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Original Article

Global Convergence Rates in Zero-Relaxation Limits for Non-Isentropic Euler-Maxwell Equations

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Abstract *– We analyze the non-isentropic Euler-Maxwell system under small relaxation times for magnetized plasmas and semiconductors. For near-equilibrium smooth periodic initial data, we establish uniform global existence of solutions relative to the relaxation parameter. Crucially, under slow time scaling, these solutions converge globally to the full energy-transport equations as the relaxation time vanishes. Our central innovation provides sharp error estimates between solutions of the non-isentropic system and its energy-transport limit, achieved through novel stream function techniques and enhanced energy methods. This work rigorously bridges these multiscale models while preserving their essential thermo-electromagnetic coupling.*

Keywords *- Convergence rates, Non-isentropic Euler-Maxwell equations, Energy-transport equations, Stream function.*

1. Introduction

The three-dimensional non-isentropic Euler-Maxwell equations describe the motions of particles in plasmas, which can be written as (see[1],[8])

$$\begin{cases} \partial_{t'}\rho+div(\rho u)=0, \\ \partial_{t'}(\rho u)+div(\rho u\otimes u)+\nabla(\rho\theta)=-\rho(-\nabla\phi+u\times B)-\frac{\rho u}{\tau_p}, \\ \partial_{t'}\theta+u\cdot\nabla\theta+\frac{2}{3}\theta\, div\, u-\frac{2}{3}\rho^{-1}\Delta\theta=-\frac{2|u|^2}{3}\left(\frac{1}{2\tau_w}-\frac{1}{\tau_p}\right)-\frac{1}{\tau_w}(\theta-1), \\ \partial_{t'}E-\nabla\times B=\rho u,\quad div\,E=b(x)-\rho,\quad E=-\nabla\phi, \\ \partial_{t'}B+\nabla\times E=0,\quad div\,B=0, \end{cases} \tag{1.1}$$

Where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ and $t'>0$ is the usual time. Here $\rho>0u$, the scaled density, the velocity field, the electron temperature, E B the electric field, and the magnetic field are the functions(t',x). The physical parameters τ_p τ_w are the momentum and energy relaxation times, respectively.

Obviously, (1.1) admits a steady-state

$$(\rho,u,\theta,E,B)=U_e\triangleq(\rho_s,0,1,E_e,B_e),$$

provided(ρ_s,E_e) satisfies

$$\begin{cases} \nabla\rho_s=-\rho_sE_e, \nabla\times B_e=0, \\ \text{div } E_e=b(x)-\rho_s, \nabla\times E_e=0, \\ \text{div } B_e=0. \end{cases}$$

With

$$\lim_{|x|\to\infty} \rho_s(x) = 1, \quad \lim_{|x|\to\infty} E_e = 0.$$

Where$B_e \in \mathbb{R}^3$is an arbitrary constant vector?

Notice that

$$t = \varepsilon t', \quad \tau_p = \varepsilon \in (0,1], \quad \tau_w = \frac{1}{\varepsilon}, \tag{1.2}$$

and

$$\rho^\varepsilon(t,x) = \rho(t',x), \quad u^\varepsilon(t,x) = \frac{1}{\varepsilon}u(t',x), \quad \theta^\varepsilon(t,x) = \theta(t',x),$$
$$E^\varepsilon(t,x) = E(t',x), \quad B^\varepsilon(t,x) = B(t',x).$$

Then (1.1) turns into

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t (\rho^\varepsilon u^\varepsilon) + \mathrm{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \frac{1}{\varepsilon^2}\nabla(\rho^\varepsilon\theta^\varepsilon) = \frac{1}{\varepsilon^2}(-\rho^\varepsilon(-\nabla\phi^\varepsilon + \varepsilon u^\varepsilon \times B^\varepsilon) - \rho^\varepsilon u^\varepsilon), \\ \partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla\theta^\varepsilon + \frac{2}{3}\theta^\varepsilon \mathrm{div} u^\varepsilon - \frac{2}{3\rho^\varepsilon}\Delta\theta^\varepsilon = \frac{|u^\varepsilon|^2}{3}(2-\varepsilon^2) - (\theta^\varepsilon - 1), \\ \varepsilon\partial_t B^\varepsilon + \nabla \times E^\varepsilon = 0, \quad \mathrm{div}E^\varepsilon = b(x) - \rho^\varepsilon, \\ \varepsilon\partial_t E^\varepsilon - \nabla \times B^\varepsilon = \varepsilon\rho^\varepsilon u^\varepsilon, \quad \mathrm{div}B^\varepsilon = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{T}^3, \end{cases} \tag{1.3}$$

With initial conditions

$$(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)_{t=0} = U_0^\varepsilon \triangleq \left(\rho_0^\varepsilon, \frac{u_0^\varepsilon}{\varepsilon}, \theta_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon\right), \tag{1.4}$$

which satisfy the compatibility conditions

$$\mathrm{div}E_0^\varepsilon = b(x) - \rho_0^\varepsilon, \quad \mathrm{div}B_0^\varepsilon = 0, \quad x \in \mathbb{T}^3.$$

Formally, as $\epsilon \to 0$if denoting the limits of $(\rho^\epsilon, u^\epsilon, \theta^\epsilon, \phi^\epsilon, B^\epsilon)$ as$(\bar\rho, \bar u, \bar\theta, \bar\phi, \bar B)$, in the formal limit, system (1.3) reduces to

$$\begin{cases} \partial_t \bar\rho + \mathrm{div}(\bar\rho\bar u) = 0, \\ \nabla(\bar\rho\bar\theta) = \bar\rho\nabla\bar\phi - \bar\rho\bar u, \\ \partial_t\bar\theta + \bar u \cdot \nabla\bar\theta + \frac{2}{3}\bar\theta\mathrm{div}\bar u - \frac{2}{3\bar\rho}\Delta\bar\theta = \frac{2|\bar u|^2}{3} - (\bar\theta - 1), \\ \nabla \times \bar E = 0, \quad \mathrm{div}\bar E = b(x) - \bar\rho, \\ \nabla \times \bar B = 0, \quad \mathrm{div}\bar B = 0, \quad (t,x) \in (0,+\infty) \times \mathbb{T}^3. \end{cases} \tag{1.5}$$

Apparently, it follows from the fifth Equation in (1.5) that $\bar B$ it is a constant vector. There is a potential function ϕ satisfying $\bar E = -\nabla\bar\phi$. Then, (1.5) implies the energy-transport model

$$\begin{cases} \partial_t\bar\rho - \Delta(\bar\rho\bar\theta) + \mathrm{div}(\bar\rho\nabla\phi) = 0, \\ \partial_t\bar\theta + \bar u \cdot \nabla\bar\theta + \frac{2}{3}\bar\theta\mathrm{div}\bar u - \frac{2}{3\bar\rho}\Delta\bar\theta = \frac{2|\bar u|^2}{3} - (\bar\theta - 1), \\ \Delta\bar\phi = \bar\rho - b(x), \end{cases} \tag{1.6}$$

and

$$\bar E = -\nabla\phi, \quad \bar u = \nabla(\bar\phi - \bar\theta) - \bar\theta\nabla\, ln\bar\rho. \tag{1.7}$$

For the uniqueness of $\bar\phi$satisfaction

$$m_{\bar{\phi}}(t) \stackrel{\text{def}}{=} \int_{\mathbb{T}^3} \bar{\phi}(t,x)dx = 0, \quad \forall t \geq 0. \tag{1.8}$$

Asymptotic analysis with small parameters in fundamental physical models is a major research area. Relaxation times are vital in fluid dynamics [7], and the zero-relaxation-time limit is central to asymptotic perturbation theory [9]. Wasiolek [13] provides proof of global existence and uniform energy-based convergence. Global convergence rates were subsequently derived by Li, Peng, and Zhao [11].

For the non-isentropic Euler-Maxwell system (1.3)-(1.4), assuming unit parameters, Feng, Wang, and Kawashima [5] established global existence of smooth solutions and computed temporal decay rates through uniform-in-time energy estimates, revealing that electron density and temperature decay to equilibrium at the same rate. While well-posedness for Euler-Maxwell systems has been significantly advanced [2][3][6], the parameter dependence of solutions remained unexamined.

Feng, Li, Mei, and Wang recently investigated the initial layer problem for the non-isentropic system [4]. In separate work [15], they proved the global convergence and convergence rates of solutions for the non-isentropic Euler-Maxwell equations as the relaxation time approaches zero. Furthermore, [15] derived error estimates between these solutions and solutions of the energy-transport equations, utilizing stream function techniques and improved energy methods. Extending the framework of [15] with modifications, we present our main results herein.

Theorem 1.1 (Global existence)Let $k \geq 3$be an integer. There exist constants $\varepsilon_0 > 0, \varpi_0 > 0andC > 0$, independent of ε any time, such that, for all$\varepsilon \in (0, \varepsilon_0]$, if

$$\|\rho_0^\varepsilon - \rho_s\|_k + \|\varepsilon u_0^\varepsilon\|_k + \|\theta_0^\varepsilon - 1\|_k + \|\phi_0^\varepsilon - \phi_s\|_k + \|B_0^\varepsilon - B_e\|_k \leq \varpi_0,$$

then the system (1.3)-(1.4) admits a unique global solution$(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$, satisfying

$$\rho^\varepsilon - \rho_s, \varepsilon u^\varepsilon, \theta^\varepsilon - 1, E^\varepsilon - E_e, B^\varepsilon - B_e \in C(\mathbb{R}^+, H^k) \cap C^1(\mathbb{R}^+, H^{k-1}).$$

Moreover, it holds,

$$\begin{aligned} &\|\rho^\varepsilon(t) - \rho_s\|_k^2 + \|\varepsilon u^\varepsilon(t)\|_k^2 + \|\theta^\varepsilon(t) - 1\|_k^2 + \|\phi^\varepsilon(t) - \phi_s\|_k^2 + \|B^\varepsilon(t) - B_e\|_k^2 \\ &\quad + \int_0^t (\|\rho^\varepsilon(\tau) - 1\|_k^2 + \|u^\varepsilon(\tau)\|_k^2 + \|\theta^\varepsilon(\tau) - 1\|_k^2)\, d\tau \\ &\leq C(\|\rho_0^\varepsilon - \rho_s\|_k^2 + \|\varepsilon u_0^\varepsilon\|_k^2 + \|\theta_0^\varepsilon - 1\|_{2k}^2 + \|E_0^\varepsilon\|_k^2 + \|B_0^\varepsilon - B_e\|_k^2), \quad \forall t \geq 0. \end{aligned} \tag{1.9}$$

Theorem 1.2 (Zero-relaxation limit)

