International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V71I7P106
Volume 71 Issue 7, 49-58, July 2025

Original Article

Cosmological Dynamics of Bianchi Type VI_0 Universe in Higher Dimensions within Lyra Geometry

Rajni Gupta[1], Riddhi Jain[2]

[1]*Department of Mathematics, IIS (Deemed to be University), Jaipur(Rajasthan), India.*
[2]*Department of Mathematics, IIS (Deemed to be University), Jaipur, India.*

[1]*Corresponding Author* : *rajni.gupta@iisuniv.ac.in*

Received: 23 May 2025 Revised: 29 June 2025 Accepted: 15 July 2025 Published: 28 July 2025

Abstract - *In this study, we analyze a higher-dimensional Bianchi Type VI_0 cosmological model within the General Relativity and Lyra geometry frameworks. The field equations are solved analytically under the assumption that the shear scalar is proportional to the scalar Expansion. Specific choices of the constants α, β, γ lead to the construction of Bianchi Type I, III, and V models, each of which is examined in detail. The physical and geometrical properties of these cosmological solutions are discussed. A fractional cosmological model is also developed, and its consequences are discussed.*

Keywords - *Energy Momentum Tensor, Fractional Cosmological Model, Higher Dimensional Bianchi Type VI_0, Last Step Modification, Lyra Geometry.*

1. Introduction

The Bianchi type models are a class of solutions to the equations of general relativity that describe the Spatially homogeneous and anisotropic universe. These models were first introduced by Luigi Bianchi, an An Italian mathematician in the early 20th century. Researchers have explored different Bianchi-type models, such as Bianchi type I, II, VI, IX, and others, in multiple theoretical frameworks, including general relativity, Lyra geometry, Scalar-Tensor theories, string cosmology, etc. Among all of them, the Lyra geometry framework is the most commonly used framework because it can include extensions with string, bulk viscosity, extra dimensions, dark energy, and fractional calculus (Lyra 1951). [12]. These models have provided insights into the early universe, inflationary scenarios, and the formation of large-scale structures. Lyra's modification of Riemannian geometry introduces a displacement vector field, which replaces the arbitrary gauge of Weyl geometry. When constant, the displacement vector acts much like the cosmological constant in Einstein's general theory of relativity. To develop an equivalent of the Einstein field equations in which the scalar and tensor fields both have inherent geometrical relevance, Lyra geometry was used by Sen and Dunn (1971). [20]. A symmetric model within the framework of Lyra's manifold mainly explores the behavior of the specific model under certain conditions, including the presence of a bulk viscous fluid and strings of a single dimension, which has been studied by Reddy and Innaiah (1986). [17]. However, this limitation of the constant displacement field has no underlying rationale and is only a matter of convenience. Many researchers studied various cosmological models within Lyra's geometry, assuming a time-dependent displacement field (Alkaoud et al, 2024; Bishi et al, 2023; Desikan K, 2020; Megied et al, 2016). [2, 4, 7, 13].

The study of higher dimensions is crucial nowadays due to significant advancements. Theoretical studies in higher dimensions offer insights into complex phenomena, driving researchers to investigate the mysteries of the universe through multi-dimensional frameworks. It offers a path towards unifying physics and provides deeper insights into the universe's structure and origins. The extended vacuum Bianchi type-VI_h universe with perfect fluid, which examines the influence and effect of extra dimensions, $D \geq 1$ $(N = 1 + d + D)$, was explored by Lorenz-Petzold (1985). [11]. Another higher-dimensional cosmological model with variable gravitational constant and bulk viscosity in Lyra geometry was presented by Singh et al (2004). [23]. Further, a spatially homogeneous Kaluza-Klein model with magnetized fluid in the scalar-tensor theory of gravitation was proposed by Katore et al (2014). [9]. Inspired by the scope in higher-dimensional studies, many researchers explored higher-dimensional cosmological models under the various theories, including Lyra geometry (Adhav et al, 2011; Daimary and Baruah, 2022; Sahu et al, 2018; Trivedi and Bhabor, 2021). [1, 6, 19, 25].

