alpha(L_p)$, $p \ge 1$ *and* $q = \infty$, *then*

$$\sup_{t<0,u\le\pi} t^{-\beta}||\phi(x,t,u)||_p = O(u^{\alpha-\beta})$$

Lemma 3.4
Let $K_n(u)$ *The kernel of the Fourier series is defined in (2.22). Then*

$$|K_n(u)| = O(n) \ for \quad 0 \le u \le \frac{\pi}{n},$$
$$|K_n(u)| = O(\frac{1}{u}) \ for \quad \frac{\pi}{n} \le u \le \pi.$$

Proof of Lemma 3.4

$$K_n(u) = (1+q)^{-n}\sum_{k=0}^{n} {}^nC_k \ q^{n-k}\frac{\sin(k+\frac{1}{2})u}{\sin\frac{u}{2}}$$

if $0 \le u \le \frac{\pi}{n}$ *and* $\sin nu = n \sin u$

$$|K_n(u)| = |(1+q)^{-n}\sum_{k=0}^{n} {}^nC_k \ q^{n-k}\frac{\sin(k+\frac{1}{2})u}{\sin\frac{u}{2}}|$$

$$|K_n(u)| = |(1+q)^{-n}\frac{2n+1}{2}\sum_{k=0}^{n} {}^nC_k \ q^{n-k}|$$

$$|K_n(u)| = O(n).$$

if $\frac{\pi}{n} \le u \le \pi$, $\sin\frac{u}{2} \ge \frac{u}{\pi}$ *And* $\sin nu \le 1$.

$$|K_n(u)| = |(1+q)^{-n}\sum_{k=0}^{n} {}^nC_k \ q^{n-k}\frac{\sin(k+\frac{1}{2})u}{\sin\frac{u}{2}}|$$

$$|K_n(u)| = O(\frac{1}{u}).$$

4. Main Theorem

Let $0 < \alpha < 2$ *and* $0 \le \beta < \alpha$. *If* $f \in B_q^\alpha(L_p)$, $p \ge 1$ *then*

$$||T_n(x)||_{B_q^\beta(L_p)} = O(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}) \quad for \ 1 < q < \infty, \tag{4.1}$$

and

$$||T_n(x)||_{B_\infty^\beta(L_p)} = O(\frac{1}{n^{\alpha-\beta}} for \ q = \infty. \tag{4.2}$$

Note:

(a) Our results generalizes the result of P. Chandra for $0 < \alpha \le 1$ *in Theorem 3.1, as Conditions (4.1) and (4.2) include condition (3.1) in both cases when* $1 < q < \infty$ *and when* $q = \infty$.

(b) Our results also generalizes the result of P. Chandra for $0 < \alpha \le 1$ *in Theorem 3.2, as Conditions (4.1) and (4.2) include condition (3.2) in both cases when* $1 < q < \infty$ *and when* $q = \infty$.

5. Proof of the Main Theorem

Case I

We consider $1 < q < \infty$. *We have* $p \ge 1$, $0 \le \beta < \alpha < 2$.

$$||T_n(.)||_{B_q^\beta(L_p)} = ||T_n(.)||_p + ||w_k(T_n,.)||_{(\beta,q)} \tag{5.1}$$

applying Lemma 4.1

$$||T_n(.)||_p \leq \frac{1}{\pi}\int_0^{\pi} ||\phi_x(u)||_p \ |K_n(u)|du$$

$$||T_n(.)||_p \leq \frac{2}{\pi}\int_0^{\pi} |K_n(u)| \ w_k(f,u)_p du$$

applying hölder inequality

$$||T_n(.)||_p \leq \frac{2}{\pi}[(\int_0^{\pi} |K_n(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du]^{1-\frac{1}{q}}.[\int_0^{\pi} (\frac{w_k(f,u)_p}{u^{\alpha+\frac{1}{p}}})^q du]^{\frac{1}{q}}$$