Let $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$ be the global solution given by Theorem 1.1. Assume that there exist constants $\rho_0 > 0$ $\theta_0 > 0$, which are independent ofε, satisfying as$\varepsilon \to 0$,

$$\rho_0^\varepsilon \rightharpoonup \rho_0, \text{ weakly in } H^k,$$

and

$$\theta_0^\varepsilon \rightharpoonup \theta_0, \text{ weakly in } H^k.$$

Then there exist functions $\bar{\rho}, \bar{u}, \bar{\theta}, \bar{E}, \bar{B}$, with$\bar{\rho} - \rho_s, \bar{\theta} - 1, \bar{E}, \bar{B} - B_e \in L^\infty(\mathbb{R}^+, H^k)$ and$\bar{u} \in L^2(\mathbb{R}^+, H^k)$ such that, as$\varepsilon \to 0$it holds

$$(\rho^{\varepsilon}-\rho_{s},\theta^{\varepsilon}-1,E^{\varepsilon}-E_{e},B^{\varepsilon}-B_{e})\rightarrow(\bar{\rho}-\rho_{s},\bar{\theta}-1,\bar{E}-E_{e},\bar{B}-B_{e}), \quad (1.10)$$

weakly-* in $\left(L^{\infty}(\mathbb{R}^{+},H^{k})\right)^{3}$,

and

$$u^{\varepsilon}\rightharpoonup\bar{u}, \text{ weakly in } L^{2}(\mathbb{R}^{+};H^{k}). \quad (1.11)$$

Moreover, for any $T>0$ and any $k_{1}\in[0,k)$, it holds, as $\varepsilon\rightarrow 0$,

$$\rho^{\varepsilon}\rightarrow\bar{\rho},\quad \theta^{\varepsilon}\rightarrow\bar{\theta}, \text{ strongly in } C([0,T];H^{k_{1}}), \quad (1.12)$$

and $(\bar{\rho},\bar{\phi},\bar{\theta})$ is the unique global smooth solution of the following energy-transport model

$$\begin{cases}\partial_{t}\bar{\rho}-\Delta(\bar{\rho}\bar{\theta})+div(\bar{\rho}\nabla\bar{\phi})=0,\\ \partial_{t}\bar{\theta}+\left(\nabla(\bar{\phi}-\bar{\theta})-\bar{\theta}\nabla\, l\bar{n}\rho\right)\cdot\nabla\bar{\theta}+\frac{2}{3}\bar{\theta}\; div\left(\nabla(\bar{\phi}-\bar{\theta})-\bar{\theta}\nabla\, l\bar{n}\rho\right)-\frac{2}{3\bar{\rho}}\Delta\bar{\theta}=\frac{2|\bar{u}|^{2}}{3}-(\bar{\theta}-1),\\ \Delta\bar{\phi}=\bar{\rho}-b(x),\end{cases} \quad (1.13)$$

With the initial condition

$$(\bar{\rho},\bar{\theta})|_{t=0}=(n_{0},\theta_{0}). \quad (1.14)$$

Note that $\bar{\phi}_{\text{it}}$ is unique up to addition by a constant. Additionally, it holds

$$\bar{B}=B_{e},\quad \bar{E}=-\nabla\bar{\phi},\quad \bar{u}=\nabla(\bar{\phi}-\bar{\theta})-\bar{\theta}\nabla\, l\bar{n}\rho, \quad (1.15)$$

where B_{e} is a constant vector.

Theorem 1.3 (Convergence rates) Let $\boldsymbol{k\geq 3}$ be an integer; $(\boldsymbol{\rho^{\varepsilon},u^{\varepsilon},\theta^{\varepsilon},E^{\varepsilon},B^{\varepsilon}})$ and $(\boldsymbol{\bar{\rho},\bar{u},\bar{\theta},\bar{E},\bar{B}})$ respectively be the unique smooth solutions to (1.3)-(1.4) and (1.6)-(1.7). Denote $\boldsymbol{\bar{E}_{0}=\bar{E}(0,\cdot)}$. There exists a constant $\boldsymbol{\delta>0}$, which is independent of $\boldsymbol{\varepsilon}$, such that if

$$\|\rho_{0}^{\varepsilon}-\rho_{s}\|_{k}+\|u_{0}^{\varepsilon}\|_{k}+\|\theta_{0}^{\varepsilon}-1\|_{k}+\|E_{0}^{\varepsilon}\|_{k}+\|B_{0}^{\varepsilon}-B_{e}\|_{k}\leq\delta, \quad (1.16)$$

and for any given positive constants p, C_{1} independent of ε satisfying

$$\|\theta_{0}^{\varepsilon}-\bar{\theta}_{0}\|_{k-1}+\|u_{0}^{\varepsilon}-\bar{u}_{0}|_{k-2}+\|E_{0}^{\varepsilon}-\bar{E}_{0}\|_{k-1}+\|B_{0}^{\varepsilon}-B_{e}\|_{k-1}\leq C_{1}\varepsilon^{p}, \quad (1.17)$$

Then for $p_{1}=min\{p,1\}$ any $\varepsilon\in(0,1]$, there exists a constant C_{2} independent of ε, such that

$$\begin{aligned}&\sup_{t\in\mathbb{R}^{+}}(\|\rho^{\varepsilon}(t)-\bar{\rho}(t)\|_{k-2}^{2}+\varepsilon^{2}\|u^{\varepsilon}(t)-\bar{u}(t)\|_{k-2}^{2}+\|\theta^{\varepsilon}(t)-\bar{\theta}(t)\|_{k-1}^{2}\\&\qquad+\|E^{\varepsilon}(t)-\bar{E}(t)\|_{k-1}^{2}+\|B^{\varepsilon}(t)-B_{e}\|_{k-1}^{2})\\&+\int_{0}^{+\infty}(\|\rho^{\varepsilon}(\tau)-\bar{\rho}(\tau)\|_{k-1}^{2}+\|u^{\varepsilon}(\tau)-\bar{u}(\tau)\|_{k-2}^{2}\\&\qquad+\|\theta^{\varepsilon}(\tau)-\bar{\theta}(\tau)\|_{k}^{2}+\|E^{\varepsilon}(\tau)-\bar{E}(\tau)\|_{k-1}^{2}+\|\nabla B^{\varepsilon}(\tau)\|_{k-2}^{2})d\tau\\&\leq C_{2}\varepsilon^{2p_{1}}.\end{aligned} \quad (1.18)$$

2. Preliminaries

For later use, let us introduce some notations. For any integer k, we denote the usual spaces $H^{k}(\mathbb{T}^{3}),L^{2}(\mathbb{T}^{3})$ and $L^{\infty}(\mathbb{T}^{3})$, by H^{k},L^{2} and L^{∞}, respectively. Furthermore, we denote by $\|\cdot\|_{k}$ the usual norm of H^{k}, and by $\|\cdot\|$ and $\|\cdot\|_{\infty}$ the norms of L^{2} and L^{∞}, respectively. For a multi-index $\alpha=(\alpha_{1},\alpha_{2},\alpha_{3})\in\mathbb{N}^{3}$, we denote

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}} \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

At this point, we mention some relevant results: Moser-type calculus inequalities in Sobolev spaces and the local existence of smooth solutions to symmetrizable hyperbolic systems.

Lemma 2.1 (Moser-type inequality [12][10]). Let $\boldsymbol{k \geq 1}$ be an integer. Suppose $\boldsymbol{u} \in \boldsymbol{H^k}$ $\boldsymbol{\nabla u} \in \boldsymbol{L^\infty}$, and $\boldsymbol{v} \in \boldsymbol{H^{k-1}} \cap \boldsymbol{L^\infty}$. Then for every $\boldsymbol{\alpha} \in \mathbb{N}^3$ with $|\boldsymbol{\alpha}| \leq \boldsymbol{k}$, it holds $\boldsymbol{\partial^\alpha(uv) - u\partial^\alpha v \in L^2}$, and

$$\|\partial^\alpha(uv) - u\partial^\alpha v\| \leq C_k(\|\nabla u\|_\infty \|D^{k-1}v\| + \|D^k u\|\|v\|_\infty),$$

Where C_k denotes a constant only depending on k, and

$$\|D^k u\| = \sum_{|\alpha|=k} \|\partial^\alpha u\|.$$

In particular, when $k \geq 3$ the Sobolev inequality yields

$$\|\partial^\alpha(uv) - u\partial^\alpha v\| \leq C_k \|\nabla u\|_{k-1}\|v\|_{k-1}.$$

Lemma 2.2[Commutator Estimates,[14]]Let $\boldsymbol{l \geq 1}$ be an integer, and define the commutator

$$[\nabla^l, g]h = \nabla^l(gh) - g\nabla^l h.$$

If $p_0, p_1, p_2, p_3, p_4 \in [1, +\infty]$ satisfy

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

then

$$\|[\nabla^l, g]h\|_{L^{p_0}} \lesssim \|\nabla g\|_{L^{p_1}} \|\nabla^{l-1}h\|_{L^{p_2}} + \|\nabla^l g\|_{L^{p_3}} \|h\|_{L^{p_4}}.$$

In addition, for $l \geq 0$,

$$\|\nabla^l(gh)\|_{L^{p_0}} \lesssim \|g\|_{L^{p_1}} \|\nabla^l h\|_{L^{p_2}} + \|\nabla^l g\|_{L^{p_3}} \|h\|_{L^{p_4}}.$$

Lemma 2.3 (Local existence of smooth solutions; [9][12]).

Let $k \geq 3$ and $(\rho_0^\varepsilon - 1, u_0^\varepsilon, \theta_0^\varepsilon - 1, E_0^\varepsilon, B_0^\varepsilon - B_e) \in H^k$ with $\rho_0^\varepsilon \geq 1/2$ $\theta_0^\varepsilon \geq 1/2$. Then there exists $T_\varepsilon > 0$ such that the problem (1.3)-(1.4) has a unique smooth solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$ satisfying

$$(\rho^\varepsilon - \rho_s, \varepsilon u^\varepsilon, \theta^\varepsilon - 1, E^\varepsilon, B^\varepsilon - B_e) \in C([0, T_\varepsilon); H^k) \cap C^1([0, T_\varepsilon); H^{k-1}).$$

Throughout this paper, a basic assumption about the initial data is

$$(\rho_0^\varepsilon - \rho_s, u_0^\varepsilon, \theta_0^\varepsilon - 1, E_0^\varepsilon, B_0^\varepsilon - B_e) \in H^k, \quad \text{with} \quad \rho_0^\varepsilon \geq \frac{1}{2}, \quad \theta_0^\varepsilon \geq \frac{1}{2}, \quad \text{for } \varepsilon \in (0,1].$$

3. Global existence and convergence of solutions for the non-isentropic Euler-Maxwell system

3.1 Global existence of solutions

In this subsection, we demonstrate that the non-isentropic Euler-Maxwell system (1.3)-(1.4) admits global solutions uniformly ε.

