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Original Article

Bitopological Harmonious Labeling of Some Star-Related Graphs

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Abstract - *Bitopological harmonious labeling for a graph $G = (V(G), E(G))$ with n vertices, is an injective function $f: V(G) \to 2^X$, where X is any non–empty set such that $|X| = m$, $m < n$ and $\{f(V(G))\}$ forms a topology on X, that induces an injective function $f^*: E(G) \to 2^{X^*}$, defined by $f^*(uv) = f(u) \cap f(v)$ for every $uv \in E(G)$ such that $\{f^*(E(G))\}$ forms a topology on X^* where $X^* = X\backslash\{1,2, \dots, m\}$. A graph that admits bitopological harmonious labeling is called a bitopological harmonious graph. In this paper, we discuss bitopological harmonious labeling of some star-related graphs.*

Keywords - *Bistar graph, Bitopological harmonious graph, Firecracker graph, Lilly graph, Spider graph.*

1. Introduction

In this paper, we consider only simple, finite and undirected graphs. The graph G has a vertex set $V = V(G)$ and edge set $E = E(G)$. For notations and terminology, we refer to Bondy and Murthy[5]. Acharya[1] established another link between graph theory and point set topology. Selestin Lina S and Asha S defined bitopological star labeling for a graph $G = (V, E)$ as X being any non-empty set if there exists an injective function $f: V(G) \to 2^X$ Which induces the function $f^*: E(G) \to 2^X$ as $f^*(v_1 v_2) = [f(v_1) \cup f(v_2)]^c$ for every $v_1, v_2 \in V(G)$, if $\{f(V(G))\}$and $\{f^*(E(G))If \cup X$ are topologies on X, then G is said to be a bitopological star graph. In this paper, we proved that some star-related graphs are bitopologically harmonious graphs.

2. Basic Definitions

2.1. Definition

Bitopological harmonious labeling of a graph $G = (V(G), E(G))$ With n vertices, an injective function $f: V(G) \to 2^X$, where X is any non–empty set such that $|X| = m$, $m < n$ and $\{f(V(G))\}$ Forms a topology on X, that induces an injective function $f^*: E(G) \to 2^{X^*}$, defined by $f^*(uv) = f(u) \cap f(v)$ for every $uv \in E(G)$ such that $\{f^*(E(G))\}$ forms a topology on X^* where $X^* = X\backslash\{1,2, \dots, m\}$. A graph that admits bitopological harmonious labeling is called a bitopological harmonious graph.

2.2. Definition

Bistar graph $B_{m,n}$ is obtained from K_2 by attaching m pendant edges to one end of K_2 and n pendant edges to the other end of K_2.

2.3. Definition

A spider graph $SP(1^n 2^{2m})$ is a star graph $K_{1,n+m}$ such that each of which m vertices is joined to a new vertex.

2.4. Definition

Lilly graph L_n, $n \geq 2$, is obtained from 2 stars $2K_{1,n}, n \geq 2$, by joining 2 paths $2P_n, n \geq 2$ with sharing a common vertex.

2.5. Definition

Firecracker graph $F_{n,k}$ Is the graph obtained by concatenation of n $k-$stars by linking one leaf from each.

3. Main Results

Theorem 3.1

The bistar graph $B_{m,n}, m, n \geq 1$ is a bitopological harmonious graph.

Proof:

Let $G = B_{m,n}$.
Let $V(G) = \{u, v\} \cup \{u_i/1 \leq i \leq m\} \cup \{v_i/1 \leq i \leq n\}$.
Let $E(G) = \{uv\} \cup \{uu_i/1 \leq i \leq m\} \cup \{vv_i/1 \leq i \leq n\}$.
$|V(G)| = m + n + 2, |E(G)| = m + n + 1$.
Let $X = \{1,2, \dots, |V(G)| - 1\}$.
Define a function $f: V(G) \rightarrow 2^X$ As follows:
$f(u_1) = \phi$;
$f(u_i) = \{1,2, \dots, i-1\}$ for $2 \leq i \leq m$;
$f(u) = \{1,2, \dots, m\}$;
$f(v_i) = \{1,2, \dots, m+i\}$ for $1 \leq i \leq n$;
$f(v) = \{1,2, \dots, m+n+1\}$.
Here, all the vertex labels are distinct and form a topology on X.
Then the induced function $f^*: E(G) \rightarrow X^*$ It is given as follows:
$f^*(uv) = f(u) \cap f(v)$ for all $uv \in E(G)$.
$f^*(uu_i) = f(u_i)$ for $1 \leq i \leq m$.
$f^*(vv_i) = f(v_i)$ for $1 \leq i \leq n$.
$f^*(uv) = f(v)$.
Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.
Hence, f is a bitopological harmonious labeling and G is a bitopological harmonious graph.