In the context of cosmology, the fractional derivative has been explored as a tool to study the evolution of the universe in a non-local or non-classical framework. The application of fractional derivatives in Bianchi-type models is an active area of research, but it is not yet widely explored. El-Nabulsi Ahmad Rami (2005) [15] developed the Fractional Friedman Dynamical Equations for a static universe. Although he faced challenges in solving these equations, his efforts raised numerous questions that intrigued researchers and motivated further exploration. Fractional calculus has been used in various applications to explain non-local spacetime effects. Incorporating fractional derivatives in the field of gravity is a demanding task, for which two approaches have been proposed: the First Step Modification (FSM) and the Last Step Modification (LSM) by Mark D. Robert (2009). [18]. The FSM entails expressing all of the geometrical elements of general relativity in terms of fractional derivatives, including the covariant derivative, connection, metric and Riemann tensor. This method substitutes a new fractional time derivative, which assumes a lower terminal at 0, for the time derivative that appears in the Friedman equation using the Riemann-Liouville fractional derivative. On the other hand, the LSM modifies the field equations of gravity by substituting a similar fractional derivative for the covariant derivative order. The classic Friedmann and scalar field equations were expanded (V. K. Shchigolev 2011) [21] by including fractional derivatives of the scale factor. He offered a number of accurate answers to these fractional cosmological questions with a focus on accurate models of accelerated cosmological Expansion and equations, and he raised the question of how useful fractional differential calculus ideas are in the field of cosmology. Recently, he presented the application of a fractional differential approach in cosmology (V. K. Shchigolev 2021). [22]. He explores cosmological theories that incorporate a fractional scalar field, and the variational principle for the Einstein-Hilbert action is used to present a novel method for fractional differentiation and integration in cosmology. The authors expect that additional research will reveal the utility of fractional differential calculus in cosmology by providing exact solutions to modified Friedmann and scalar field equations that could be helpful for present cosmology. Using the Last Step Modification method, Ernesto Barrientos et al (2021) [3] provided an experimental model that expanded the Friedmann equations by incorporating fractional derivatives. Some researchers (Calcagni et al. 2020; Jalalzadeh et al. 2022; Micolta et al. 2023).[5, 8, 14] have proposed using fractional derivatives to study the dynamics and stability of anisotropic cosmological models. These investigations seek to comprehend how anisotropic perturbations behave and how the evolution of the cosmos is affected by fractional derivatives.

The study of higher-dimensional cosmological models is vital because it provides deeper insights into the structure and origins of the universe. Adding extra dimensions gives cosmologists more flexibility to construct models consistent with observational data and leads to new constraints on physics beyond the standard models. Motivated by this, the current work introduces a higher-dimensional Bianchi type VI_0 cosmological model based on Lyra geometry parameterized by a set of a few constants, allowing continuous reduction to Bianchi type I, III and V. On top of this unified structure, fractional derivative dynamics is employed to construct fractional field equations, which is a modern approach shown to address cosmological tensions. The physical and geometrical properties of the newly constructed model are also discussed here.

2. Field Equations and Solution

Let us consider the cosmological model Bianchi type VI_0 in higher dimensions as:

$$ds^2 = -dt^2 + [A(t)]^2 dx^2 + [B(t)]^2 e^{\alpha x} dy^2 + [C(t)]^2 e^{\beta x} dz^2 + [D(t)]^2 e^{\gamma x} dm^2 \tag{1}$$

Where A, B, C and D are all assumed to be metric functions of cosmic time. The fifth coordinate 'm' is taken to be space-like in this context.

The energy-momentum tensor is given as

$$T_{ij} = \varphi_{,i}\varphi_{,j} - \left(\frac{1}{2}\varphi_{,i}\varphi_{j}^{,} + V(\varphi)\right) g_{ij},$$

Choose $v_i v^j = -1, V(\varphi) = \beta(constant)$ and $\varphi_i^j = -\frac{dV}{d\varphi}$

Assuming a scalar field region with potential $V(\varphi)$, the action of the field of gravitation coupled minimally is given by

$$S = \int \left(R - \frac{1}{2}\varphi_{,i}\varphi_{,j}\, g^i j - V(\varphi)\right)\sqrt{-g}\, dx^5$$

This is a variation on S with respect to the dynamical field provided by Einstein's field equations as follows:

$$R_i^j - \frac{1}{2} R g_i^j = -8\pi T_i^j,$$

gives

$$\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{D}}{BD} + \frac{\dot{C}\dot{A}}{CA} + \frac{\dot{C}\dot{D}}{CD} + \frac{\dot{A}\dot{D}}{AD} - \frac{\alpha^2+\beta^2+\gamma^2+\alpha\beta+\beta\gamma+\alpha\gamma}{4A^2} = 8\pi\left(-\frac{1}{2}\dot{\varphi}^2 + \beta\right) \tag{2}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{D}}{D} + \frac{\dot{B}\dot{D}}{BD} + \frac{\dot{C}\dot{D}}{CD} - \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{4A^2} = 8\pi\left(\frac{1}{2}\dot{\phi}^2 + \beta\right) \tag{3}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{C}\dot{A}}{CA} + \frac{\ddot{D}}{D} + \frac{\dot{A}\dot{D}}{AD} + \frac{\dot{C}\dot{D}}{CD} - \frac{\beta^2+\gamma^2+\beta\gamma}{4A^2} = 8\pi\left(\frac{1}{2}\dot{\phi}^2 + \beta\right) \tag{4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{D}}{D} + \frac{\dot{B}\dot{D}}{BD} + \frac{\dot{A}\dot{D}}{AD} - \frac{\alpha^2+\gamma^2+\alpha\gamma}{4A^2} = 8\pi\left(\frac{1}{2}\dot{\phi}^2 + \beta\right) \tag{5}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{\alpha^2+\beta^2+\alpha\beta}{4A^2} = 8\pi\left(\frac{1}{2}\dot{\phi}^2 + \beta\right) \tag{6}$$

$$(\alpha+\beta+\gamma)\frac{\dot{A}}{A} = \left(\alpha\frac{\dot{B}}{B} + \beta\frac{\dot{C}}{C} + \gamma\frac{\dot{D}}{D}\right) \tag{7}$$