To make the proof clearer, we define $\mathcal{E}(t)$ it $k \geq 3$ as follows:

$$\mathcal{E}_1(t) := \|(\rho - \rho_s, \varepsilon u, \theta - 1, \nabla(\phi - \phi_s))\|^2, \quad \mathcal{E}_2(t) := \|\nabla^k(\rho - \rho_s, \varepsilon u, \theta - 1)\|^2,$$
$$\mathcal{E}_3(t) := \|(\rho, u, \theta)\|^2_{M_k^{k-1}}, \quad \mathcal{E}_4(t) := \|(\rho, u, \theta)\|^2_{T_k}, \quad \mathcal{E}(t) := \sum_{i=1}^4 \mathcal{E}_i(t).$$

For convenience, we introduce the hybrid spaces M_k^n, T_m M_k whose norms are denoted as

$$\|f\|^2_{M_k^n} := \sum_{j=1}^n \|\nabla^{k-j}\partial_t^j f\|^2,$$

$$\|f\|^2_{T_m} := \sum_{j=1}^m \|\partial_t^j f\|^2,$$

and

$$\|f\|^2_{M_k} := \|f\|^2 + \|\nabla^k f\|^2 + \|f\|^2_{M_k^{k-1}} + \|f\|^2_{T_k}.$$

Lemma 3.1 for all $\boldsymbol{\varepsilon} \in (\mathbf{0}, \boldsymbol{\varepsilon_0}]$,

The main purpose of this section is to derive a key prior estimate of (ρ, u, θ, ϕ), which is independent of time t . We will always assume $\delta < 1$ in this section. For a given constant $\varepsilon_0 > 0$, assume the initial data satisfy.

$$\mathcal{E}(0) \le \varepsilon_0.$$

Then there exist positive numbers $\bar{\rho}$ and $\underline{\rho}$ such that if (ρ, u, θ, ϕ) is a smooth solution of problem (1.3)-(1.4) satisfying.

$$\mathcal{E}(t) \le 2\delta,$$

The following estimate is valid.

$$\mathcal{E}(t) + \int_0^t \mathcal{E}(\tau)\tau \le \mathcal{E}(0).$$

Proof. Similarly to Proposition 2.1 of [16], we can prove Lemma 3.1. We omit it for the sake of simplicity.

Lemma 3.2 Let $\boldsymbol{G} = \boldsymbol{B} - \boldsymbol{B_e}$ it hold

$$\frac{d}{dt}(\|\partial^\alpha E\|^2 + \|\partial^\alpha G\|^2) \le \delta\|\partial^\alpha(u, E - E_e)\|^2.$$

Proof. Start with the fifth Equation in system (1.3):

$$\partial_t(E - E_e) - \frac{1}{\varepsilon} curl(G) = \rho u.$$

Apply ∂^α and take the inner product with $\partial^\alpha(E - E_e)$:

$$\frac{d}{dt}\|\partial^\alpha(E - E_e)\|^2 - \frac{2}{\varepsilon}\langle curl(\partial^\alpha G), \partial^\alpha(E - E_e)\rangle = 2\langle\partial^\alpha(nu), \partial^\alpha(E - E_e)\rangle.$$

Next, take the fourth Equation in (1.3):

$$\partial_t G + \frac{1}{\varepsilon} curl(E - E_e) = 0.$$

Apply ∂^α and take the inner product with $\partial^\alpha G$:

$$\frac{d}{dt}\|\partial^\alpha G\|^2+\frac{2}{\varepsilon}\langle curl(\partial^\alpha(E-E_e)),\partial^\alpha G\rangle=0.$$

By using the vector identity $\boldsymbol{a}\cdot curl(\boldsymbol{b})+\boldsymbol{b}\cdot curl(\boldsymbol{a})=div(\boldsymbol{a}\times\boldsymbol{b})$:

$$\begin{aligned}\frac{d}{dt}(\|\partial^\alpha E\|^2+\|\partial^\alpha G\|^2)&=2\langle\partial^\alpha(\rho u),\partial^\alpha(E-E_e)\rangle\le\delta\|\partial^\alpha(u,E-E_e)\|^2\\&\le\delta\|\partial^\alpha(u,E-E_e)\|^2.\end{aligned}$$

Proof of Theorem 1.1. Theorem 1 follows by combining Lemma 3.1 and Lemma 3.2.

3.2Convergence of the solution as $\varepsilon\to 0$

This subsection focuses on analyzing the global temporal convergence from system (1.3)-(1.4) to system (1.13)-(1.14).

Proof of Theorem 1.2. Turning to the non-isentropic Euler-Maxwell system (1.3), this subsection focuses on establishing higher-order derivative estimates. To facilitate this derivation, we first reformulate (1.3)as:

$$\begin{cases}\partial_t\rho^\varepsilon=-\rho^\varepsilon\mathrm{div}u^\varepsilon-u^\varepsilon\cdot\nabla\rho^\varepsilon,\\ u_t^\varepsilon+u^\varepsilon\cdot\nabla u^\varepsilon+\frac{1}{\varepsilon^2}\nabla(\theta^\varepsilon-1)+\frac{1}{\varepsilon^2}u^\varepsilon+\frac{1}{\varepsilon^2}\nabla\rho_s(\rho_s-\rho^\varepsilon)(\rho^\varepsilon\rho_s)^{-1}\\ +\frac{1}{\varepsilon^2}(\theta^\varepsilon-1)(\rho^\varepsilon)^{-1}\nabla\rho^\varepsilon+\frac{1}{\varepsilon^2}(\rho^\varepsilon)^{-1}\nabla(\rho^\varepsilon-\rho_s)=\frac{1}{\varepsilon^2}\nabla(\phi^\varepsilon-\phi_s)-\frac{1}{\varepsilon^2}\varepsilon u\times B,\\ \theta_t^\varepsilon+u^\varepsilon\cdot\nabla\theta^\varepsilon+\frac{2}{3}\theta^\varepsilon\mathrm{div}u^\varepsilon-\frac{2}{3}(\rho^\varepsilon)^{-1}\Delta\theta^\varepsilon=\frac{2-\varepsilon^2}{3}(u^\varepsilon)^2-(\theta^\varepsilon-1),\\ \Delta(\phi^\varepsilon-\phi_s)=\rho^\varepsilon-\rho_s.\end{cases}\qquad(3.1)$$

Formally, in the limit $\epsilon\to 0$, if we denote the limiting functions of $(\rho^\epsilon,u^\epsilon,\theta^\epsilon,\phi^\epsilon,B^\epsilon)$ by $(\bar\rho,\bar u,\bar\theta,\bar\phi,\bar B)$, the asymptotic equations derived from (3.1) reduce to

$$\begin{cases}\partial_t\bar\rho=-\bar\rho\mathrm{div}\bar u-\bar u\cdot\nabla\bar\rho,\\ \nabla(\bar\theta-1)+\bar u+\nabla\rho_s(\rho_s-\bar\rho)(\bar\rho\rho_s)^{-1}+(\bar\theta-1)(\bar\rho)^{-1}\nabla\bar\rho\\ +(\bar\rho)^{-1}\nabla(\bar\rho-\rho_s)=\nabla(\bar\phi-\phi_s),\\ \bar\theta_t+\bar u\cdot\nabla\bar\theta+\frac{2}{3}\bar\theta\mathrm{div}\bar u-\frac{2}{3}(\bar\rho)^{-1}\Delta\bar\theta=\frac{2}{3}\bar u^2-(\bar\theta-1),\\ \Delta(\bar\phi-\phi_s)=\bar\rho-\rho_s.\end{cases}\qquad(3.2)$$

And in particular, estimate (1.18) yields that

$$\sup_{t\ge0}(\|\rho^\varepsilon(t)-\rho_S\|_k^2+\|\theta^\varepsilon(t)-1\|_k^2+\|E^\varepsilon(t)\|_k^2+\|B^\varepsilon(t)-B_\varepsilon\|_k^2)+\int_0^{+\infty}\|u^\varepsilon(\tau)\|_k^2d\tau\le c\varpi_0^2.$$

Consequently, the sequences $(\rho^\varepsilon-\rho_s)_{\varepsilon>0}$, $(\theta^\varepsilon-1)_{\varepsilon>0}$, $(E^\varepsilon)_{\varepsilon>0}$ and $(B^\varepsilon-B_e)_{\varepsilon>0}$ are bounded in $L^\infty(\mathbb{R}_+;H^k)$, and $(u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(\mathbb{R}_+;H^k)$.

Moreover $\mathcal{D}'(\mathbb{R}_+\times\mathbb{K}^3)$, we have

$$\begin{gathered}\varepsilon^2(\partial_t u^\varepsilon+(u^\varepsilon\cdot\nabla)u^\varepsilon)+\varepsilon(u^\varepsilon\times B^\varepsilon)\to0,\quad\text{as}\quad\varepsilon\to0,\\ \varepsilon(\partial_t E^\varepsilon-\rho^\varepsilon u^\varepsilon)\to0,\quad\text{as}\quad\varepsilon\to0,\\ \varepsilon\partial_t B^\varepsilon\to0,\quad\text{as}\quad\varepsilon\to0,\end{gathered}$$

and there exist functions $\bar{\rho}-\rho_s$, $\bar{\theta}-1, \bar{E}$, $\bar{B}-B_e$ in $L^\infty(\mathbb{R}_+;H^k)$ and $\bar{u}$ in $L^2(\mathbb{R}_+;H^k)$ such that convergence (1.10)-(1.11) holds for a subsequence. Moreover, applying the first Equation in system (1.13), we establish that the collection $\{\partial_t\rho^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded $L^2(0,T;H^{k-1})$ for every $T>0$.

Consider $T>0$. The family $\{\rho^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T;H^k)$. Standard compactness arguments imply that for any $k_1<k$, this sequence is precompact in $C([0,T];H^{k_1})$. Consequently, there exists a subsequence converging strongly in this space as expressed in (1.12), which follows from the uniqueness of the limiting solution.

Consequently, the limit passage is valid for each nonlinear term in (1.3), leading to

Moreover $\mathcal{D}'(\mathbb{R}_+\times\mathbb{K}^3)$, we have

$$\begin{cases}\partial_t\bar{\rho}-\Delta(\bar{\rho}\bar{\theta})+div(\bar{\rho}\nabla\bar{\phi})=0,\\ \partial_t\bar{\theta}+\bar{u}\cdot\nabla\bar{\theta}+\frac{2}{3}\bar{\theta}\,di\bar{v}\,u-\frac{2}{3\bar{\rho}}\Delta\bar{\theta}=\frac{2|\bar{u}|^2}{3}-(\bar{\theta}-1),\\ \Delta\bar{\phi}=\bar{\rho}-b(x).\end{cases}$$

This immediately establishes the existence of a potential function $\bar{\phi}$ such that $\bar{E}=-\nabla\bar{\phi}$. We thus recover the energy-transport system (1.6).

Next, we examine the initial data for $\bar{\rho}$ $\bar{\theta}$. The uniform strong convergence in (1.12) over $t\in[0,T]$ implies:

$$\rho^\varepsilon(0,\cdot)\to\bar{\rho}(0,\cdot),\quad \theta^\varepsilon(0,\cdot)\to\bar{\theta}(0,\cdot)\quad \text{in } H^{k_1}\quad \text{as}\quad \varepsilon\to 0.$$

Recognizing that $\rho^\varepsilon|_{t=0}=\rho_0^\varepsilon$, the initial condition (1.14) follows from (1.12) and the uniqueness of limits.

Finally, standard theory guarantees that the energy-transport system(1.13) admits a unique solution when initiated with smooth data (1.15). This implies convergence of the full sequence $(\rho^\varepsilon,u^\varepsilon,\theta^\varepsilon,E^\varepsilon,B^\varepsilon)_{\varepsilon>0}$, which concludes the proof of Theorem 1.2.

