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Original Article

Stability of Two-Dimensional Incompressible MHD Equations with Mixed Partial Dissipation

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Abstract - *This paper investigates the stabilization effect of the two-dimensional incompressible MHD system with partial dissipation near a background magnetic field. Employing time-global uniform a priori estimates, we establish the global well-posedness of the nonlinear MHD equations in* $H^3(\mathbb{R}^2)$. *Building upon this fundamental result, we derive the large-time asymptotic behavior of the solutions.*

Keywords *- MHD equations, Stability, Large-time behavior.*

AMS Subject Classifications (2020) - *35A01, 35B40, 76E25*

1. Introduction

The MHD equations combine the Navier-Stokes equations for fluid dynamics and the Maxwell equations for electromagnetism. They control the movement of electrically conductive fluids, such as liquid metals, and have numerous applications in astrophysics, geophysics, cosmology, and engineering fields (D.Biskamp,1993; H. Qiu, M Chen,1995) [5,14]. Extensive experimental and numerical studies have consistently shown that the applied background magnetic field exerts a significant stabilizing influence on magnetohydrodynamic (MHD) turbulence (A. Alemany, R. Moreau, P.L. Sulem, U. Frisch,1979; A. Alexakis, 2011; H. Alfv´en,1942; C. Bardos, C. Sulem, P.L. Sulem,1988; P. Burattini, O Zikanov, B. Knaepen,2010) [1-4,6]. This paper focuses on the following 2D MHD system with anisotropic dissipation,

$$\begin{cases} \partial_t U + U \cdot \nabla U + \nabla P = (\nu\partial_2^2 U_1,\ 0)^\top + B \cdot \nabla B, \\ \partial_t B + U \cdot \nabla B + \eta\partial_1^2 B = B \cdot \nabla U, \\ \nabla \cdot U = \nabla \cdot B = 0. \end{cases} \tag{1.1}$$

Where $U = (U_1, U_2)^\top$ [?] denotes the velocity field, [?] denotes the magnetic field, and[?] denotes the pressure. The nonnegative constants $\nu$and $\eta$are the viscosity coefficients. Without the magnetic field equation, the global stability and large-time behavior of (1.1) MHD is an outstanding open problem. The major difficulty is that dissipation in only one direction is too weak to control all the nonlinear terms in the whole space $\mathbb{R}^2$.

Recent studies have shown a growing interest in the stability of incompressible MHD equations with partial dissipation. Paicu and Zhang (2019) [23] established strong global solutions to the 3-D Navier-Stokes system with strong dissipation in one direction. Wu-Wu-Xu (2015) [25] analyzed the stability of a two-dimensional MHD system with only velocity damping. Boardman-Lin-Wu (2020) [8] established the stability of the two-dimensional MHD system while the velocity field involves only one direction of damping. Lin-Ji-Wu-Li (2020) [19] investigated the stability of perturbations near a background magnetic field of the two-dimensional incompressible MHD equations with mixed dissipation. Subsequently, in the two-dimensional periodic domain, Feng-Hafeez-Wu (2021,2023) [10,11] focused on the following 2D MHD system with anisotropic dissipation,

$$\begin{cases} \partial_t U + U \cdot \nabla U + \nabla P = \nu \partial_2^2 U + B \cdot \nabla B, \\ \partial_t B + U \cdot \nabla B + \eta (\sigma B_1, B_2)^\top = B \cdot \nabla U, \\ \nabla \cdot U = \nabla \cdot B = 0. \end{cases}$$

For $\sigma = 1$, they proved that any perturbation near the background magnetic field is asymptotically stable and established certain explicit large-time behaviors. Lin-Chen-Bai-Zhang (2023) [17] studied the stability of a system of 2D incompressible anisotropic magnetohydrodynamic equations. Qiu and Chen (2025) [14]establish the large-time behavior with explicit decay rates to linearized two-dimensional MHD equations. For more works, the readers can also refer to (E. Priest, T. Forbes,2000; S.H. Lai, J.H. Wu, J.W. 2024.) [13,16] and the references therein.