The law of conservation of energy-momentum tensor,

$$\frac{1}{\sqrt{-g}}\partial_i\left[\sqrt{-g}\partial_i\phi\right] = -\frac{dV}{d\phi} \tag{8}$$

where

$$\partial_l\phi = g^{lv}\partial_v\phi = g^{lv}\frac{\partial\phi}{\partial x^v} \tag{9}$$

which gives

$$\ddot{\phi} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D}\right)\dot{\phi} = -\frac{dV}{d\phi} \tag{10}$$

The average scale factor is given as:

$$a = \left[ABCDe^{\frac{(\alpha+\beta+\gamma)x}{2}}\right]^{\frac{1}{4}} \tag{11}$$

From equation (7), it is found that

$$A = l\left(B^{\alpha}C^{\beta}D^{\gamma}\right)^{\frac{1}{\alpha+\beta+\gamma}} \tag{12}$$

Additionally, by taking the components of the shear tensor σ as proportional to the expansion scalar), the following relations can be established.

$$C = B^{\eta} \text{ and } D = B^{\zeta} \tag{13}$$

Putting these values, equation (12) gives

$$A = lB^{L} \tag{14}$$

Where $L = \left(\frac{\alpha+\eta\beta+\gamma\varsigma}{\alpha+\beta+\gamma}\right)$ And l is the constant of proportionality. Again, from equations (4) and (5), it is obtained.

$$\frac{\ddot{C}}{C} - \frac{\ddot{B}}{B} + \frac{\dot{C}\dot{A}}{CA} + \frac{\dot{C}\dot{D}}{CD} - \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{D}}{BD} + \frac{(\alpha-\beta)(\alpha+\beta+\gamma)}{4A^2} = 0 \tag{15}$$

The result obtained by solving is:

$$A = l[L(Kt + c_1)] \tag{16}$$

$$B = [L(Kt + c_1)]^{\frac{1}{L}} \tag{17}$$

$$C = [L(Kt + c_1)]^{\frac{\eta}{L}} \tag{18}$$

$$D = [L(Kt + c_1)]^{\frac{\zeta}{L}} \tag{19}$$

where c_1 is the constant of integration and $K = \left[\frac{(\alpha-\beta)(\alpha+\beta+\gamma)}{4l^2(\eta-1)(1+\eta+\zeta)}\right]^{\frac{1}{2}}$,
Using the above results, the metric (1) reduces to

$$ds^2 = -dt^2 + l^2T^2dx^2 + T^{\frac{2}{L}}e^{\alpha x}dy^2 + T^{\frac{2\eta}{L}}e^{\beta x}dz^2 + T^{\frac{2\zeta}{L}}e^{\gamma x}dm^2 \tag{20}$$

where $T = [L(Kt + c_1)]$.

3. Physical and Geometrical Properties

In this section, a few geometrical and physical features of the model obtained in Lyra's geometry have been discussed. The Average scale factor, Hubble parameter and the Deceleration parameter are derived as:

$$a = \left[l\,T^{\frac{1+\eta+\zeta+L}{L}}\; e^{\frac{(\alpha+\beta+\gamma)x}{2}}\right]^{\frac{1}{4}} \tag{21}$$

$$H = \frac{\dot{a}}{a} = \frac{1+\eta+\zeta+L}{4LT} \tag{22}$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\left[1 - \frac{4L}{1+\eta+\zeta+L}\right] \tag{23}$$

The other cosmological parameters, such as Expansion(θ), Shera tensor (σ) are obtained as:

$$\theta = 4H = \frac{1+\eta+\zeta+L}{LT} \tag{24}$$

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{4} H_i^2 - 4H^2\right) = \frac{1}{2L^2T^2}\left[(1+\eta^2+\zeta^2+L^2) - \frac{(1+\eta+\zeta+L)^2}{4}\right] \tag{25}$$

Using equation (10), the following result is derived

$$\phi = -\frac{c_2}{K(1+\eta+\zeta)}T^{-\frac{1+\eta+\zeta}{L}} \tag{26}$$

Since $\frac{\sigma}{\theta}$ = constant, which shows that this general model does not approach isotropy for a large value of T. The model stops expanding as $T\to\infty$ Moreover, starts expanding at $T > 0$.