By the definition of Besov space,

$$||T_n(.)||_p \leq O(1)[\int_0^{\pi} (|K_n(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du]^{1-\frac{1}{q}}$$

$$||T_n(.)||_p \leq O(1)[(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi})(|K_n(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du]^{1-\frac{1}{q}}$$

[By the inequality $(x+y)^r \leq x^r + y^r for \quad 0 < r \leq 1$*]*

$$||T_n(.)||_p = O(1)[I+J] \quad (let)$$

Now

$$I = [\int_0^{\frac{\pi}{n}} (|K_n(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du]^{1-\frac{1}{q}}$$

from Lemma 4.4 (case I)

$$I = O(n)[\int_0^{\frac{\pi}{n}} (u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du]^{1-\frac{1}{q}}$$

$$I = O(\frac{1}{n^{\alpha}}) \tag{5.2}$$

$$J = [\int_{\frac{\pi}{n}}^{\pi} (|K_n(u)|u^{\alpha+\frac{1}{q}})^{\frac{q}{q-1}}du]^{1-\frac{1}{q}}$$

$$J = [\int_{\frac{\pi}{n}}^{\pi} (u^{\alpha+\frac{1}{q}-1})^{\frac{q}{q-1}}du]^{1-\frac{1}{q}}$$

$$J = O(\frac{1}{n^{\alpha}}) \tag{5.3}$$

By (5.2) and (5.3),

$$||T_n(.)||_p = O(\frac{1}{n^{\alpha}}) \tag{5.4}$$

Now, by using the Besov space,

$$||w_k(T_n,t)||_{(\beta,q)} = [\int_0^{\pi} (t^{-\beta}w_k(T_n,t)_p)^q \frac{dt}{t}]^{\frac{1}{q}}$$

$$= [\int_0^{\pi} (\frac{w_k(T_n,t)_p}{t^{\beta}})^q \frac{dt}{t}]^{\frac{1}{q}}$$

From the definition of $w_k(T_n,t)_p$*,*

$$w_k(T_n,t)_p = ||T_n(x,t)||_p$$

$$||w_k(T_n,t)||_{(\beta,q)} = [\int_0^{\pi} (\frac{||T_n(x,t)||_p}{t^{\beta}})^q \ \frac{dt}{t}]^{\frac{1}{q}}$$

$$= [\int_0^\pi (\int_0^\pi |T_n(x,t)|^p dx)^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}]^{\frac{1}{q}}$$

Since $T_n(x,t) = \frac{1}{\pi}\int_0^\pi \phi(x,t,u)K_n(u)du,$

$$||w_k(T_n,t)||_{(\beta,q)} \le [\int_0^\pi (\int_0^\pi |\frac{1}{\pi}\int_0^\pi \phi(x,t,u)K_n(u)du|^p dx)^{\frac{q}{p}} \frac{dt}{t^{\beta q+1}}]^{\frac{1}{q}}$$

By repeated application of generalized Minkowski inequality,

$$= \frac{1}{\pi}[\int_0^\pi (\int_0^\pi (\int_0^\pi |\phi(x,t,u)|^p|K_n(u)|^p dx)^{\frac{1}{p}} du)^q \frac{dt}{t^{\beta q+1}}]^{\frac{1}{q}}$$

$$\le \frac{1}{\pi}[\int_0^\pi (\int_0^\pi |K_n(u)|||\phi(x,t,u)||_p du)^q \frac{dt}{t^{\beta q+1}}]^{\frac{1}{q}}$$

$$\le \frac{1}{\pi}\int_0^\pi (\int_0^\pi |K_n(u)|^q||\phi(x,t,u)||_p^q \frac{dt}{t^{\beta q+1}})^{\frac{1}{q}} du$$

$$\le \frac{1}{\pi}\int_0^\pi |K_n(u)|du(\int_0^\pi \frac{||\phi(x,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t})^{\frac{1}{q}}$$

$$\le \frac{1}{\pi}\int_0^\pi |K_n(u)|du[(\int_0^u + \int_u^\pi)\frac{||\phi(x,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t}]^{\frac{1}{q}}$$

$$= \frac{1}{\pi}\int_0^\pi |K_n(u)|du[\int_0^u \frac{||\phi(x,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t}]^{\frac{1}{q}} + \frac{1}{\pi}\int_0^\pi |K_n(u)|du[\int_u^\pi \frac{||\phi(x,t,u)||_p^q}{t^{\beta q}}\frac{dt}{t}]^{\frac{1}{q}}$$