The stability and large time behavior of the incompressible 3D MHD equations with partial dissipation has attracted more and more attention, and significant progress has been made (J.H.Wu,2001) [24]. Lin-Wu-Suo(2023) [20]refined the result of (Boardman, Lin, Wu,2020) [8] by establishing the large-time behavior of solutions in H^4space with only horizontal magnetic diffusion and vertical velocity damping required. Lin-Wu-Zhu (2025) [22] established the stability and large-time behavior of 3D incompressible MHD equations with partial dissipation near a background magnetic field. Recently, Lai-Wu-Zhang-Zhao (2024) [16] extended the global well-posedness of (2023) [21] and (2020) [26] $H^3(\mathbb{R}^3)$, and established the optimal decay rate.
the equilibrium state (U^0, B^0) $\mathbb{R}^2$. It $U^0 = 0, B^0 = e_1 = (1,0)$ is a steady solution of (1.1). Then $u = U - U^0, b = B - B^0$, $P = p$, and $\nu = \eta = 1$satisfy

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = (\partial_2^2 u_1, 0)^\top + b \cdot \nabla b + \partial_1 b, \\ \partial_t b + u \cdot \nabla b + \partial_1^2 b = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \mathrm{b}(x,0) = b_0(x). \end{cases} \tag{1.2}$$

Our main results are as follows:

Theorem 1.1. Suppose that the initial data $(u_0, b_0) \in H^3(\mathbb{R}^2)$ satisfy $\nabla \cdot u_0 = 0 \nabla \cdot b_0 = 0$. There exists a small enough constant $\varepsilon > 0$such that.

$$\|(u_0, b_0)\|_{H^3}^2 \le \varepsilon,$$

Then the system (1.2) has a unique global solution $(u, b) \in C\big(0, \infty); H^3(\mathbb{R}^2)\big)$, satisfying

$$\|(u,b)(t)\|_{H^3}^2 + \int_0^t \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_1 u_2\|_{H^2}^2 d\tau \le C\varepsilon^2 \tag{1.3}$$

for some uniform constant C and any $t \ge 0$. In addition, $(\nabla u, \nabla b)$ obeys the following large-time behavior:

$$\left\| \left(\nabla u, \partial_1 b \right)(\mathrm{t}) \right\|_{H^1} \to 0 \quad \text{as} \quad t \to 0. \tag{1.4}$$

The bootstrapping argument is applied to prove Theorem 1.1. The global a priori estimates of the solutions must be established. Due to the lack of horizontal dissipation $u_2$in the velocity field, we need to construct a suitable energy function. Since the initial data $(\mathrm{u}_0, \mathrm{b}_0) \in H^3(\mathbb{R}^2)$, a natural part of the energy function is

$$\mathcal{E}_1(t) := \sup_{0 \le t \le t} \|(u,b)(t)\|_{H^3}^2 + 2\int_0^t \|\partial_2 u_1(t)\|_{H^3}^2 + \|\partial_1 b(t)\|_{H^3}^2 \, dt,$$

Due to the lack of horizontal dissipation u_2 in the velocity field, we define

$$\mathcal{E}_2(t) := \int_0^t \|\partial_1 u_2(t)\|_{H^2}^2 \, dt.$$

Let

$$\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t).$$

The centrepiece is to establish the inequality

$$\mathcal{E}(t) \le C\mathcal{E}(0) + C\mathcal{E}^{3/2}(t), \tag{1.5}$$

where C is a generic positive constant. The verification of estimate (1.5) fundamentally reduces to the following two inequalities:

$$\mathcal{E}_1(t) \le C\mathcal{E}_1(0) + C\mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t), \tag{1.6}$$

$$\mathcal{E}_2(t) \le C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t). \tag{1.7}$$

For any $t \ge 0$ combination of (1.6) and (1.7) with judiciously chosen multiplicative constants yields the targeted estimate for (1.5). The bootstrapping argument implies that if

$$\mathcal{E}(0) = \|(u_0, b_0)\|_{H^3}^2 < \varepsilon^2.$$

For some small enough $\varepsilon > 0$, then $\mathcal{E}(t)$ it remains small for,

$$\mathcal{E}(t) \le C\varepsilon^2$$

For some pure constant $C > 0$. This will be proved in more detail in section 3.3.