4. Results and Discussion

Here are a few cases for the different choices of α, β *and* γ are discussed as follows:

4.1. Case 1: When $(\alpha = \beta = \gamma = 0)$

With the above assumption, metric (1) reduces to the Bianchi Type-I cosmological model defined as per the above equation:

$$ds^2 = -dt^2 + [A(t)^2dx^2] + [B(t)^2dy^2] + [C(t)^2dz^2] + [D(t)^2dm^2] \tag{27}$$

And the field equations are:

$$\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{D}}{BD} + \frac{\dot{C}\dot{A}}{CA} + \frac{\dot{C}\dot{D}}{CD} + \frac{\dot{A}\dot{D}}{AD} = 8\pi\left(-\frac{1}{2}\dot{\phi}^2 + \beta\right) \tag{28}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{D}}{D} + \frac{\dot{B}\dot{D}}{BD} + \frac{\dot{C}\dot{D}}{CD} = 8\pi\left(\frac{1}{2}\dot{\phi}^2 + \beta\right) \tag{29}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{D}}{D} + \frac{\dot{A}\dot{D}}{AD} + \frac{\dot{C}\dot{D}}{CD} = 8\pi\left(\frac{1}{2}\dot{\phi}^2 + \beta\right) \tag{30}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{D}}{D} + \frac{\dot{B}\dot{D}}{BD} + \frac{\dot{A}\dot{D}}{AD} = 8\pi\left(\frac{1}{2}\dot{\phi}^2 + \beta\right) \tag{31}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{C}}{D} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} = 8\pi\left(\frac{1}{2}\dot{\phi}^2 + \beta\right) \tag{32}$$

By assuming Expansion scalar θ proportional to shear tensorσ, and also by assuming B=C D, it follows that $A = B^3$Using this in equations (30) and (31) provides the following results:

$$A = \left[\frac{2}{7}(c_1t + c_2)\right]^{6/7},$$

$$B = C = D = \left[\frac{2}{7}(c_1 t + c_2)\right]^{2/7},$$

Where c_1 and c_2 are the constants of integration.

The other physical parameters are obtained as:

$$a = \left[\frac{2}{7}(c_1 t + c_2)\right]^{3/7} \tag{33}$$

$$H = \frac{\theta}{4} = \frac{3}{7}\left(\frac{c_1}{c_1 t + c_2}\right) \tag{34}$$

$$q = -\frac{4}{3} < -1 \tag{35}$$

$$\sigma^2 = \frac{6}{49}\left(\frac{c_1}{c_1 t + c_2}\right)^2 \tag{36}$$

$$\phi = -\frac{2c_1 c_3}{5}\left[\frac{2}{7}(c_1 t + c_2)\right]^{-5/7} \tag{37}$$

The obtained value of the deceleration parameter, $-\frac{4}{3}$, suggests a very rapid acceleration of the universe's Expansion, much stronger than the rate observed in currently available cosmological data. Specifically, this value could suggest a scenario where dark energy is far more dominant than in our current models. As ϕ depends on time and decreases in magnitude as t increases (assuming c > 0), it asymptotically approaches zero from the negative side.

The form of ϕ could correspond to a dynamic dark energy field with significant early influence, driving a rapid expansion (with a very negative q) that could slow over time. If this field has a phantom-like behaviour ($\omega < - \, - -1$), it could even lead to a Big Rip. This scalar field could play a role in early-universe inflation or another epoch of rapid Expansion. As t approaches zero (near the Big Bang), φ might have a large, negative value, which could impact the early dynamics strongly. The results obtained in the particular case are consistent with those reported by KP Singh et al. (2021) [24]

4.2. Case 2: (One out of α, β, γ *is* non-zero, the other two are zero)

Based on the above assumptions, the metric is converted into a Bianchi-type III curved spacetime. If $\beta \neq 0$ is chosen while α, γ are zero, then all the physical properties contain imaginary values as the value of $K = \frac{\sqrt{-\beta^2}}{4}$ is imaginary. Another two possibilities are discussed in the table shown below:

Table 1. Physical parameters for the Bianchi type-III metric space for specific values $l = 1, c_1 = 1, \eta = 2, \zeta = 1, c_2 = 1$.

Case	$(\alpha \neq 0, \beta = \gamma = 0)$	$(\gamma \neq 0, \alpha = \beta = 0)$
a	$e^{\frac{\alpha x}{8}}\left(1 + \frac{\alpha}{4} t\right)^{5/4}$	$e^{\frac{\gamma x}{8}}$
H	$\frac{5}{4}\left(1 + \frac{\alpha t}{4}\right)^{-1}$	$\frac{5}{4}$
q	$-\frac{1}{5}$	$-\frac{1}{5}$
σ^2	$\frac{3}{8}\left(1 + \frac{\alpha t}{4}\right)^{-2}$	$\frac{3}{8}$
φ	$-\frac{1}{\alpha}\left(1 + \frac{\alpha t}{4}\right)^{-4}$	Complex infinity

Here, it is observed that the scale factor is dependent on x and t, both of which show that different regions of space are expanding or contracting at different rates. This could represent a universe with inhomogeneities or anisotropies, which may be a novel contribution, as no published article has presented a true inhomogeneous Bianchi type III model with scale factor depending on both x and t. The 'average' Hubble parameter is adjusted over time, without accounting for spatial variations. Lastly, the condition $q = -\frac{1}{5}$ shows the evolution of the universe at a constant rate, and its age is equal to the Hubble time.

4.3. Case 3: (one out of α, β, γ is zero, the other two are equal and non-zero)

Based on the above assumption, the metric is converted into a Bianchi type V curved spacetime. Out of three possible cases ($\alpha = \beta, \gamma = 0$) and ($\alpha = \gamma, \beta = 0$) are discussed in the above table, but for ($\beta = \gamma, \alpha = 0$), the physical quantities contain imaginary values.