Example 3.2

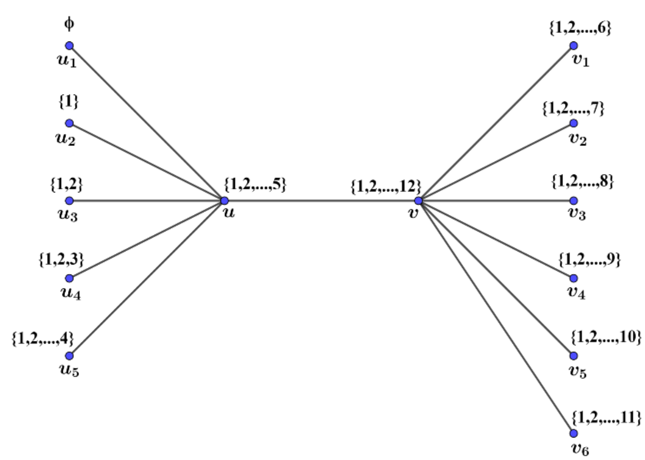

Fig. 1 Bitopological harmonious labelling of $B_{5,6}$

Theorem 3.3

The Spider graph $SP(1^n2^m), m, n \geq 1$ is a bitopological harmonious graph.

Proof:

Let $G = SP(1^n 2^m)$.
Let $V(G) = \{v_i, u_j/0 \leq i \leq n, 1 \leq j \leq 2m\}$ where v_0 be the centre vertex.
Let $E(G) = \{v_0v_i/1 \leq i \leq n\} \cup \{v_0u_{2i-1}/1 \leq i \leq m\}\} \cup \{u_{2i-1}u_{2i}/1 \leq i \leq m\}$. .
$Then$ $|V(G)| = n + 2m + 1, |E(G)| = n + 2m$.
Let $X = \{1,2, \dots, |V(G)| - 1\}$.

Define a function $f: V(G) \rightarrow 2^X$ As follows:
$f(v_1) = \phi$;
$f(v_i) = \{1,2,\ldots,i-1\}$ for $2 \leq i \leq n$;
$f(u_{2i}) = \{1,2,\ldots,n+2i-2\}$ for $1 \leq i \leq m$;
$f(u_{2i-1}) = \{1,2,\ldots,n+2i-1\}$ for $1 \leq i \leq m$;
$f(v_0) = \{1,2,\ldots,n+2m\}$.
Here, all the vertex labels are distinct and form a topology on X.
Then the induced function $f^*: E(G) \rightarrow 2^{X^*}$ It is given as follows:
$f^*(uv) = f(u) \cap f(v)$ for all $uv \epsilon E(G)$.
Here $f^*(v_0 v_i) = f(v_i)$ for $1 \leq i \leq n$;
$f^*(v_0 u_{2i-1}) = f(u_{2i-1})$ for $1 \leq i \leq m$;
$f^*(v_{2i-1} u_{2i}) = f(u_{2i})$ for $1 \leq i \leq m$.
Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.
Hence, f is a bitopological harmonious labeling and G is a bitopological harmonious graph.

Example 3.4

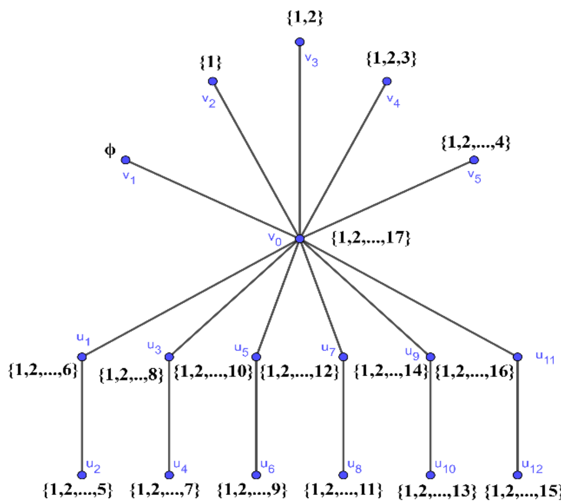

Fig. 2 Bitopological harmonious labelling of SP$(1^5 2^6)$

Theorem 3.5
Lilly graph $L_n, n \geq 2$ is a bitopological harmonious graph.
Proof:
Let $G = L_n$.
Let $V(G) = \{u_i / 1 \leq i \leq 2n-1\} \cup \{v_i / 1 \leq i \leq 2n\}$.
Let $E(G) = \{v_i u_n / 1 \leq i \leq 2n\} \cup \{u_i u_{i+1} / 1 \leq i \leq 2n-2\}$.
$|V(G)| = 4n-1, |E(G)| = 4n-2$.
Let $X = \{1,2,\ldots,|V(G)|-1\}$.
Define a function $f: V(G) \rightarrow 2^X$ As follows:
$f(v_1) = \phi$;
$f(v_i) = \{1,2,\ldots,i-1\}$ for $2 \leq i \leq 2n$;
$f(u_i) = \{1,2,\ldots,2n+i-1\}$ for $1 \leq i \leq 2n-1$.
Here, all the vertex labels are distinct and form a topology on X.
Then the induced function $f^*: E(G) \rightarrow 2^{X^*}$ It is given as follows:
$f^*(uv) = f(u) \cap f(v)$ for all $uv \epsilon E(G)$.
Here $f^*(v_i u_n) = f(v_i)$ for $1 \leq i \leq 2n$;
$f^*(u_i u_{i+1}) = f(u_i)$ for $1 \leq i \leq 2n-1$.
Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.
Hence, f is a bitopological harmonious labeling and G is a bitopological harmonious graph.