The structure of this paper unfolds as follows. Section 2 presents several useful mathematical tools that will be used frequently later. In Section 3, we rigorously prove Theorem 1.1.

2. Preliminaries

In this section, we establish the proofs of Theorem 1.1. The following essential tool is introduced. Lemma 2.1(Cao and Wu,2011) [7] and Lemma 2.2 (Feng-Hafeez-Wu,2021) [10] are some basic anisotropic upper bounds of the triple products. Lemma 2.3 (Doering-Wu-Zhao-Zheng,2018) [9] can help us obtain the asymptotic behavior.

Lemma 2.1. Assume $\boldsymbol{f}$ h $\boldsymbol{\partial_1 g}$ $\boldsymbol{\partial_2 h}$ all are in $L^2(\mathrm{R}^2)$. It holds that $\int |\boldsymbol{fgh}|\, \boldsymbol{dx} <$

$$\boldsymbol{C}\|\boldsymbol{f}\|_{L^2}\|\boldsymbol{g}\|_{L^2}^{\frac{1}{2}}\|\boldsymbol{\partial_1 g}\|_{L^2}^{\frac{1}{2}}\|\boldsymbol{h}\|_{L^2}^{\frac{1}{2}}\|\boldsymbol{\partial_2 h}\|_{L^2}^{\frac{1}{2}}.$$

Lemma 2.2 The following estimates hold when the right-hand sides are all bounded in R^2 $\|\boldsymbol{f}\|_{L^\infty} <$

$\boldsymbol{C}\|\boldsymbol{f}\|_{L^2}^{\frac{1}{4}}\|\boldsymbol{\partial_1 f}\|_{L^2}^{\frac{1}{4}}\|\boldsymbol{\partial_2 f}\|_{L^2}^{\frac{1}{4}}\|\boldsymbol{\partial_1\partial_2 f}\|_{L^2}^{\frac{1}{4}}$,

which implies that

$$\|f\|_{L^\infty} < C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_1 f\|_{H^1}^{\frac{1}{2}}$$

$$\|f\|_{L^\infty} < C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_2 f\|_{H^1}^{\frac{1}{2}}$$

Lemma 2.3 Let $f = f(\mathrm{t})$ f $\boldsymbol{0 \le t < \infty}$ be a nonnegative continuous function. Assume that

$$\int_0^\infty f(\mathrm{t})\,\mathrm{dt} < \infty$$

For a constant $C > 0$, and any $0 \le t < \infty$,

$$|f(\mathrm{t}) - \mathrm{f(s)}| \le C|t - s|.$$

Then

$$f(\mathrm{t}) \to 0 \quad \text{as } t \to 0.$$

3. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. For the sake of clarity, we establish (1.6) and (1.7) respectively.

3.1 Proof of (1.6)

Lemma 3.1. Assume $(\boldsymbol{u}, \boldsymbol{b})$ that is a solution to (1.2). Then we have

$$\|(u,b)(\mathrm{t})\|_{H^3}^2 + \int_0^t \|\partial_2 u_1\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2\, dt \le C\mathrm{E}_1(0) + C\mathrm{E}_1^{3/2}(\mathrm{t}) + C\mathrm{E}_2^{3/2}(\mathrm{t})$$

Proof: First, taking the L^2 inner product of (1.2) with (u, b), one can get

$$\frac{1}{2}\frac{d}{dt}\|(u,b)\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 b\|_{L^2}^2 = 0. \tag{3.1}$$

Applying ∇ to the equations (1.2) and taking the L^2 inner product of the resulting equations with $(\nabla u, \nabla b)$,

$$\frac{1}{2}\frac{d}{dt}\|(\nabla u, \nabla b)\|_{L^2}^2 + \|\nabla\partial_2 u_1\|_{L^2}^2 + \|\nabla\partial_1 b\|_{L^2}^2$$

$$= -\int \nabla u\cdot\nabla b\cdot\nabla b dx + \int \nabla b\cdot\nabla b\cdot\nabla u dx + \int \nabla b\cdot\nabla u\cdot\nabla b dx$$