Table 2. Physical parameters for Bianchi type V metric space for specific values $l = 1, c_1 = 1, \eta = 1, \zeta = 1, c_2 = 1$.

Case	$(\alpha = \beta, \gamma = 0)$	$(\alpha = \gamma, \beta = 0)$
a	$\left(\frac{3}{2}\right)^{\frac{11}{12}} e^{\frac{\alpha x}{4}}$	$e^{\frac{\alpha x}{4}}\left(1 + \frac{\alpha t}{2\sqrt{2}}\right)^{5/4}$
H	$\frac{11}{18}$	$\frac{5}{4}\left[1 + \frac{\alpha t}{2\sqrt{2}}\right]^{-1}$
q	$\frac{1}{11}$	$-\frac{1}{5}$
σ^2	$\frac{11}{162}$	$\frac{3}{8}\left[1 + \frac{\alpha t}{2\sqrt{2}}\right]^{-2}$
φ	Infinity	$-\frac{1}{\sqrt{2\alpha}}\left[1 + \frac{\alpha t}{2\sqrt{2}}\right]^{-4}$

In the first case, H, q and σ are constant, the scale factor a(x) would suggest that the metric components allow for a homogeneous expansion (as far as H and q are concerned), while still incorporating spatial variations through a(x). This could lead to a type of cosmological model where spatial inhomogeneities exist, but the overall expansion dynamics remain uniform. A small positive value of q suggests the influence of another component, like dark energy, is already present but not yet dominant. In the other case, the results are similar to case 2.

4.4. Case 4

Conversion of the metric into Bianchi type VI_h Curved space given by:

$$ds^2 = -dt^2 + [A(t)]^2 dx^2 + [B(t)]^2 e^{\alpha x} dy^2 + e^{\alpha h x}([C(t)]^2 dz^2 + [D(t)]^2 dm^2) \tag{38}$$

by assuming $\beta \to \alpha h, \gamma \to \alpha h$. The obtained physical quantities are:

$$a = e^{\frac{\alpha x(1+2h)}{8}}\left(\frac{1+3h}{1+2h} * \left(1 + \frac{\alpha t\sqrt{(1-h)(1+2h)}}{4}\right)\right)^{\frac{(5+11h)}{4(1+3h)}} \tag{39}$$

$$H = \frac{(1+2h)(5+11h)}{4(1+3h)\left[1+\frac{\alpha t\sqrt{(1-h)(1+2h)}}{4}\right]} \tag{40}$$

$$q = -1 + \frac{4(1+3h)}{(5+11h)} \tag{41}$$

$$\sigma^2 = \frac{(1+2h)^2(11h^2+10h+3)}{8(1+3h)^4\left[1+\frac{\alpha t\sqrt{(1-h)(1+2h)}}{4}\right]^2} \tag{42}$$

$$\phi = -\left[\frac{\left(\left(\frac{1+3h}{1+2h}\right)\left(1+\frac{\alpha t\sqrt{(1-h)(1+2h)}}{4}\right)\right)^{\frac{-4(1+2h)}{1+3h}}}{\alpha\sqrt{(1-h)(1+2h)}}\right] \tag{43}$$

4.5. Case 5

In this case, the fractional cosmological model by converting the Einstein field equations into equations with fractional derivatives has been discussed, and for this purpose, B has been assumed as:

$$B = B_0 t^k \tag{44}$$

Now, using the concept of Fractional derivatives [18], it is obtained that

$$\dot{B} = [\mathbb{D}_t^s B_0 t^k] = B_0 \frac{\Gamma(k+1)}{\Gamma(k+1-s)} t^{k-s} \tag{45}$$

$$\ddot{B} = \mathbb{D}_t^s[\mathbb{D}_t^s B_0 t^k] = B_0 \frac{\Gamma(k+1)}{\Gamma(k+1-2s)} t^{k-2s} \tag{46}$$

After using the above B and its derivatives in equations (13) to (15), the values of A, B, C and D are obtained as

$$A = \ell \left[\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4\ell^2 R}\right]^{\frac{1}{2}} t^s \tag{47}$$

$$B = \left[\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4\ell^2 R}\right]^{\frac{1}{2L}} t^{\frac{s}{L}} \tag{48}$$

$$C = \left[\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4\ell^2 R}\right]^{\frac{\eta}{2L}} t^{\frac{sn}{L}} \tag{49}$$

$$D = \left[\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4\ell^2 R}\right]^{\frac{\zeta}{2L}} t^{\frac{s\zeta}{L}} \tag{50}$$

where

$$R = \left\{\frac{(\eta-1)\Gamma(\mathrm{k}+1)}{\Gamma(\mathrm{k}+1-2\mathrm{s})} + [(\eta-1)(\eta+\zeta+2)] * \frac{\Gamma(\mathrm{k}+1))^2}{\Gamma(\mathrm{k}+1-\mathrm{s})^2}\right\} \tag{51}$$

k be any constant and s be a fractional parameter having values between 0 and 1. Using the above results, the metric (1) reduces to

$$ds^2 = -dt^2 + \ell^2 \left[\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4\ell^2 R}\right] t^{2s} dx^2 + \left[\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4\ell^2 R}\right]^{\frac{1}{L}} t^{\frac{2s}{L}} e^{\alpha x} dy^2 + \left[\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4\ell^2 R}\right]^{\frac{\eta}{L}} t^{\frac{2s\eta}{L}} e^{\beta x} dz^2 + \left[\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4\ell^2 R}\right]^{\frac{\zeta}{L}} t^{\frac{2s\zeta}{L}} e^{\gamma x} dm^2 \tag{52}$$