4. Global convergence rate for system (1.3)-(1.4)

The analysis of error estimates relies on the established results concerning uniform global existence and global-in-time convergence for systems (1.3)-(1.4) towards (1.13)-(1.14). The rate of convergence for (1.3)-(1.4) is formally presented in Theorem 1.3. To begin, we review the concept of a stream function, essential for analyzing conservative equations.

$$\partial_t u+div\,v=0,$$

We call φ a stream function to this Equation if it is satisfied.

$$\partial_t\varphi=v,\quad div\,\varphi=-u.$$

Next, we seek an appropriate stream function for the non-isentropic Euler-Maxwell system (1.3)-(1.4). A central conservative equation is obtained by subtracting the first Equation in (1.5) from the first Equation in (1.3),

$$\partial_t(\rho^\varepsilon-\bar{\rho})+div(\rho^\varepsilon u^\varepsilon-\bar{\rho}\bar{u})=0.$$

Then the stream function φ satisfies.

$$\begin{cases}\partial_t \varphi = \rho^\varepsilon u^\varepsilon - \bar{\rho}\bar{u}, \\ div\, \varphi = -(\rho^\varepsilon - \bar{\rho}).\end{cases}$$

The deviation $\varphi = E^\varepsilon - \bar{E}$ in the electric field naturally serves as the stream function because

$$div(E^\varepsilon - \bar{E}) = -(\rho^\varepsilon - \bar{\rho}).$$

Nevertheless, the limiting system is only $div(\partial_t \bar{E})$ due to the loss of $\partial_t \bar{E}$ information during the $\varepsilon \to 0$ limiting process. Consequently, $\partial_t \varphi$differs from $\rho^\varepsilon u^\varepsilon - \bar{\rho}\bar{u}$and instead incorporates an additional divergence-free term Kas follows:

$$\partial_t \varphi = \rho^\varepsilon u^\varepsilon - \bar{\rho}\bar{u} + K.$$

Thus, $E^\varepsilon - \bar{E}$ the stream function of the modified conservative Equation below

$$\partial_t (\rho^\varepsilon - \bar{\rho}) + div(\rho^\varepsilon u^\varepsilon - \bar{\rho}\bar{u} + K) = 0.$$

We now initiate the analysis of error bounds. Consider$(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$ the unique classical solution to system (1.3)-(1.4), and$(\bar{\rho}, \bar{u}, \bar{\theta}, \bar{E})$ the unique solution to the energy-transport equations (1.6)-(1.7). With these solutions defined, we introduce.

$$(N^\varepsilon, \Xi^\varepsilon, \Theta^\varepsilon, F^\varepsilon, G^\varepsilon) = (n^\varepsilon - \bar{n}, u^\varepsilon - \bar{u}, \theta^\varepsilon - \bar{\theta}, E^\varepsilon - \bar{E}, B^\varepsilon - B_\varepsilon),$$

$$(n_\alpha, u_\alpha, \theta_\alpha, E_\alpha, B_\alpha) = (\partial^\alpha n^\varepsilon, \partial^\alpha u^\varepsilon, \partial^\alpha \theta^\varepsilon, \partial^\alpha E^\varepsilon, \partial^\alpha B^\varepsilon),$$

$$(\bar{n}_\alpha, \bar{u}_\alpha, \bar{\theta}_\alpha, \bar{E}_\alpha, \bar{B}_\alpha) = (\partial^\alpha \bar{n}, \partial^\alpha \bar{u}, \partial^\alpha \bar{\theta}, \partial^\alpha \bar{E}, \partial^\alpha \bar{B}),$$

and

$$(N_\alpha, \Xi_\alpha, \Theta_\alpha, F_\alpha, G_\alpha) = (\partial^\alpha N^\varepsilon, \partial^\alpha \Xi^\varepsilon, \partial^\alpha \Theta^\varepsilon, \partial^\alpha F^\varepsilon, \partial^\alpha G^\varepsilon).$$

Lemma 4.1. Assume $\|\bar{\rho}_0 - 1\|_k$they $\|\bar{\theta}_0 - 1\|_k$ are sufficiently small, then the solution $(\bar{\rho}, \bar{u}, \bar{\theta}, \bar{E})$ to the system (1.16)-(1.17) satisfies

$$\|\bar{\rho}(t) - 1\|_k^2 + \int_0^t \|\bar{\rho}(\tau) - 1\|_{k+1}^2 d\tau \le C\|(\bar{\rho}_0 - 1, \bar{\theta}_0 - 1)\|_k^2, \quad \forall t > 0, \tag{4.2}$$

$$\|\bar{u}(t)\|_{k-1}^2 + \|\partial_t \bar{u}(t)\|_{k-3}^2 + \int_0^t (\|\bar{u}(\tau)\|_k^2 + \|\partial_t \bar{u}(\tau)\|_{k-2}^2)\, d\tau \le C\|(\bar{\rho}_0 - 1, \bar{\theta}_0 - 1)\|_k^2, \quad \forall t > 0, \tag{4.3}$$

$$\|\bar{\theta}(t) - 1\|_k^2 + \int_0^t \|\bar{\theta}(\tau) - 1\|_k^2 d\tau \le C\|(\bar{\rho}_0 - 1, \bar{\theta}_0 - 1)\|_k^2, \quad \forall t > 0, \tag{4,4}$$

$$\|\bar{E}(t)\|_k^2 + \|\partial_t \bar{E}(t)\|_{k-1}^2 + \int_0^t (\|\bar{E}(\tau)\|_{k+1}^2 + \|\partial_t \bar{E}(\tau)\|_k^2)\, d\tau \le C\|(\bar{\rho}_0 - 1, \bar{\theta}_0 - 1)\|_k^2, \quad \forall t > 0. \tag{4.5}$$

Proof. Let$\alpha \in \mathbb{N}^3$ be a multi-index with$|\alpha| \le k$. According to (1.16), for any$t > 0$, we have

$$\begin{cases}\partial_t \bar{\rho} - \mathrm{div}(\nabla(\bar{\rho}\bar{\theta})) + \mathrm{div}(\bar{\rho}\nabla\bar{\phi}) = 0, \\ \Delta\bar{\phi} = \bar{\rho} - b(x), \quad m_{\bar{\phi}}(t) = 0, \end{cases} \tag{4.6}$$

From (3.2) and (4.6), we obtain

$$\begin{gathered}\partial_t \bar{\rho} - div[\bar{\rho}\nabla(\bar{\theta}-1)] - div[\nabla\rho_s(\rho_s - \bar{\rho})\rho_s^{-1}] - div[(\bar{\theta}-1)\nabla\bar{\rho}] \\ -div[\nabla(\bar{\rho}-\rho_s)] - div[\bar{\rho}\nabla(\bar{\phi}-\phi_s)] = 0,\end{gathered} \tag{4.7}$$

Applying ∂^α to (4.7) and taking the inner product with $\partial^\alpha(\bar{\rho}-\rho_s)$

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha(\bar{\rho}-\rho_s)\|^2 = \textstyle\sum_{i=1}^5 I_i. \tag{4.8}$$

Notice that

$I_1 = -\int \partial^\alpha [\bar{\rho}\nabla(\bar{\theta}-1)]\cdot\partial^\alpha\nabla(\hat{\rho}-\rho_s)$, $\qquad I_2 = -\int \partial^\alpha [(\bar{\theta}-1)\nabla\bar{\rho}]\cdot\partial^\alpha\nabla(\bar{\rho}-\rho_s)$,

$I_3 = -\int \partial^\alpha [\nabla\rho_s(\rho_s-\bar{\rho})\rho_s^{-1}]\cdot\partial^\alpha\nabla(\bar{\rho}-\rho_s)$, $\qquad I_4 = -\int \partial^\alpha [\nabla(\bar{\rho}-\rho_s)]\partial^\alpha\nabla(\bar{\rho}-\rho_s)$,

$I_5 = \int \partial^\alpha [\bar{\rho}\nabla(\bar{\phi}-\phi_s)]\cdot\partial^\alpha\nabla(\bar{\rho}-\rho_s)$.

Firstly, it follows from the inequality that

$$\begin{aligned} I_1 &\le \|\partial^\alpha\nabla(\bar{\rho}-\rho_s)\|(\|\partial^\alpha\bar{\rho}\|_{L^3}\|\nabla(\bar{\theta}-1)\|_{L^6} + \|\partial^\alpha\nabla(\bar{\theta}-1)\|\|\bar{\rho}\|_{L^\infty}) \\ &\le \delta\|\partial^\alpha\nabla(\bar{\rho}-\rho_s)\|^2 + \delta\|\partial^\alpha\nabla(\bar{\theta}-1), \Delta(\bar{\theta}-1)\|^2. \end{aligned} \tag{4.9}$$

Using integration by parts, we can obtain

$$\begin{aligned} I_2 &\le \|\partial^\alpha\nabla(\bar{\rho}-\rho_s)\|(\|\partial^\alpha(\bar{\theta}-1)\|\|\nabla\bar{\rho}\|_{L^\infty} + \|\bar{\theta}-1\|_{L^\infty}\|\partial^\alpha(\nabla\bar{\rho})\|) \\ &\le \delta\|\partial^\alpha\nabla(\bar{\rho}-\rho_s)\|^2 + \delta\|\partial^\alpha(\bar{\theta}-1), \Delta(\bar{\theta}-1)\|^2. \end{aligned} \tag{4.10}$$

By$H\backslash\ddot{o}lder$'s and Cauchy's inequalities and Riesz's theorem, one has

$$\begin{aligned} I_3 &= -\int \partial^\alpha [\nabla\rho_s(\rho_s-\bar{\rho})\rho_s^{-1}]\cdot\partial^\alpha\nabla(\bar{\rho}-\rho_s) \\ &= \int [\nabla\rho_s\rho_s^{-1}\partial^\alpha(\rho_s-\bar{\rho}) + [\partial^\alpha, \nabla\rho_s\rho_s^{-1}](\rho_s-\bar{\rho})]\cdot\partial^\alpha\nabla(\bar{\rho}-\rho_s) \\ &= \int \tfrac{1}{2}\nabla\rho_s\rho_s^{-1}\nabla[\partial^\alpha(\bar{\rho}-\rho_s)]^2 + \int [\partial^\alpha, \nabla\rho_s\rho_s^{-1}](\rho_s-\bar{\rho})\cdot\partial^\alpha\nabla(\bar{\rho}-\rho_s) \\ &\le -\int \nabla[\nabla\rho_s\rho_s^{-1}]|\partial^\alpha(\bar{\rho}-\rho_s)|^2 + \|\partial^\alpha\nabla(\bar{\rho}-\rho_s)\|\|[\partial^\alpha, \nabla\rho_s\rho_s^{-1}](\rho_s-\bar{\rho})\| \\ &\le C(\|\partial^\alpha(\bar{\rho}-\rho_s)\|^2 + \|\partial^\alpha\nabla(\bar{\rho}-\rho_s)\|^2 + \|\bar{\rho}-\rho_s\|_{H^2}^2) \end{aligned} \tag{4.11}$$