$$:= A_1 + A_2 + A_3.$$

Where we used the significant fact that

$$-\int \nabla u \cdot \nabla u \cdot \nabla u dx = 0.$$

By Höölder's inequality and Lemma 2.1 and Lemma 2.2, we have

$$A_1 = -\int \partial_1 u \cdot \nabla b \cdot \partial_1 b dx - \int \partial_2 u_1 \partial_1 b \cdot \partial_2 b dx - \int \partial_2 u_2 \partial_2 b \cdot \partial_2 b dx$$

$$= -\int \partial_1 u \cdot \nabla b \cdot \partial_1 b dx - \int \partial_2 u_1 \partial_1 b \cdot \partial_2 b dx - 2\int u_1 \partial_1 \partial_2 b \cdot \partial_2 b dx$$

$$\leq C\left(\|\nabla b\|_{L^\infty} \|\partial_1 u\|_{L^2} \|\partial_1 b\|_{L^2} + \|\partial_2 b\|_{L^\infty} \|\partial_2 u_1\|_{L^2} \|\partial_1 b\|_{L^2} + \|\partial_2 \partial_1 b\|_{L^2} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 b\|_{L^2}^{\frac{1}{2}} \right)$$

$$\leq C\left(\|b\|_{H^3} + \|u\|_{L^2}\right)\left(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 b\|_{H^1}^2\right)$$

In similar calculations

$$A_2 = \int \partial_1 b \cdot \nabla b \cdot \partial_1 u dx + \int \partial_2 b_1 \partial_1 b \cdot \partial_2 u dx + \int \partial_2 b_2 \partial_2 b \cdot \partial_2 u dx$$

$$= \int \partial_1 b \cdot \nabla b \cdot \partial_1 u dx + \int \partial_2 b_1 \partial_1 b \cdot \partial_2 u dx + \int \partial_2 b_2 \partial_2 b \cdot \partial_2 u dx$$

$$\leq C\left(\|\nabla b\|_{L^\infty} \|\partial_1 b\|_{L^2} \|\partial_1 u\|_{L^2} + \|\partial_2 b_1\|_{L^\infty} \|\partial_2 u\|_{L^2} \|\partial_1 b\|_{L^2} + \|\partial_2 b\|_{L^\infty} \|\partial_2 u\|_{L^2} \|\partial_2 b_2\|_{L^2}\right)$$

$$\leq C\|b\|_{H^3}\left(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 b\|_{L^2}^2\right)$$

And

$$A_3 = \int \partial_1 b \cdot \nabla u \cdot \partial_1 b dx + \int \partial_2 b_1 \partial_1 u_1 \cdot \partial_2 b_1 dx + \int \partial_2 b_1 \partial_1 u_2 \cdot \partial_2 b_2 dx + \int \partial_2 b_2 \partial_2 u \cdot \partial_2 b dx$$

$$= \int \partial_1 b \cdot \nabla u \cdot \partial_1 b dx - 2\int u_1 \partial_2 \partial_1 b_1 \partial_2 b_1 dx + \int \partial_2 b_1 \partial_1 u_2 \partial_2 b_2 dx + \int \partial_2 b_2 \partial_2 u \cdot \partial_2 b dx$$

$$\leq C\Big(\|\nabla u\|_{L^\infty} \|\partial_1 b\|_{L^2}^2 + \|\partial_2 \partial_1 b_1\|_{L^2} \|u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 b_1\|_{L^2}^{\frac{1}{2}}$$

$$+\|\partial_2 b_1\|_{L^\infty} \|\partial_1 u_2\|_{L^2} \|\partial_2 b_2\|_{L^2} + \|\partial_2 b\|_{L^\infty} \|\partial_2 u\|_{L^2} \|\partial_2 b_2\|_{L^2}\Big)$$

$$\leq C\left(\|b\|_{H^3}+\|u\|_{H^3}\right)\left(\|\partial_1 u_2\|_{L^2}^2+\|\partial_2 u_1\|_{L^2}^2+\|\partial_1 b\|_{H^1}^2\right)$$