The obtained results for the new model as a function of fractional parameter s are as follows:

$$a = \ell^{\frac{1}{4}} \left[\left(\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4L^2 R}\right)^{\frac{1+L+\eta+\zeta}{8L}}\right] * t^{\frac{(1+L+\eta+\zeta)s}{4L}} * e^{\frac{(\alpha+\beta+\gamma)x}{2}} \tag{53}$$

$$H = \frac{s(1+L+\eta+\zeta)}{4Lt} \tag{54}$$

$$q = -\left[\frac{(1+\eta+\zeta-3L)}{s^2(1+L+\eta+\zeta)}\right] \tag{55}$$

$$\sigma^2 = \frac{5s^2(1+L^2+\eta^2+\zeta^2)+s^2(L+\eta+\zeta)+2s^2(L\eta+\eta\zeta+L\zeta)}{8L^2t^2} \tag{56}$$

$$\phi = \frac{KL}{-s(1+L+\eta+\zeta)+L}\left(\frac{(\beta-\alpha)(\alpha+\beta+\gamma)}{4L^2 R}\right)^{-(1+L+\eta+\zeta)} * t^{\frac{-(1+L+\eta+\zeta)s+L}{L}} \tag{57}$$

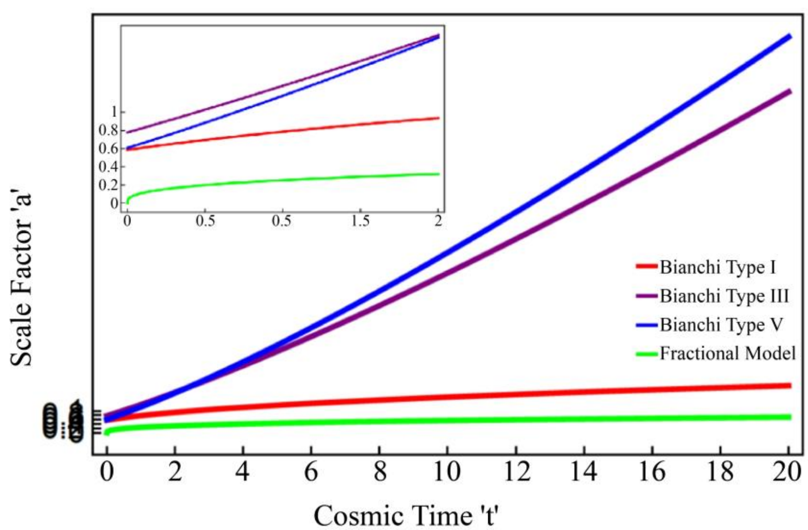

Fig. 1 Graph between Scale factor (a) with respect to cosmic time assuming $s = 0.5, k = 0.5, \alpha = 1.2, \gamma = -1, \beta = -1.2, \eta = 2, \zeta = 1, R = 1, c_1 = 1, c_2 = 1, l = 1, x = 1$, with the different values of fractional parameter s between 0 and 1.

This paper presents some key findings from research on the Bianchi type VI_0 cosmological model and demonstrates that, for the different choices of α, β and γ, the constructed model can be converted into a Bianchi type model of I, III and V kind, with a model of fractional derivatives. Also, it was found that all of the general theories of gravitation are satisfied, which can be seen in the graphs shown below.

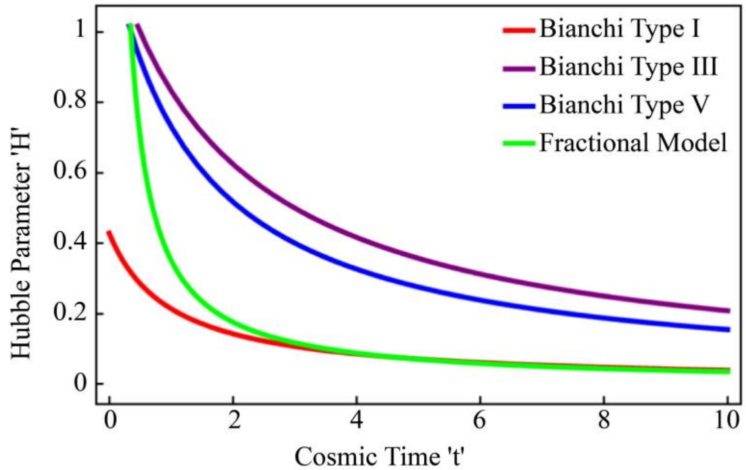

Fig. 2 Graph between Hubble parameter (H) and cosmic time (t)