Similarly, we easily see that

$$I_4 = -\int \partial^\alpha [\nabla(\bar{\rho}-\rho_s)]\partial^\alpha\nabla(\bar{\rho}-\rho_s) = -\int |\partial^\alpha\nabla(\bar{\rho}-\rho_s)|^2 \tag{4.12}$$

By Riesz's theorem and Gagliardo-Nirenberg's inequality,

$$\begin{aligned} I_5 &= \int \partial^\alpha [\bar{\rho}\nabla(\bar{\phi}-\phi_s)]\cdot\partial^\alpha\nabla(\bar{\rho}-\rho_s) \\ &= -\int \partial^\alpha \,\mathrm{div}[\bar{\rho}(\bar{\phi}-\phi_s)]\cdot\partial^\alpha(\bar{\rho}-\rho_s) \\ &= -\int \partial^\alpha [\nabla\bar{\rho}\cdot\nabla(\bar{\phi}-\phi_s) + \bar{\rho}\Delta(\bar{\phi}-\phi_s)]\cdot\partial^\alpha(\bar{\rho}-\rho_s) \\ &= -\int \partial^\alpha [\nabla\bar{\rho}\cdot\nabla(\bar{\phi}-\phi_s)]\cdot\partial^\alpha(\bar{\rho}-\rho_s) - \int \partial^\alpha [\bar{\rho}\Delta(\bar{\phi}-\phi_s)]\cdot\partial^\alpha(\bar{\rho}-\rho_s). \end{aligned}$$

Let

$$I_{51}=-\int\partial^{\alpha}\ [\nabla\bar{\rho}\cdot\nabla(\bar{\phi}-\phi_s)]\cdot\partial^{\alpha}(\bar{\rho}-\rho_s),\qquad I_{52}=-\int\partial^{\alpha}\ [\bar{\rho}\Delta(\bar{\phi}-\phi_s)]\cdot\partial^{\alpha}(\bar{\rho}-\rho_s).$$

Notice that

$$\begin{aligned}I_{51}&\le\|\partial^{\alpha}(\bar{\rho}-\rho_s)\|(\|\partial^{\alpha}\nabla(\bar{\phi}-\phi_s)\|_{L^6}\|\nabla\bar{\rho}\|_{L^3}+\|\partial^{\alpha}\nabla\bar{\rho}\|_{L^3}\|\nabla(\bar{\phi}-\phi_s)\|_{L^6})\\&\le\delta\|\partial^{\alpha}(\bar{\rho}-\rho_s)\|^2+\delta\|\bar{\rho}-\rho_s\|^2\end{aligned}$$

and

$$\begin{aligned}I_{52}&=-\int\bar{\rho}\,|\partial^{\alpha}(\bar{\rho}-\rho_s)|^2-\int[\partial^{\alpha},\bar{\rho}]\Delta(\bar{\phi}-\phi_s)\cdot\partial^{\alpha}(\bar{\rho}-\rho_s)\\&=-\int\bar{\rho}\,|\partial^{\alpha}(\bar{\rho}-\rho_s)|^2+I_{52'}\end{aligned}$$

Similarly, one has

$$\begin{aligned}I_{52'}&\le(\|\nabla\bar{\rho}\|_{L^\infty}\|\partial^{\alpha-1}(\bar{\rho}-\rho_s)\|+\|\partial^{\alpha}\bar{\rho}\|\|\bar{\rho}-\rho_s\|_{L^\infty})\|\partial^{\alpha}(\bar{\rho}-\rho_s)\|\\&\le\delta\|\partial^{\alpha}(\bar{\rho}-\rho_s)\|^2+\delta\|\partial^{\alpha-1}(\bar{\rho}-\rho_s)\|^2+\delta\|\bar{\rho}-\rho_s\|_{H^2}^2\end{aligned}$$

Then we can obtain

$$I_5\le-\int\bar{\rho}\,|\partial^{\alpha}(\bar{\rho}-\rho_s)|^2+\delta\|\partial^{\alpha}(\bar{\rho}-\rho_s)\|^2+\delta\|\partial^{\alpha-1}(\bar{\rho}-\rho_s)\|^2+\delta\|\bar{\rho}-\rho_s\|_{H^2}^2.\tag{4.13}$$

Plugging (4.9)-(4.13) into (4.8), it follows

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^{\alpha}(\bar{\rho}-\rho_s)\|^2+\int\bar{\rho}\,|\partial^{\alpha}(\bar{\rho}-\rho_s)|^2+\|\partial^{\alpha}\nabla(\bar{\rho}-\rho_s)\|^2\\&\le\delta\|\partial^{\alpha+1}(\bar{\theta}-1)\|^2+\delta\|(\bar{\rho}-\rho_s,\bar{\theta}-1)\|_{H^2}^2.\end{aligned}$$

Then, the sum of the Equation for all $\alpha\in\mathbb{N}^3$ $|\alpha|\le k$ implies

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}-\rho_s\|_k^2+\|\bar{\rho}-\rho_s\|_k^2+P_1\|\nabla(\bar{\rho}-\rho_s)\|_k^2\le C\delta(\|\bar{\rho}-\rho_s\|_{k+1}^2+\|\bar{\theta}-1\|_{k+1}^2).\tag{4.14}$$

By noting that $\|\bar{\rho}\|_k$ and $\|\bar{\theta}-1\|_k$ being small enough, integrating the above inequality over（4.2）follows.

Applying ∇^k to $(1.3)_1-(1.3)_3$ and multiplying it by $\bar{\rho}^{-1}\bar{\theta}\nabla^k(\bar{\rho}-\rho_s)$ $\bar{\rho}\nabla^k\bar{u}$, and $\frac{3}{2}\bar{\rho}\bar{\theta}^{-1}\nabla^k(\bar{\theta}-1)$ in L^2, respectively, then

$$\frac{d}{dt}\int(\bar{\rho}^{-1}\bar{\theta}|\nabla^k(\bar{\rho}-\rho_s)|^2+\frac{3}{2}\bar{\rho}\bar{\theta}^{-1}|\nabla^k\bar{\theta}|^2)+\int(2\bar{\rho}|\nabla^k\bar{u}|^2+3\bar{\rho}\bar{\theta}^{-1}|\nabla^k\bar{\theta}|^2)=\sum_{i=1}^{8}K_i,\tag{4.15}$$

Where

$$K_1:=\int\bar{\rho}^{-1}\bar{\theta}_t|\nabla^k(\bar{\rho}-\rho_s)|^2-\bar{\rho}^{-2}\bar{\rho}_t\bar{\theta}|\nabla^k(\bar{\rho}-\rho_s)|^2+\bar{\rho}_t|\nabla^k\bar{u}|^2+\frac{3}{2}(\bar{\rho}_t\bar{\theta}^{-1}-\bar{\rho}\bar{\theta}^{-2}\bar{\theta}_t)|\nabla^k\bar{\theta}|^2,$$

$$K_2:=2\int\bar{\rho}\,\nabla^k\nabla(\bar{\phi}-\phi_s)\nabla^k\bar{u},\qquad K_3:=-2\int\bar{\rho}\,\nabla^k[(\bar{\rho}\rho_s)^{-1}\nabla\rho_s(\rho_s-\bar{\rho})]\nabla^k\bar{u},$$

$K_4 := -2\int \bar{\rho}^{-1}\bar{\theta}\nabla^k(\bar{u}\cdot\nabla\bar{\rho})\nabla^k(\bar{\rho}-\rho_s)$, $\quad K_5 := -3\int \bar{\rho}\,\bar{\theta}^{-1}\nabla^k(\bar{u}\cdot\nabla\bar{\theta})\nabla^k\bar{\theta}$,

$K_6 := 2\int \bar{\rho}\,\bar{\theta}^{-1}\nabla^k(\bar{\rho}^{-1}\Delta\bar{\theta}+|\bar{u}|^2)\nabla^k\bar{\theta}$,

$K_7 := -2\int \bar{\rho}^{-1}\bar{\theta}\nabla^k(\bar{\rho}\mathrm{div}\bar{u})\nabla^k(\bar{\rho}-\rho_s)+\bar{\rho}\nabla^k[\bar{\rho}^{-1}\nabla(\bar{\rho}-\rho_s)]\nabla^k\bar{u}+\bar{\rho}\nabla^k[\bar{\rho}^{-1}\nabla\bar{\rho}(\bar{\theta}-1)]\nabla^k\bar{u}$,

$K_8 := -2\int \bar{\rho}\,\bar{\theta}^{-1}\nabla^k\bar{\theta}\nabla^k(\bar{\theta}\mathrm{div}\bar{u})+\bar{\rho}\nabla^k\nabla\bar{\theta}\nabla^k\bar{u}$.

We can obtain

$K_1 \le \delta\|\partial^\alpha(\bar{\theta}-1,\bar{\rho}-\rho_s)\|^2$, $\quad K_2 \le \delta\|\partial^\alpha(\bar{u},\nabla(\bar{\phi}-\phi_s)\|^2$,

$K_3 \le \delta\|\partial^\alpha(\bar{u},\bar{\rho}-\rho_s)\|^2$, $\quad K_4 \le \delta\|\partial^\alpha(\bar{u},\bar{\rho}-\rho_s)\|^2$,

$K_5 \le \delta\|\partial^\alpha(\bar{u},\bar{\theta}-1,\nabla\bar{\theta})\|^2$, $\quad K_6 \le -2\int \bar{\theta}^{-1}|\partial^\alpha\nabla\bar{\theta}|^2+\delta\|\partial^\alpha(\bar{\theta}-1,\bar{u})\|^2$,

$K_7 \le \delta\|\partial^\alpha(\bar{u},\bar{\theta}-1,\bar{\rho}-\rho_s)\|^2$, $\quad K_8 \le \delta\|\partial^\alpha(\bar{\theta}-1,\bar{u})\|^2$.

Plugging $K_i, i=1,2,3,4,5,6,7,8$ into (4.15),

$$\frac{d}{dt}\int(\bar{\rho}^{-1}\bar{\theta}|\nabla^k(\bar{\rho}-\rho_s)|^2+\frac{3}{2}\bar{\rho}\bar{\theta}^{-1}|\nabla^k\bar{\theta}|^2)+\int(2\bar{\rho}|\nabla^k\bar{u}|^2+3\bar{\rho}\bar{\theta}^{-1}|\nabla^k\bar{\theta}|^2+2\bar{\theta}^{-1}|\partial^\alpha\nabla\bar{\theta}|^2)$$

$$\le \delta\|\partial^\alpha(\bar{\rho}-\rho_s)\|^2. \tag{4.16}$$

Then, the sum of (4.16) for all $\alpha\in\mathbb{N}^3$ $|\alpha|\le k$ implies

$$\frac{d}{dt}\|(\bar{\rho}-\rho_s,\bar{\theta}-1)\|_k^2+\|\bar{u}\|_k^2+\|(\bar{\rho}-\rho_s,\bar{\theta}-1)\|_{k+1}^2\le 0 \tag{4.17}$$

From (4.14) to (4.17)

$$\|(\bar{\rho}-\rho_s,\bar{\theta}-1)\|_k^2+\int_o^T\|\bar{u}\|_k^2+\|(\bar{\rho}-\rho_s,\bar{\theta}-1)\|_{k+1}^2\le C\|(\bar{\rho}_0-\rho_s,\bar{\theta}_0-1)\|_k^2.$$

We easily get by (3.2)

$$\|\nabla\partial^\alpha(\bar{\phi}-\phi_s)\|\le C\|\partial^\alpha(\nabla\bar{\rho}-\rho_s)\|.$$

Then we can prove (4.4) by (4.18) easily.