Combining A_1, A_2 with A_3, we can get

$$\frac{1}{2}\frac{d}{dt}\|(\nabla u,\nabla b)\|_{L^2}^2+\|\nabla\partial_2 u_1\|_{L^2}^2+\|\nabla\partial_1 b\|_{L^2}^2$$

$$\leq C\left(\|b\|_{H^3}+\|u\|_{H^3}\right)\left(\|\partial_1 u_2\|_{L^2}^2+\|\partial_2 u_1\|_{L^2}^2+\|\partial_1 b\|_{H^1}^2\right) \tag{3.2}$$

Similarly

$$\frac{1}{2}\frac{d}{dt}\|(\nabla^2 u,\nabla^2 b)\|_{L^2}^2+\|\nabla^2\partial_2 u_1\|_{L^2}^2+\|\nabla^2\partial_1 b\|_{L^2}^2$$

$$\leq C\left(\|b\|_{H^3}+\|u\|_{H^3}\right)\left(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^2}^2\right) \tag{3.3}$$

Applying $\partial_i^3(\mathrm{i}=1,2)$ to the equations (1.2)and taking the L^2 inner product of the resulting equations with $(\partial_i^3 u,\partial_i^3 \mathrm{b})$,

$$\frac{1}{2}\frac{d}{dt}\|(\nabla^3 u,\nabla^3 b)\|_{L^2}^2+\|\nabla^3\partial_2 u_1\|_{L^2}^2+\|\nabla^3\partial_1 b\|_{L^2}^2$$

$$=-\int\partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\,dx+\int\partial_i^3(\mathrm{b}\cdot\nabla b)\cdot\partial_i^3 u\,dx-\int\partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\,dx+\int\partial_i^3(\mathrm{b}\cdot\nabla u)\cdot\partial_i^3 b\,dx$$

$$:=B_1+B_2+B_3+\mathrm{B}_4. \tag{3.4}$$

Due to the Newton-Leibniz formula and the fact that $\nabla\cdot u=0$ it follows

$$B_1=-\sum_{i=1}^{2}\int\partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,dx-\sum_{i=1}^{2}\int\partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\,dx$$

$$-\sum_{i=1}^{2}\int\partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,dx$$

$$\leq C\sum_{i=1}^{2}\left(\|\nabla u\|_{L^\infty}\left\|\partial_i^3 u\right\|_{L^2}^2+\left\|\partial_i^3 u\right\|_{L^2}\|\partial_i^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_i^2\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_i u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_i\partial_2 u\|_{L^2}^{\frac{1}{2}}\right)$$

$$\le C\|u\|_{H^3}\left(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2\right) \tag{3.5}$$

By Höolder's inequality and Lemma 2.1 and Lemma 2.2, we have

$$B_3 = -\sum_{i=1}^{2}\int \partial_i^3 u\cdot\nabla b\cdot\partial_i^3 b\,dx - \sum_{i=1}^{2} 3\int \partial_i^2 u\cdot\nabla\partial_i b\cdot\partial_i^3 b\,dx - \sum_{i=1}^{2} 3\int \partial_i u\cdot\nabla\partial_i^2 b\cdot\partial_i^3 b\,dx$$

$$= -\int \partial_1^3 u\cdot\nabla b\cdot\partial_1^3 b\,dx - 3\int \partial_1^2 u\cdot\nabla\partial_1 b\cdot\partial_1^3 b\,dx - 3\int \partial_1 u\cdot\nabla\partial_1^2 b\cdot\partial_1^3 b\,dx$$

$$-\int \partial_2^3 u_1\partial_1 b\cdot\partial_1^3 b\,dx - \int \partial_2^3 u_2\partial_2 b\cdot\partial_2^3 b\,dx - 3\int \partial_2^2 u_1\partial_1\partial_2 b\cdot\partial_2^3 b\,dx$$

$$-3\int \partial_2^2 u_2\partial_2^2 b\cdot\partial_2^3 b\,dx - 3\int \partial_2 u_1\partial_1\partial_2^2 b\cdot\partial_2^3 b\,dx - 3\int \partial_2 u_2\partial_2^3 b\cdot\partial_2^3 b\,dx$$