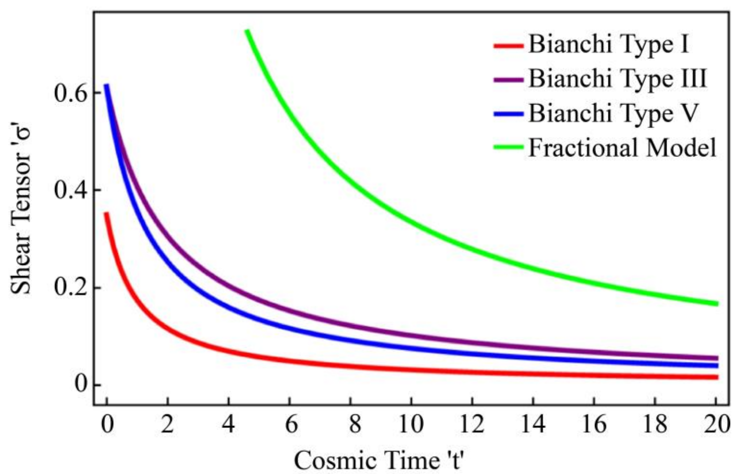

Fig. 3 Graph between Shear tensor (σ) and cosmic time (t)

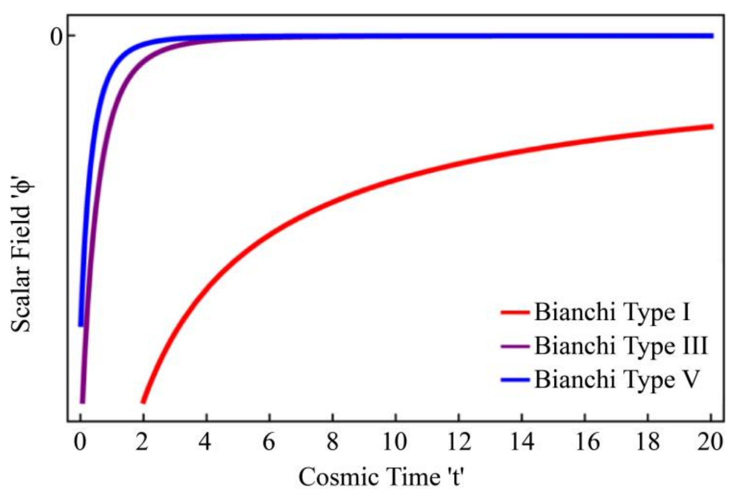

Fig. 4 Graph between Scalar field (φ) and cosmic time (t)

5. Conclusion

In the present study, it was found that the universe started in an anisotropic, possibly extra-dimensional phase and is evolving toward a more isotropic and lower-dimensional FRW-like state. The scalar field played a role in the early evolution but is becoming negligible, possibly stabilizing or freezing out. Indicates a specific phase in scalar field evolution, and the negative value of the scalar field is acceptable in the field theories if the potential remains positive. The Hubble parameter and shear decreasing indicate a transition toward a universe dominated by matter or radiation, rather than a continued inflationary phase. The negative value of the deceleration parameter in each case shows an accelerating expansion of the universe, which also agrees well with the results obtained earlier for a four-dimensional model. (PM Lambat et al, 2022; Shri Ram et al, 2022) [10, 16]. This model suggests that early higher-dimensional effects were significant, but these effects faded with time, leading to a universe that behaves more like standard 4D cosmology in the late times. In this study, constructed models are truly inhomogeneous.

Acknowledgments

Author 1 and Author 2 contributed equally to this work.

References

[1] K.S. Adhav et al., "Higher Dimensional Bianchi Type-V Cosmological Models with Perfect Fluid and Dark Energy," *International Journal of Theoretical Physics*, vol. 50, pp. 2573-2581, 2011. [CrossRef] [Google Scholar] [Publisher Link]

[2] A. Alkaoud et al., "A Cosmological Model Formulated in Lyra Geometry Incorporating Ricci Dark Energy and a Deceleration Parameter that Varies Linearly," *Modern Physics Letters A*, vol. 39, no. 15, pp. 2024. [CrossRef] [Google Scholar] [Publisher Link]

[3] Ernesto Barrientos, Sergio Mendoza, and Pablo Padilla, "Extending Friedmann Equations using Fractional Derivatives using a Last Step Modification Technique, The Case of a Matter Dominated Accelerated Expanding Universe," *Symmetry*, vol. 13, no. 2, pp. 1-18, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[4] Binaya Kumar Bishi, Pratik Vijay Lepse, and Aroonkumar Beesham, "Dark Energy Cosmological Models in Lyra Geometry for Bianchi-I Space Time," *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, vol. 93, pp. 645-659, 2023. [CrossRef] [Google Scholar] [Publisher Link]