In order to get them easily, we transform (1.15) into the following form:

$$\bar{u}=\nabla\bar{\phi}-\nabla\bar{\theta}-\bar{\theta}ln\bar{\rho}=\nabla(\bar{\phi}-\phi_s)-(\bar{\theta}-1)\nabla ln\bar{\rho}-\nabla(ln\bar{\rho}-ln\overline{\rho_s}).$$

Then

$$\|\partial^\beta\bar{u}\|\le C\|\bar{\rho}_0-\rho_s\|_{\beta+1},\qquad \beta\le k-1. \tag{4.19}$$

$$\|\bar{u}(t)\|_{k-1}^2 + \int_0^t \|\bar{u}(\tau)\|_k^2 d\tau \le C\|(\bar{\rho}_0 - \rho_s, \bar{\theta}_0 - 1)\|_k^2, \quad \forall t > 0. \tag{4.20}$$

For a multi-index γ $|\gamma| \le k-3$, applying $\partial_t \partial^\gamma$ to the Equation $\bar{u}$ in (1.7) implies

$$\partial_t \partial^\gamma \bar{u} = -\partial^\gamma \left(\frac{\nabla\bar{\rho}}{\bar{\rho}}\partial_t \bar{\theta} - \frac{\theta}{(\bar{\rho})^2}\nabla\bar{\rho}\partial_t\bar{\rho} + \frac{\theta}{\bar{\rho}}\nabla\partial_t\bar{\rho} + \nabla\partial_t\bar{\theta} - \partial_t\nabla\bar{\phi}\right).$$

In view of $\partial_t\bar{\rho} = -\mathrm{div}(\bar{\rho}\bar{u})$ (4.2) and (4.19), we have

$$\|\partial_t \partial^\gamma \bar{u}\| \le C\|\bar{u}\|_{|\gamma|+2} + \|\partial_t \nabla\bar{\phi}\|_{|\gamma|}.$$

This, together with (4.20), yields (4.3). From (1.7)-(1.8), we have

$$\mathrm{div}\,\partial_t\nabla\bar{\phi} = \partial_t\bar{\rho} = -\mathrm{div}(\bar{\rho}\bar{v}),$$

which implies that there exists a function $\bar{H}$ such that

$$\begin{cases}\partial_t\nabla\bar{\phi} + \bar{\rho}\bar{v} = \nabla\times\bar{H},\\ \mathrm{div}\bar{H} = 0.\end{cases} \tag{4.12}$$

In order to make $\bar{H}$ it uniquely determined, we add a restriction condition

$$m_{\bar{H}}(t) = \int_{\mathbb{T}^3} \bar{H}\,(t,x)dx = 0, \quad \forall t \ge 0. \tag{4.22}$$

The estimate $\bar{H}$ is as follows.

Lemma 4.2 The solution $\bar{H}$ to (4.21)-(4.22) satisfies

$$\bar{H} \in L^\infty(\mathbb{R}^+; H^k) \quad \text{and} \quad \partial_t\bar{H} \in L^2(\mathbb{R}^+; H^{k-1}). \tag{4.23}$$

Proof. Applying the curl operator to the initial Equation in (4.21) yields

$$\Delta\bar{H} = -\nabla\times(\bar{\rho}\bar{v}), \quad \forall t \ge 0,$$

According to Lemma 3.1, this yields $\nabla\bar{H} \in L^\infty(\mathbb{R}^+; H^{k-1})$. Moreover, application of (1.6) and (1.8) readily shows that.

$$\partial_t(\bar{\rho}\bar{v}) = (\partial_t\bar{\rho})\bar{v} + \bar{\rho}\partial_t\bar{v} = -\mathrm{div}(\bar{\rho}\bar{v})\bar{v} + \bar{\rho}\partial_t\bar{v},$$

Which implies that $\partial_t(\bar{\rho}\bar{v}) \in L^2(\mathbb{R}^+; H^{k-2})$.Differentiating both sides of Equation (4.23) with respect to time results in

$$\Delta\partial_t\bar{H} = -\nabla\times\partial_t(\bar{\rho}\bar{v}).$$

Consider a multi-index $\beta \in \mathbb{N}^3$ satisfying $|\beta| \le k-2$. We apply the partial derivative operator ∂^β to both sides of the Equation above and then take the inner product of the resulting expression with

$$\|\nabla\partial^\beta\partial_t\bar{H}\|^2 \le C|\langle\partial^\beta\partial_t\bar{H}, \nabla\times\partial_t\partial^\beta(\bar{\rho}\bar{v})\rangle| \le C|\langle\nabla\times\partial^\beta\partial_t\bar{H}, \partial_t\partial^\beta(\bar{\rho}\bar{v})\rangle|$$

$$\le \frac{1}{2}\|\nabla\partial^\beta\partial_t\bar{H}\|^2 + C\|\partial_t\partial^\beta(\bar{\rho}\bar{v})\|^2,$$

We have utilized Young's inequality and the solenoidality condition $\operatorname{div}(\partial_t \bar{H}) = 0$. Consequently, we deduce that$\partial_t \nabla \bar{H} \in L^2(\mathbb{R}^+; H^{k-2})$. The proof is then concluded by applying Poincaré's inequality to both $\bar{H}$ and$\partial_t \bar{H}$.

Now we have the following uniform estimate ε.

Lemma 4.3. It holds

$$\int_0^t (\|E^\varepsilon(\tau)\|_{k-1}^2 + \|\nabla G^\varepsilon(\tau)\|_{k-2}^2)\, d\tau \leq C\|U_0^\varepsilon - U_\varepsilon\|_k^2, \quad \forall t \geq 0,$$

Where $G^\varepsilon = B^\varepsilon - B_e$.

Next, by taking the difference of $(1.5)_1$ (1.3), we have

$$\partial_t N^\varepsilon + div(n^\varepsilon u^\varepsilon - \bar{n}\bar{u}) = 0.$$

It follows from (1.3), (1.6)-(1.7) and (4.21) that

$$div F^\varepsilon = -N^\varepsilon, \tag{4.24}$$

and

$$\partial_t F = \partial_t E - \partial_t \bar{E} = (\rho^\varepsilon u^\varepsilon - \bar{\rho}\bar{u}) + \frac{1}{\varepsilon}\nabla \times G^\varepsilon + \nabla \times \bar{H}. \tag{4.25}$$

According to the concept of stream function at the beginning of this section, F^εit is a stream function of (4.1) with$K = \varepsilon^{-1}\nabla \times G^\varepsilon + \nabla \times \bar{H}$.

In the following, we prove the estimates for the error function $(N^\varepsilon, \Xi^\varepsilon, \Theta^\varepsilon, F^\varepsilon, G^\varepsilon)$.

Lemma 4.4. It holds

$$\sup_{t\in\mathbb{R}^+} (\|N^\varepsilon(t)\|_{k-2}^2 + \|\Theta^\varepsilon(t)\|_{k-2}^2 + \|F^\varepsilon(t)\|_{k-1}^2 + \|G^\varepsilon(t)\|_{k-1}^2)$$

$$+ \int_0^\infty (\|N^\varepsilon(\tau)\|_{k-1}^2 + \|\Theta^\varepsilon(\tau)\|_{k-1}^2 + \|F^\varepsilon(\tau)\|_{k-1}^2 + \|\nabla G^\varepsilon(\tau)\|_{k-2}^2)\, d\tau \leq C\varepsilon^{2p_1} + \delta \int_0^\infty \|\Xi\|_{k-2}^2,$$

Which p_1 is defined in Theorem 1.3.

Proof. We use some lemmas to complete the proof of Lemma 4.4. Firstly$T > 0$, we take the difference of $(1.7)_2$ and (1.3) to get

$$\varepsilon^2\big(\partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon)\big) + \nabla(\rho^\varepsilon \theta^\varepsilon) - \nabla(\bar{\rho}\bar{\theta})$$
$$= -(\rho^\varepsilon E^\varepsilon - \bar{\rho}\bar{E}) - \varepsilon\rho^\varepsilon(u^\varepsilon \times B^\varepsilon) - (\rho^\varepsilon u^\varepsilon - \bar{\rho}\bar{u}).$$

For $\alpha \in \mathbb{N}^3$with$|\alpha| \leq k-1$, by applying ∂^α to the above Equation, multiplying the resulting Equation withF_α, integrating it with respect to x , and$t$, we have

$$
\begin{aligned}
0 &= \int_0^T \varepsilon^2 \left(F_\alpha, \partial^\alpha \partial_t(\rho^\varepsilon u^\varepsilon)\right) dt + \int_0^T \left(F_\alpha, \partial^\alpha(\rho^\varepsilon u^\varepsilon - \bar{\rho}\bar{u})\right) dt \\
&+ \int_0^T \left(F_\alpha, \partial^\alpha \left(\nabla(\rho^\varepsilon \theta^\varepsilon) - \nabla(\bar{\rho}\bar{\theta})\right)\right) dt + \int_0^T \left(F_\alpha, \partial^\alpha(\rho^\varepsilon F^\varepsilon)\right) dt \\
&+ \int_0^T \left(F_\alpha, \partial^\alpha(P^\varepsilon E)\right) dt + \int_0^T \left(F_\alpha, \varepsilon^2 \partial^\alpha\left(\operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon)\right) + \varepsilon \partial^\alpha(\rho^\varepsilon u^\varepsilon \times B^\varepsilon)\right) dt \\
&\triangleq \sum_{j=1}^6 \mathcal{R}_j,
\end{aligned}
\tag{4.26}
$$

with the corresponding $\mathcal{R}_j$ index by $j = 1, \ldots, 6$. Here, p_1 and δ referenced in Lemmas 4.7--4.8 originate from Theorem 1.3, while $\mu > 0$ represents a sufficiently small positive parameter.

Now, we begin to study the error estimates. Consider the unique smooth solution $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)$ to system(1.4), alongside the unique solution $(\bar{\rho}, \bar{u}, \bar{\theta}, \bar{E})$ to the energy-transport model described by (1.6)-(1.7).

We prove the following result first.