Example 3.6

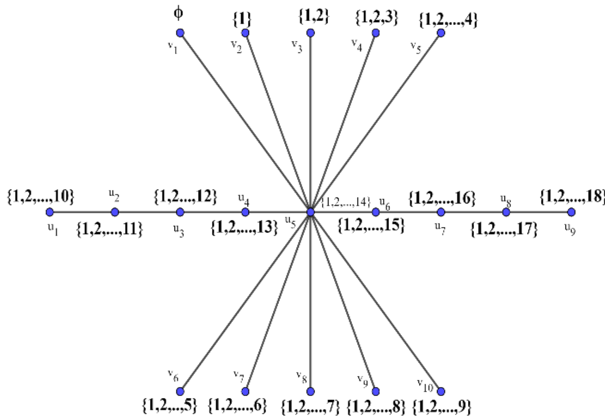

Fig. 3 Bitopological harmonious labelling of L_5

Theorem 3.7

The firecracker graph $F_{n,k}, n, k \geq 1$ is a bitopological harmonious graph.

Proof:

Let $G = F_{n,k}$.

Let $V(G) = \{v_{ij} / 1 \leq i \leq n, 1 \leq j \leq k\}$.

Let $E(G) = \{v_{i1}v_{ij} / 1 \leq i \leq n,\ 2 \leq j \leq k\} \cup \{v_{ik}v_{i+1k} / 1 \leq i \leq n-1\}$.

$|V(G)| = nk, |E(G)| = nk - 1$.

Let $X = \{1,2,\ldots,|V(G)| - 1\}$.

Define a function $f: V(G) \rightarrow 2^X$ As follows:

$f(v_{12}) = \phi$;

$f(v_{1j}) = \{1,2,\ldots,j-2\}$ for $3 \leq j \leq k-1$;

$f(v_{i1}) = \{1,2,\ldots,ki-2\}$ for $1 \leq i \leq n$;

$f(v_{ik}) = \{1,2,\ldots,ki-1\}$ for $1 \leq i \leq n$;

$f(v_{ij}) = \{1,2,\ldots,k(i-1)+j-2\}$ for $2 \leq i \leq n,\ 2 \leq j \leq k-1$.

Here, all the vertex labels are distinct, and they form a topology on X.

Then the induced function $f^*: E(G) \rightarrow 2^{X^*}$ It is given as follows:

$f^*(uv) = f(u) \cap f(v)$ for all $uv \in E(G)$.

Here $f^*(v_{i1}v_{ij}) = f(v_{ij})$ for $1 \leq i \leq n,\ 2 \leq j \leq k$;

$f^*(v_{ik}v_{i+1k}) = f(v_{ik})$ for $1 \leq i \leq n-1$.

Since f is 1-1 and so f^*. Also $\{f^*(E(G))\}$ forms a topology on X^*.

Hence, f is a bitopological harmonious labeling and G is a bitopological harmonious graph.

Example 3.8

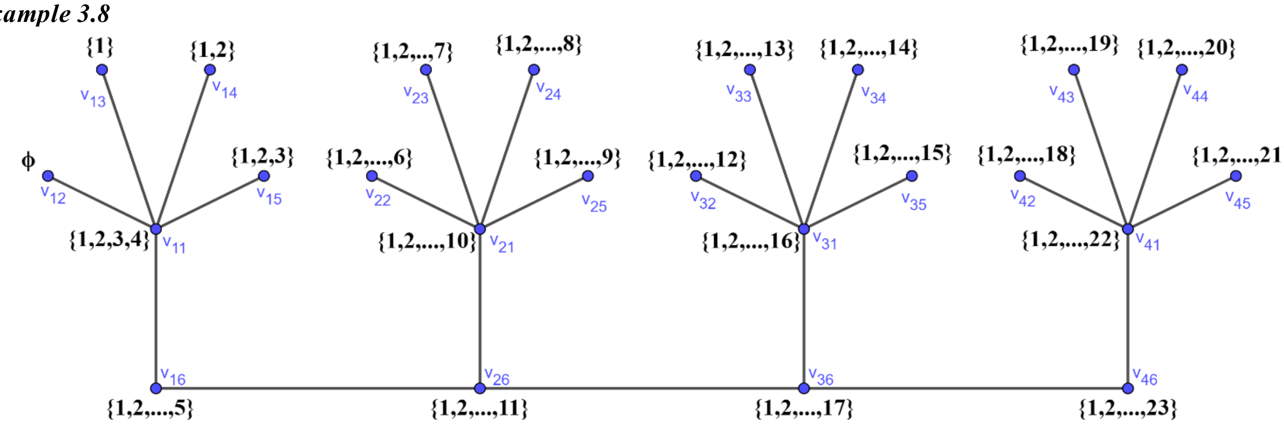

Fig. 3 Bitopological harmonious labelling of $F_{4,6}$

4. Conclusion

In this paper, we proved that some star-related graphs, such as the bistar, spider graph, lilly graph and firecracker graph, are bitopologically harmonious graphs.

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