[5] Gianlica Calcagni, and Antonio De Felice, "Dark Energy in Multifractional Spacetimes," *Physical Review D*, vol. 102, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[6] Jagat Daimary, and Rajshekhar Roy Baruah, "Five Dimensional FRW Radiating Cosmological Model in Presence of Bulk Viscous Fluid in Scalar Tensor Theory of Gravitation," *Trends in Sciences*, vol. 19, no. 20, pp. 1-10, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[7] Kalyani Desikan, "Cosmological Models in Lyra Geometry with Time-Varying Displacement Field," *African Review of Physics*, vol. 15, pp. 32-34, 2020. [Google Scholar] [Publisher Link]

[8] S. Jalalzadeh, E.W. Oliveira Costa, and P.V. Moniz, "De Sitter Fractional Quantum Cosmology," *Physical Review*, vol. 105, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[9] S.D. Katore, M.M Sancheti, and N.K. Shekate, "Kaluza-Klein Anisotropic Magnetized Dark Energy Cosmological Model in Brans-Dicke Theory of Gravitation," *Astrophysics*, vol. 57, no. 3, pp. 384-400, 2014. [CrossRef] [Google Scholar] [Publisher Link]

[10] P.M. Lambat, and A.M. Pund, "Bianchi type -VI_0 Inflationary Model in Lyra Geometry," *Journal of Scientific Research*, vol. 14, no. 2, pp. 435-442, 2022. [Google Scholar]

[11] D. Lorenz-Petzold, "Higher-Dimensional Bianchi Type VI_h Cosmologies," *Journal of Astrophysics and Astronomy*, vol. 6, pp. 131-135, 1985. [CrossRef] [Google Scholar] [Publisher Link]

[12] Gerhard Lyra, "On a Modification of Riemannian Geometry," *Mathematische Zeitschrift*, vol. 54, pp. 52-64, 1951. [CrossRef] [Google Scholar] [Publisher Link]

[13] M. Abdel-Megied, and E.A. Hegazy, "Bianchi Type VI Cosmological Model with Electromagnetic Field in Lyra Geometry," *Canadian Journal of Physics*, vol. 94, pp. 992-1000, 2016. [CrossRef] [Google Scholar] [Publisher Link]

[14] Bayron Micolta-Riascos et al., "Revisiting Fractional Cosmology," *Fractal and Fractional*, vol. 7, no. 2, pp. 1-32, 2023. [CrossRef] [Google Scholar] [Publisher Link]

[15] El-Nabulsi Ahmad Rami, "Fractional Unstable Euclidean Universe," *Electronic Journal of Theoretical Physics*, vol. 2, no. 8, pp. 1-11, 2005. [Google Scholar]

[16] Shri Ram, and M.K. Verma, "Dynamics of Bianchi Type-VI_0 Anisotropic Dark Energy Cosmological Model with Massive Scalar Field in Lyra Manifold," *Indian Journal of Physics*, vol. 96, pp. 1269-1275, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[17] D.R.K. Reddy, and P. Innaiah, "A Plane Symmetric Cosmological Model in Lyra Manifold," *Astrophysics and Space Science*, vol. 123, pp. 49-52, 1986. [CrossRef] [Google Scholar] [Publisher Link]

[18] Mark D. Roberts, "Fractional Derivative Cosmology," *SOP Transactions on Theoretical Physics*, no. 2, pp. 1-8, 2014. [CrossRef] [Google Scholar] [Publisher Link]

[19] Subrata Kumar Sahu, Samuel Ganiamo Ganebo, and Gebretsadik Gidey Weldemariam, "Kaluza-Klein Tilted Cosmological Model in Lyra Geometry," *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 42, pp. 1451-1457, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[20] D.K. Sen, and A.K. Dunn, "A Scalar-Tensor Theory of Gravitation in a Modified Riemannian Manifold," *Journal of Mathematical Physics*, vol. 12, pp. 578-586, 1971. [CrossRef] [Google Scholar] [Publisher Link]

[21] V.K. Shchigolev, "Cosmological Models with Fractional Derivatives and Fractional Action Functional," *Communications in Theoretical Physics*, vol. 56, no. 2, pp. 389-396, 2011. [CrossRef] [Google Scholar] [Publisher Link]

[22] V.K. Shchigolev, "Fractional-Order Derivatives in Cosmological Models of Accelerated Expansion," *Modern Physics Letters*, vol. 36, no. 14, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[23] G.P. Singh, R.V. Deshpande, and T. Singh, "Higher-Dimensional Cosmological Model with Variable Gravitational Constant and Bulk Viscosity in Lyra Geometry," *Pramana-Jour of* Physics, vol. 63, no. 5, pp. 937-945, 2004. [CrossRef] [Google Scholar] [Publisher Link]

[24] Kangujam Priyokumar Singh, Jiten Baro, and Asem Jotin Meitei, "Higher Dimensional Bianchi Type-I Cosmological Models with Massive String in General Relativity," *Frontiers in Astronomy and Space Sciences*, vol. 8, pp. 1-6, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[25] Diksha Trivedi, and A.K. Bhabor, "Higher Dimensional Bianchi Type-III String Cosmological Models with Dark Energy in Brans-Dicke Scalar-Tensor Theory of Gravitation," *New Astronomy*, vol. 89, 2021. [CrossRef] [Google Scholar] [Publisher Link]