By Lemma 4.2

$$= O(1)\left[\left(\int_0^\pi (|K_n(u)|u^{\alpha-\beta})^{\frac{q}{q-1}} du\right)^{1-\frac{1}{q}} + \left(\int_0^\pi \left(|K_n(u)|u^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}} du\right)^{1-\frac{1}{q}}\right]$$

$$||W_k(T_n,t)||_{(\beta,q)} = O(1)[I' + J'] \quad (let) \tag{5.5}$$

Now, proceed to estimate I'

$$I' = [\int_0^\pi (|K_n(u)|u^{\alpha-\beta})^{\frac{q}{q-1}} du]^{1-\frac{1}{q}}$$

$$= [(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^\pi)(|K_n(u)|u^{\alpha-\beta})^{\frac{q}{q-1}} du]^{1-\frac{1}{q}}$$

$$= O(n)(\int_0^{\frac{\pi}{n}} (u^{\alpha-\beta})^{\frac{q}{q-1}} du)^{1-\frac{1}{q}} + O(1)(\int_{\frac{\pi}{n}}^\pi (u^{\alpha-\beta-1})^{\frac{q}{q-1}} du)^{1-\frac{1}{q}}$$

$$I' = I'_1 + I'_2 \tag{5.6}$$

Let

$$I'_1 = O(n)(\int_0^{\frac{\pi}{n}} (u^{\alpha-\beta})^{\frac{q}{q-1}} du)^{1-\frac{1}{q}}$$

$$I'_1 = O(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}})$$

$$I'_2 = O(1)(\int_{\frac{\pi}{n}}^{\pi} (u^{\alpha-\beta-1})^{\frac{q}{q-1}} du)^{1-\frac{1}{q}}$$

$$I'_2 = O(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}})$$

Put the value of I'_1 *and* I'_2 *In equation (5.6),*

$$I' = O(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}) \tag{5.7}$$

Now

$$J' = (\int_0^{\pi} (|K_n(u)|u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du)^{1-\frac{1}{q}}$$

$$J' = [(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi})(|K_n(u)|u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du]^{1-\frac{1}{q}}$$

$$\leq O(n)[\int_0^{\frac{\pi}{n}} (u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du]^{1-\frac{1}{q}} + O(\frac{1}{u})[\int_{\frac{\pi}{n}}^{\pi} (u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du]^{1-\frac{1}{q}}$$

$$J' = J'_1 + J'_2 \tag{5.8}$$

Let

$$J'_1 = O(n)[\int_0^{\frac{\pi}{n}} (u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du]^{1-\frac{1}{q}}$$

$$J'_1 = O(n)(\frac{1}{n^{\alpha-\beta+1}})$$

$$J'_1 = O(\frac{1}{n^{\alpha-\beta}})$$

Let

$$J'_2 = O(\frac{1}{u})[\int_{\frac{\pi}{n}}^{\pi} (u^{\alpha-\beta+\frac{1}{q}})^{\frac{q}{q-1}} du]^{1-\frac{1}{q}}$$

$$J'_2 = O(\frac{1}{u})(\frac{1}{n^{\alpha-\beta+1}})$$

$$J'_2 = O(\frac{1}{n^{\alpha-\beta}})$$

Put the value of J'_1 *and* J'_2 *In equation (5.8), we have*

$$J' = O(\frac{1}{n^{\alpha-\beta}}) \tag{5.9}$$

from equation (5.5)

$$||w_k(T_n,t)||_{(\beta,q)} = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\frac{1}{n^{\alpha-\beta}}\right) \tag{5.10}$$

by equations (5.1) ,(5.4) and (5.9)

$$||T_n(.)||_{B_q^\beta(L_p)} = O\left(\frac{1}{n^{\alpha}}\right) + O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\frac{1}{n^{\alpha-\beta}}\right)$$

$$||T_n(.)||_{B_q^\beta(L_p)} = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) \quad for \quad 1 < q < \infty \tag{5.11}$$