$$\le C\Big(\|\nabla b\|_{L^\infty}\|\partial_1^3 u\|_{L^2}\|\partial_1^3 b\|_{L^2} + \|\partial_1^3 b\|_{L^2}\|\partial_1^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_1\partial_2 b\|_{L^2}^{\frac{1}{2}}$$

$$+\|\partial_1 u\|_{L^\infty}\|\nabla\partial_1^2 b\|_{L^2}\|\partial_1^3 b\|_{L^2} + \|\partial_1 b\|_{L^\infty}\|\partial_2^3 u_1\|_{L^2}\|\partial_1^3 b\|_{L^2}$$

$$+\|\partial_2^3 b\|\|\partial_2^2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^3 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_1^3 b\|_{L^2}^{\frac{1}{2}} + \|\partial_2 u_1\|_{L^\infty}\|\partial_1\partial_2^3 b\|_{L^2}\|\partial_2^3 b\|_{L^2}\Big)$$

$$-\int \partial_2^3 u_2\partial_2 b\cdot\partial_2^3 b\,dx - 3\int \partial_2^2 u_2\partial_2^2 b\cdot\partial_2^3 b\,dx - 3\int \partial_2 u_2\partial_2^3 b\cdot\partial_2^3 b\,dx$$

$$\le C\left(\|u\|_{H^3} + \|b\|_{H^3}\right)\left(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b\|_{H^2}^2\right) + B_{31} + B_{32} + B_{33}. \tag{3.6}$$

By applying $\nabla\cdot u = 0$ integration by parts, we have

$$B_{31} + B_{32} + B_{33} = -\int \partial_2^2 u_1\partial_1\partial_2 b\cdot\partial_2^3 b\,dx - \int \partial_2^2 u_1\partial_2 b\cdot\partial_2^3\partial_1 b\,dx$$

$$-3\int \partial_2 u_1\partial_1\partial_2^2 b\cdot\partial_2^3 b\,dx - 3\int \partial_2 u_1\partial_2^2 b\cdot\partial_2^3\partial_1 b\,dx - 6\int u_1\partial_2^3\partial_1 b\cdot\partial_2^3 bx$$

$$\le C\Big(\|\partial_2^3 b\|_{L^2}\|\partial_2^2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 b\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^2 b\|_{L^2}^{\frac{1}{2}} + \|\partial_2 b\|_{L^\infty}\|\partial_2^2 u_1\|_{L^2}\|\partial_2^3\partial_1 b\|_{L^2}$$

$$+\|\partial_2 u_1\|_{L^\infty}\|\partial_2^2\partial_1 b\|_{L^2}\|\partial_2^3 b\|_{L^2} + \|\partial_2 u_1\|_{L^\infty}\|\partial_2^2 b\|_{L^2}\|\partial_2^3\partial_1 b\|_{L^2}$$

$$+\|\partial_2^3\partial_1 b\|_{L^2}\|u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^3 b\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3 b\|_{L^2}^{\frac{1}{2}}\Big)$$

$$\le C\|b\|_{H^3}\left(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b\|_{H^3}^2\right). \tag{3.7}$$

Putting (3.7) into (3.6), one can get

$$B_3 \le C\left(\|u\|_{H^3} + \|b\|_{H^3}\right)\left(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b\|_{H^3}^2\right). \tag{3.8}$$

Due to the Newton-Leibniz formula,

$$\begin{aligned}
B_2 + B_4 &= \sum_{i=1}^{2}\int \partial_i^3 b\cdot\nabla b\cdot\partial_i^3 u dx + \sum_{i=1}^{2} 3\int \partial_i^2 b\cdot\nabla\partial_i b\cdot\partial_i^3 u dx \\
&\quad + \sum_{i=1}^{2} 3\int \partial_i b\cdot\nabla\partial_i^2 b\cdot\partial_i^3 u\, dx \\
&\quad + \sum_{i=1}^{2}\int \partial_i^3 b\cdot\nabla u\cdot\partial_i^3 b dx + \sum_{i=1}^{2} 3\int \partial_i^2 b\cdot\nabla\partial_i u\cdot\partial_i^3 b dx \\
&\quad + \sum_{i=1}^{2} 3\int \partial_i b\cdot\nabla\partial_i^2 u\cdot\partial_i^3 b\, dx \\
&:= B_{21} + B_{22} + B_{23} + B_{41} + B_{42} + B_{43}.
\end{aligned} \tag{3.9}$$