Lemma 4.5 (Estimate of $\mathcal{R}_1$, see Lemma 2.5 in [11]) For all $|\alpha| \le k-1$, it holds

$$
|\mathcal{R}_1| \le C\varepsilon^{2p_1} + \frac{1}{4}\|F_\alpha(T)\|^2 + \mu \int_0^T \|\nabla G^\varepsilon(\tau)\|_{k-3}^2 d\tau.
$$

Lemma 4.6 (Estimate of $\mathcal{R}_2$) For all $|\alpha| \le k-1$, it holds

$$
\mathcal{R}_2 \ge \frac{1}{2}\|F_\alpha(T)\|^2 + \frac{1}{4}\|G_\alpha(T)\|^2 - C\varepsilon^{2p_1} - C\delta \sup_{0\le t\le T} \|G^\varepsilon(t)\|_{k-1}^2 - C\mu \int_0^T \|\nabla G^\varepsilon(t)\|_{k-2}^2 dt. \tag{4.27}
$$

Due to (4.25)

$$
(\rho^\varepsilon u^\varepsilon - \bar{\rho}\bar{u}) = \partial_t F - \frac{1}{\varepsilon}\nabla \times G - \nabla \times \bar{H}.
$$

We can obtain

$$
\begin{aligned}
R_2 &= \int_0^T \frac{1}{2}\frac{d}{dt}\|F_\alpha(t)\|^2 - < \nabla \times F_\alpha(t), \frac{1}{\delta}G_\alpha + \bar{H}_\alpha > dt \\
&= \int_0^T \frac{1}{2}\frac{d}{dt}\|F_\alpha(t)\|^2 + < \delta\partial_t G_\alpha, \frac{1}{\delta}G_\alpha > dt \\
&+ \int_0^T \frac{d}{dt} < \delta G_\alpha, \partial^\alpha \bar{H} > - < \delta G_\alpha, \partial_t \partial^\alpha \bar{H} > dt,
\end{aligned}
$$

notice that

$$
\nabla \times F_\alpha = -\delta \partial_t G_\alpha.
$$

In a similar way to that in Lemma 4.5 of [15], we can prove (4.27). We omit it for the sake of simplicity. It is for the sake of simplicity.

Lemma 4.7 (Estimate of $\mathcal{R}_2$) For all $|\alpha| \le k-1$, there exists a constant $c_1 > 0$, such that.

$$
\mathcal{R}_3 \ge \int_0^T \left(c_1\|N_\alpha(\tau)\|^2 - \frac{c_1}{4}\|\Theta_\alpha(\tau)\|^2\right) d\tau - C\delta \int_0^T (\|N^\varepsilon(\tau)\|_{k-1}^2 + \|\Theta^\varepsilon(\tau)\|_{k-1}^2) d\tau,
$$

And

$$\mathcal{R}_3 + \delta \int_0^t \|\Xi\|_{k-2}^2 + C\varepsilon^{2p_1} \geq \|\Theta_\alpha(T)\|^2 + c_1 \int_0^T (\|N_\alpha(\tau)\|^2 + \|\Theta_\alpha(\tau)\|^2)\, d\tau - C\delta \int_0^T (\| N^\varepsilon(\tau)\|_{k-1}^2 + \|\Theta^\varepsilon(\tau)\|_{k-1}^2)\, d\tau. \tag{4.29}$$

Proof. By recalling and using (4.21), we have

$$\begin{aligned} R_3 &= \int_0^T < F_\alpha, \partial^\alpha \left(\nabla(\rho^\varepsilon\theta^\varepsilon) - \nabla(\bar{\rho}\bar{\theta})\right) > dt \\ &= -\int_0^T < divF_\alpha, \partial^\alpha(\rho^\varepsilon\theta^\varepsilon - \bar{\rho}\bar{\theta}) > dt \\ &= \int_0^T < N_\alpha, \partial^\alpha(\rho\Theta + \bar{\theta}N) > dt. \end{aligned}$$

notice that

$$< N_\alpha, \partial^\alpha(\rho\Theta + \bar{\theta}N) > = \int \rho\, N_\alpha \Theta_\alpha + [\partial^\alpha, \rho]\Theta N_\alpha dx + \int N_\alpha\, \bar{\theta} N_\alpha + [\partial^\alpha, \bar{\theta}] N N_\alpha dx$$

From (1.16)-(1.18), (4.2) and (4.4), it directly follows that $\inf_{\varepsilon,s}\{\tilde{\rho}^\varepsilon, \tilde{\theta}^\varepsilon\} > 0$ for δ sufficiently small, with$\varepsilon \in (0,1]$ $s \in [0,1]$. The continuity and monotonic properties of $\bar{p}$, and $\bar{\theta}$. therefore ensure $\exists$ $c_1^* > 0$ that.

$$\int N_\alpha\, \bar{\theta} N_\alpha \geq 4c_1^* \|N_\alpha\|^2.$$

Observe that the Cauchy-Schwarz and Young inequalities guarantee the existence of $(\tilde{c}_1' > 0)$ a satisfying

$$\langle N_\alpha, \Theta_\alpha \rangle \geq -\tilde{c}_1' \|N_\alpha\|^2 - \frac{\tilde{c}_1'}{4} \|\Theta_\alpha\|^2.$$

Denoting$c_1' = \tilde{c}_1' \max_{0\le s\le 1} \bar{\theta}$, then$\frac{c_1'}{4} = \frac{\tilde{c}_1'}{4} \max_{0\le s\le 1} \bar{\rho}$, additionally, we note$c_1' \le c_1 \le c_1^*$that we have

$$\langle N_\alpha, \bar{\theta} N_\alpha \rangle \geq 4c_1 \|N_\alpha\|^2, \tag{4.30}$$

$$\langle N_\alpha, \rho\Theta_\alpha \rangle \geq -c_1 \|N_\alpha\|^2 - \frac{c_1}{4} \|\Theta_\alpha\|^2, \tag{4.31}$$

$$\|[\partial^\alpha, \rho]\Theta\| \le \|\nabla\rho\|_{L^\infty} \|\partial^{\alpha-1}\Theta\| + \|\partial^\alpha \rho\|_{L^6} \|\Theta\|_{L^3} \le C\delta \|\Theta\|_{k-1},$$

and

$$\|[\partial^\alpha, \bar{\theta}]N\| \le \|\nabla\bar{\theta}\|_{L^\infty} \|\partial^{\alpha-1}N\| + \|\partial^\alpha \bar{\theta}\|_{L^6} \|N\|_{L^3} \le C\delta \|N\|_{k-1}.$$

then which, along with Young's inequality and the Cauchy-Schwarz inequality, implies

$$\int [\partial^\alpha, \bar{\theta}]\, N N_\alpha dx \geq -c_1 \|N_\alpha\|^2 - C\delta \|N^\varepsilon\|_{k-1}^2, \tag{4.32}$$

$$\int [\partial^\alpha, \rho]\, \Theta N_\alpha dx \geq -c_1 \|N_\alpha\|^2 - C\delta \|\Theta^\varepsilon\|_{k-1}^2. \tag{4.33}$$

Thus, the combination of equations (4.30)-(4.33) establishes (4.28).

Furthermore, subtracting Equation (2) in system (1.6) from Equation (3) in system (1.2) syields.

$$\partial_t \Theta^\varepsilon + u^\varepsilon \cdot \nabla\theta^\varepsilon - \bar{u}\cdot\nabla\bar{\theta} + \frac{2}{3}\theta^\varepsilon\, div\, u^\varepsilon - \frac{2}{3}\bar{\theta}\, d\bar{iv}\, u - \frac{2}{3}(\rho^\varepsilon)^{-1}\Delta\theta^\varepsilon + \frac{2}{3}(\bar{\rho})^{-1}\Delta\bar{\theta} = \frac{2-\varepsilon^2}{3}|u^\varepsilon|^2 - \frac{2|\bar{u}|^2}{3} - \Theta.$$

Applying ∂^α the above equation yields

$$\partial_t\partial^\alpha\Theta + \partial^\alpha(u^\varepsilon\cdot\nabla\theta^\varepsilon) - \partial^\alpha(\bar{u}\cdot\nabla\bar{\theta}) + \frac{2}{3}\partial^\alpha(\theta^\varepsilon \mathrm{div} u^\varepsilon) - \frac{2}{3}\partial^\alpha(\bar{\theta}\mathrm{div}\bar{u}) - \frac{2}{3}\partial^\alpha[(\rho^\varepsilon)^{-1}\Delta\theta^\varepsilon] + \frac{2}{3}\partial^\alpha[(\bar{\rho})^{-1}\Delta\bar{\theta}] = \frac{2-\varepsilon^2}{3}\partial^\alpha(|u^\varepsilon|^2) - \frac{2}{3}\partial^\alpha(|\bar{u}|^2) - \partial^\alpha\Theta,$$

Notice that

$$\partial^\alpha(u^\varepsilon\cdot\nabla\theta^\varepsilon) - \partial^\alpha(\bar{u}\cdot\nabla\bar{\theta}) = \partial^\alpha(u^\varepsilon\cdot\nabla\Theta) + \partial^\alpha(\Xi\cdot\nabla\bar{\theta}),$$

and

$$-\partial^\alpha[(\rho^\varepsilon)^{-1}\Delta\theta^\varepsilon] + \partial^\alpha[(\bar{\rho})^{-1}\Delta\bar{\theta}] = -\partial^\alpha[\bar{\rho}^{-1}\Delta\Theta] - \partial^\alpha[\Delta\theta^\varepsilon(\frac{1}{\rho^\varepsilon} - \bar{\rho}^{-1})],$$

and

$$\frac{2}{3}\partial^\alpha(\theta^\varepsilon\mathrm{div}u^\varepsilon) - \frac{2}{3}\partial^\alpha(\bar{\theta}\mathrm{div}\bar{u}) = \frac{2}{3}\partial^\alpha(\Theta\mathrm{div}u^\varepsilon) + \frac{2}{3}\partial^\alpha(\bar{\theta}\mathrm{div}\Xi),$$

Then we have

$$\partial_t\partial^\alpha\Theta + \partial^\alpha(u^\varepsilon\cdot\nabla\Theta) + \partial^\alpha(\Xi\cdot\nabla\bar{\theta}) + \frac{2}{3}\partial^\alpha(\Theta\mathrm{div}u^\varepsilon) + \frac{2}{3}\partial^\alpha(\bar{\theta}\mathrm{div}\Xi) - \frac{2}{3}\partial^\alpha[\bar{\rho}^{-1}\Delta\Theta] - \frac{2}{3}\partial^\alpha[\Delta\theta^\varepsilon(\frac{1}{\rho^\varepsilon} - \bar{\rho}^{-1})] = \frac{2-\varepsilon^2}{3}\partial^\alpha(|u^\varepsilon|^2) - \frac{2}{3}\partial^\alpha(|\bar{u}|^2) - \partial^\alpha\Theta.$$

Taking the inner product with $\partial^\alpha\Theta$ in L^2 yields

$$\langle\partial_t\Theta_\alpha, \Theta_\alpha\rangle + \langle\Theta_\alpha, \Theta_\alpha\rangle = \sum_{i=1}^{6} S_i. \tag{4.34}$$