This completes the proof in case I.

case II

For $q = \infty$,
Now, consider the case $q = \infty$.

$$||T_n(.)||_{B_\infty^\beta(L_p)} = ||T_n(.)||_p + ||w_k(T_n,.)||_{(\beta,\infty)} \tag{5.12}$$

$$T_n(x) = \frac{1}{\pi}\int_0^\pi \phi_x(u)\ K_n(u)du$$

applying Lemma 4.1 (iii) in $T_n(x)$,

$$||T_n(.)||_p \leq \frac{1}{\pi}\int_0^\pi ||\phi_x(u)||_p\ |K_n(u)|du$$

$$||T_n(.)||_p \leq \frac{2}{\pi}\int_0^\pi |K_n(u)|\ w_k(f,u)_p du$$

$$\leq O(1)[\int_0^\pi |K_n(u)|\ u^\alpha du]$$

$$\leq O(1)[(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^\pi)|K_n(u)|\ u^\alpha du]$$

$$\leq O(n)\int_0^{\frac{\pi}{n}} u^\alpha du + O(1)\int_{\frac{\pi}{n}}^\pi u^{\alpha-1} du$$

$$\therefore \quad ||T_n(.)||_p = O\left(\frac{1}{n^\alpha}\right) \qquad for \quad q = \infty \tag{5.13}$$

Now

$$||w_k(T_n,t)||_{(\beta,\infty)} = \sup_{t>0} \frac{||T_n(x,t)||_p}{t^\beta}$$

$$= \sup_{t>0} \frac{t^{-\beta}}{\pi}[\int_0^\pi |\int_0^\pi \phi(x,t,u)\ K_n(u)du|^p dx]^{\frac{1}{p}}$$

applying generalized Minkowski's inequality,

$$||w_k(T_n,t)||_{(\beta,q)} = \sup_{t>0} \int_0^{\pi} du[\int_0^{\pi} |\phi(x,t,u)|^p |K_n(u)|^p dx]^{\frac{1}{p}}$$

$$= \sup_{t>0} \frac{t^{-\beta}}{\pi} \int_0^{\pi} |K_n(u)| \ ||\phi(x,t,u)||_p \ du$$

$$\leq \frac{1}{\pi} \int_0^{\pi} |K_n(u)| du (\sup_{t>0} t^{-\beta} ||\phi(x,t,u)||_p)$$

Using Lemma 4.3,

$$||w_k(T_n,.)||_{(\beta,\infty)} \leq O(1) \int_0^{\pi} u^{\alpha-\beta} |K_n(u)| \ du$$

$$= O(1)(\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi}) u^{\alpha-\beta} |K_n(u)| \ du$$

$$= O(1) \int_0^{\frac{\pi}{n}} u^{\alpha-\beta} |K_n(u)| du + O(1) \int_{\frac{\pi}{n}}^{\pi} u^{\alpha-\beta} |K_n(u)| du$$

$$= O(n) \int_0^{\frac{\pi}{n}} u^{\alpha-\beta} du + O(1) \int_{\frac{\pi}{n}}^{\pi} u^{\alpha-\beta-1} du$$

$$||w_k(T_n,.)||_{(\beta,\infty)} = O(\frac{1}{n^{\alpha-\beta}}) \quad (5.14)$$

using equations (5.12), (5.13) and (5.14),

$$||T_n(.)||_{B_{\infty}^{\beta}(L_p)} = O(\frac{1}{n^{\alpha}}) + O(\frac{1}{n^{\alpha-\beta}})$$

$$||T_n(.)||_{B_{\infty}^{\beta}(L_p)} = O(\frac{1}{n^{\alpha-\beta}}) \quad (5.15)$$

This completes the proof of Case II.

Hence, the proof of the main theorem.

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