Due to the lack of dissipation $\partial_2 b_1$, integration by parts is required for each term in (3.9)

$$\begin{aligned}
B_{21} + B_{22} + B_{23} &= \int \partial_1^3 b\cdot\nabla b\cdot\partial_1^3 u dx + 3\int \partial_1^2 b\cdot\nabla\partial_1 b\cdot\partial_1^3 u dx + 3\int \partial_1 b\cdot\nabla\partial_1^2 b\cdot\partial_1^3 u\, dx \\
&\quad + \int \partial_2^3 b_1\partial_1 b\cdot\partial_1^3 u dx + 3\int \partial_2^2 b_1\partial_2\partial_1 b\cdot\partial_2^3 u dx + 3\int \partial_2 b_1\partial_1\partial_2^2 b\cdot\partial_2^3 u\, dx \\
&\quad + \int \partial_2^3 b_2\partial_2 b\cdot\partial_2^3 u dx + 3\int \partial_2^2 b_2\partial_2^2 b\cdot\partial_2^3 u dx + 3\int \partial_2 b_2\partial_2^3 b\cdot\partial_2^3 u\, dx \\
&\le C\Big(\|\nabla b\|_{L^\infty}\|\partial_1^3 u\|_{L^2}\|\nabla\partial_1^2 b\|_{L^2} + \|\nabla^3 u\|_{L^2}\|\nabla\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_2\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla^2 b\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1 b\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}\|\partial_2^3 u\|_{L^2} + \|\partial_2 b_1\|_{L^\infty}\|\partial_1\partial_2^2 b\|_{L^2}\|\partial_2^3 u\|_{L^2} + \|\partial_2 b\|_{L^\infty}\|\partial_2^3 b_2\|_{L^2}\|\partial_1^3 u\|_{L^2} \\
&\quad + \|\partial_2^3 u\|_{L^2}\|\partial_2^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\partial_2^2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1 b\|_{L^2}^{\frac{1}{2}} + \|\partial_2 b_2\|_{L^\infty}\|\partial_2^3 b\|_{L^2}\|\partial_2^3 u\|_{L^2}\Big)
\end{aligned}$$

$$\leq C\left(\|u\|_{H^3}+\|b\|_{H^3}\right)\left(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^2}^2\right). \tag{3.10}$$

And

$$B_{41}+B_{42}+B_{43}=\int \partial_1^3\, b\cdot\nabla u\cdot\partial_1^3 b dx+3\int \partial_1^2\, b\cdot\nabla\partial_1 u\cdot\partial_1^3 b dx+3\int \partial_1 b\cdot\nabla\partial_1^2 u\cdot\partial_1^3 b\, dxdx$$

$$+\int \partial_2^3\, b_1\partial_1 u\cdot\partial_2^3 b dx+3\int \partial_2^2\, b_1\partial_2\partial_1 u\cdot\partial_2^3 b dx+3\int \partial_2 b_1\partial_1\partial_2^2 u\cdot\partial_2^3 b\, dx$$

$$+\int \partial_2^3\, b_2\partial_2 u\cdot\partial_2^3 b dx+3\int \partial_2^2\, b_2\partial_2^2 u\cdot\partial_2^3 b dx+3\int \partial_2 b_2\partial_2^3 u\cdot\partial_2^3 b\, dx$$

$$\leq C\Big(\|\nabla u\|_{L^\infty}\|\partial_1^3 b\|_{L^2}^2+\|\partial_1^3 b\|_{L^2}\|\partial_1^2 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\nabla\partial_1 u\|_{L^2}^{\frac12}\|\nabla\partial_1\partial_2 u\|_{L^2}^{\frac12}$$

$$+\|\partial_1 b\|_{L^\infty}\|\nabla\partial_1^2 u\|_{L^2}\|\partial_1^3 b\|_{L^2}+\|\partial_2 u\|_{L^\infty}\|\partial_2^3 b_2\|_{L^2}\|\partial_2^3 b\|_{L^2}+\|\partial_2 b_2\|_{L^\infty}\|\partial_2^3 u\|_{L^2}\|\partial_2^3 b\|_{L^2}$$