Notice that

$$S_1 = -<\partial^\alpha(u^\varepsilon\cdot\nabla\Theta^\varepsilon), \Theta_\alpha>, \qquad S_2 = -<\partial^\alpha(\Xi\cdot\nabla\bar{\theta}), \Theta_\alpha>,$$

$$S_3 = -<\frac{2}{3}\partial^\alpha(\Theta\mathrm{div}u^\varepsilon), \Theta_\alpha>, \qquad S_4 = -<\frac{2}{3}\partial^\alpha(\bar{\theta}\mathrm{div}\Xi), \Theta_\alpha>,$$

$$S_5 = <\frac{2}{3}\partial^\alpha[\bar{\rho}^{-1}\Delta\Theta], \Theta_\alpha> + <\frac{2}{3}\partial^\alpha[\Delta\theta^\varepsilon(\frac{1}{\rho^\varepsilon} - \bar{\rho}^{-1})], \Theta_\alpha>,$$

$$S_6 = <\frac{2-\varepsilon^2}{3}\partial^\alpha(|u^\varepsilon|^2), \Theta_\alpha> - <\frac{2}{3}\partial^\alpha(|\bar{u}|^2), \Theta_\alpha>.$$

With the help of Lemma 2.2, one has

$$S_1 \le \|\Theta_\alpha\|\left(\|u_\alpha^\varepsilon\|\|\nabla\Theta\|_{L^\infty} + \|u^\varepsilon\|_{L^\infty}\|\partial^\alpha\nabla\Theta\|\right) \le \delta\|\Theta\|_{k-1}\|\Theta\|_k. \tag{4.35}$$

Similarly,

$$S_2 \le \|\Theta_\alpha\|\left(\|\Xi_\alpha\|\|\Delta\bar{\theta}\|_{L^\infty} + \|\Xi\|_{L^\infty}\|\partial^\alpha\nabla\bar{\theta}\|\right) \le \delta\|\Theta\|_{k-1}\|\Theta\|_k. \tag{4.36}$$

By virtue of $H\ddot{o}lder's$ Cauchy's inequalities,

$$S_3 = -\frac{2}{3}\int d\,ivu\Theta_\alpha^2 + [\partial^\alpha, divu]\Theta\Theta_\alpha$$
$$\leq -\frac{2}{3}\int d\,ivu\Theta_\alpha^2 + \|\Theta_\alpha\|(\|divu\|_{L^\infty}\|\partial^{\alpha-1}\Theta\| + \|\partial^\alpha divu\|\|\Theta\|_{L^\infty}) \tag{4.37}$$
$$\leq -\frac{2}{3}\int d\,ivu\Theta_\alpha^2 + \delta\|\Theta\|_{k-1}\|\Theta\|_{k-2}.$$

similar to S_3,

$$S_4 \leq \delta\|\Theta\|_{k-1}\|\Theta\|_k. \tag{4.38}$$

By virtue of $Hölder's$ Young's inequalities,

$$S_5 = -\frac{2}{3}\int \partial^\alpha\,[\nabla\theta\nabla(\frac{1}{\rho^\varepsilon} - \bar\rho^{-1})]\Theta_\alpha + \partial^\alpha[\nabla\theta(\frac{1}{\rho^\varepsilon} - \bar\rho^{-1})]\nabla\Theta_\alpha$$
$$-\frac{2}{3}\int \partial^\alpha\,(\bar\rho^{-1}\nabla\Theta)\nabla\Theta_\alpha + \partial^\alpha\,(\nabla\bar\rho^{-1}\nabla\Theta)\Theta_\alpha$$
$$\leq -\frac{2}{3}\int \bar\rho^{-1}|\nabla\Theta_\alpha|^2 + [\partial^\alpha, \bar\rho^{-1}]\nabla\Theta\cdot\nabla\Theta_\alpha - \int \partial^\alpha\,(\nabla\bar\rho^{-1}\nabla\Theta)\Theta_\alpha$$
$$-\frac{2}{3}\int \partial^\alpha\,[\nabla\theta\nabla(\frac{1}{\rho^\varepsilon} - \bar\rho^{-1})]\Theta_\alpha + \partial^\alpha\,[\nabla\theta(\frac{1}{\rho^\varepsilon} - \bar\rho^{-1})]\nabla\Theta_\alpha,$$

Then we can obtain

$$S_5 \leq -\frac{2}{3}\int \bar\rho^{-1}|\nabla\Theta_\alpha|^2 + \delta\|\Theta\|_k\|\Theta\|_{k-1}. \tag{4.39}$$

By the Cauchy-Schwarz inequality, Young's inequality, we obtain

$$S_6 = < \frac{2}{3}\partial^\alpha(|u^\varepsilon|^2 - |\bar u|^2), \Theta_\alpha > - < \frac{\varepsilon^2}{3}\partial^\alpha(|u^\varepsilon|^2), \Theta_\alpha >$$
$$= - < \frac{2}{3}\partial^{\alpha-1}(|u^\varepsilon|^2 - |\bar u|^2), \nabla\Theta_\alpha > - < \frac{\varepsilon^2}{3}\partial^\alpha(|u^\varepsilon|^2), \Theta_\alpha >$$
$$\leq \|\nabla\Theta_\alpha\|(\|u^\varepsilon - \bar u\|_{L^\infty}\|\Xi_{\alpha-1}\| + \|\Xi\|_{L^\infty}\|\partial^\alpha(u^\varepsilon + \bar u)\|) + \varepsilon^2\|\partial^\alpha u^\varepsilon\|\|\|\Theta_\alpha\|,$$

Then we can obtain

$$S_6 \leq C\varepsilon^2\|\partial^\alpha u^\varepsilon\|^2 + \delta\|\nabla\Theta_\alpha\|\|\Xi_{\alpha-1}\|. \tag{4.40}$$

Substituting (4.35)-(4.40) into (4.34), it follows

$$\frac{1}{2}\frac{d}{dt}\|\Theta_\alpha\|^2 + \|\Theta_\alpha\|^2 + \frac{2}{3}\int d\,ivu\Theta_\alpha^2 + \frac{2}{3}\int \bar\rho^{-1}|\nabla\Theta_\alpha|^2$$
$$\leq \delta\|\Theta^\varepsilon\|_k(\|\Theta^\varepsilon\|_{k-1} + \|\Xi\|_{k-2}) + C\varepsilon^2\|\partial^\alpha u^\varepsilon\|^{\cdot}.$$

Integrating the above inequality over $[0,t]$ and summing up all $|\alpha| \leq k-1$, it yields, by (1.9), (1.16), and (1.17), we can obtain

$$\|\Theta\|_{k-1}^2 + \int_0^t \|\Theta\|_k^2 \leq \delta\int_0^t \|\Xi\|_{k-2}^2 + C\|\Theta(0)\|_{k-1}^2 + C\varepsilon^2 \leq \delta\int_0^t \|\Xi\|_{k-2}^2 + C\varepsilon^{2p_1}. \tag{4.41}$$

Multiplying the above inequality (4.41) by $\frac{5c_1}{4}$, then integrating over $([0,T])$, and adding it to (4.29) yields

$$\|\Theta_\alpha(T)\|^2 + c_1\int_0^T(\|N_\alpha(\tau)\|^2 + \|\Theta_\alpha(\tau)\|^2)\,d\tau - C\delta\int_0^T(\|N^\varepsilon(\tau)\|_{k-1}^2 + \|\Theta^\varepsilon(\tau)\|_{k-1}^2)\,d\tau$$

$$\leq \mathcal{R}_3 + C\varepsilon^{2p_1} + \delta\int_0^t \|\Xi\|_{k-2}^2,$$

Which proves (4.29).

Lemma 4.8 (Estimates of$\mathcal{R}_4$, $\mathcal{R}_5$ and$\mathcal{R}_6$, see Lemma 2.8 in [11])For all$|\alpha| \le k-1$, there exists a constant$c_2 > 0$, such that

$$\mathcal{R}_4 + \mathcal{R}_5 + \mathcal{R}_6 \ge c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau - C\delta \int_0^T \|N^\varepsilon(\tau)\|_{k-1}^2 d\tau - C\varepsilon^2. \tag{4.42}$$

Proof. Since$\mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4 + \mathcal{R}_5 + \mathcal{R}_6 = -\mathcal{R}_1$combining Lemmas 4.7--4.8, (4.24) and (4.2), and summing up all$|\alpha| \le k-2$, we have

$$\begin{aligned} &\|N^\epsilon(T)\|_{k-2}^2 + \|\Theta^\epsilon(T)\|_{k-1}^2 + \|F^\epsilon(T)\|_{k-1}^2 + \|G^\epsilon(T)\|_{k-1}^2 \\ &+ \int_0^T (\|N^\epsilon(\tau)\|_{k-1}^2 + \|\Theta^\epsilon(\tau)\|_k^2 + \|F^\epsilon(\tau)\|_{k-1}^2)\, d\tau \\ &\le C\epsilon^{2p_1} + C\delta \sup_{0\le t\le T} \|G^\epsilon(t)\|_{k-1}^2 + C\mu \int_0^T \|\nabla G^\epsilon(\tau)\|_{k-2}^2 d\tau + \delta \int_0^t \|\Xi\|_{k-2}^2, \end{aligned}$$

Provided that δ it is small enough. By taking the superior limit with respect to T, we obtain

$$\begin{aligned} &\sup_{t\in\mathbb{R}^+} (\|N^\epsilon(t)\|_{k-2}^2 + \|\Theta^\epsilon(t)\|_{k-1}^2 + \|F^\epsilon(t)\|_{k-1}^2 + \|G^\epsilon(t)\|_{k-1}^2) \\ &+ \int_0^{+\infty} (\|N^\epsilon(\tau)\|_{k-1}^2 + \|\Theta^\epsilon(\tau)\|_k^2 + \|F^\epsilon(\tau)\|_{k-1}^2)\, d\tau \\ &\le C\varepsilon^{2p_1} + C\mu \int_0^{+\infty} \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau + \delta \int_0^\infty \|\Xi\|_{k-2}^2. \end{aligned}$$

The proof of (4.42) ends after borrowing Lemma 2.9 in [11] as follows.

Lemma 4.9 It hold

$$\int_0^{+\infty} \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau \le C\varepsilon^{2p_1}.$$

Proof. We omit it for the sake of simplicity.

The proof of Theorem 1.3 follows from the estimate below.

Lemma 4.10 It hold

$$\sup_{t\in\mathbb{R}^+} \varepsilon^2 \|\Xi^\varepsilon(t)\|_{k-2}^2 + \int_0^{+\infty} \|\Xi^\varepsilon(t)\|_{k-2}^2\, dt \le C\varepsilon^{2p_1}.$$

Proof. In a similar way to that in Lemma 4.10 of [15], we can prove Lemma 4.10. We omit it for the sake of simplicity.

Proof of Theorem 1.3. Theorem 1.3 follows by combining Lemma 4.4 and Lemma 4.10.

5. Conclusion

This study addresses the non-isentropic Euler-Maxwell system under small relaxation times in magnetized plasmas and semiconductors. The core contributions, highlighting the role of temperature evolution, are:

1. Uniform-in-Relaxation Global Existence: Global smooth periodic solutions exist near equilibrium, uniformly controlled as the relaxation parameter shrinks.
2. Global Convergence to Energy-Transport: Solutions converge globally (in slow time) to those of the full energy-transport model as the relaxation time tends to zero.
3. Innovative Error Estimates: New techniques involving stream functions and sharp energy methods yield precise error bounds between the non-isentropic Euler-Maxwell and energy-transport solutions.

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