$$+\|\partial_2^3 b\|_{L^2}\|\partial_2^2 b_2\|_{L^2}^{\frac12}\|\partial_2^2\partial_1 b_2\|_{L^2}^{\frac12}\|\partial_2^2 u\|_{L^2}^{\frac12}\|\partial_2^3 u\|_{L^2}^{\frac12}\Big)+\int \partial_2^3\, b_1\partial_1 u\cdot\partial_2^3 b dx$$

$$+3\int \partial_2^2\, b_1\partial_2\partial_1 u\cdot\partial_2^3 b dx+3\int \partial_2 b_1\partial_1\partial_2^2 u\cdot\partial_2^3 b\, dx. \tag{3.11}$$

By integration by parts, we can obtain

$$\int \partial_2^3\, b_1\partial_1 u\cdot\partial_2^3 b dx+3\int \partial_2^2\, b_1\partial_2\partial_1 u\cdot\partial_2^3 b dx+3\int \partial_2 b_1\partial_1\partial_2^2 u\cdot\partial_2^3 b\, dx$$

$$=-\int \partial_2^3\,\partial_1 b_1 u\cdot\partial_2^3 b dx-\int \partial_2^3\, b_1 u\cdot\partial_2^3\partial_1 b dx-3\int \partial_2^2\,\partial_1 b_1\partial_2 u\cdot\partial_2^3 b dx$$

$$-3\int \partial_2^2\, b_1\partial_2 u\cdot\partial_2^3\partial_1 b dx-3\int \partial_2\partial_1 b_1\partial_2^2 u\cdot\partial_2^3 b\, dx-3\int \partial_2 b_1\partial_2^2 u\cdot\partial_2^3\partial_1 b\, dx$$

$$\leq C\Big(\|\partial_2^3\partial_1 b\|_{L^2}\|u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_2^3\partial_1 b\|_{L^2}^{\frac12}+\|\partial_2 u\|_{L^\infty}\|\partial_2^2\partial_1 b_1\|_{L^2}\|\partial_2^3 b\|_{L^2}$$

$$+\|\partial_2 u\|_{L^\infty}\|\partial_2^2 b_1\|_{L^2}\|\partial_2^3\partial_1 b\|_{L^2}+\|\partial_2^3 b\|_{L^2}\|\partial_2^2 u\|_{L^2}^{\frac12}\|\partial_2^2\partial_1 u\|_{L^2}^{\frac12}\|\partial_2\partial_1 b_1\|_{L^2}^{\frac12}\|\partial_2^2\partial_1 b\|_{L^2}^{\frac12}$$

$$+\|\partial_2 b_1\|_{L^\infty}\|\partial_2^2 u\|_{L^2}\|\partial_2^3\partial_1 b\|_{L^2}\Big)$$

$$\leq C\left(\|u\|_{H^3}+\|b\|_{H^3}\right)\left(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^3}^2\right) \tag{3.12}$$

Putting (3.12) into (3.11), one can get

$$B_{41}+B_{42}+B_{43} \le C\left(\|u\|_{H^3}+\|b\|_{H^3}\right)\left(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^3}^2\right) \tag{3.13}$$

Now, substituting (3.5), (3.8), (3.10), and (3.13) into (3.4), we find

$$\frac{1}{2}\frac{d}{dt}\left\|\left(\nabla^3 u,\nabla^3 b\right)\right\|_{L^2}^2+\left\|\nabla^3\partial_2 u_1\right\|_{L^2}^2+\left\|\nabla^3\partial_1 b\right\|_{L^2}^2 \le C\left(\|u\|_{H^3}+\|b\|_{H^3}\right)\left(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^3}^2\right)$$

(3.14)

Combining (3.1) - (3.3) with (3.14) and integrating it over $[0,t]$, one can deduce

$$\begin{aligned}
&\|(u,b)(t)\|_{H^3}^2+2\int_0^t\|\partial_2 u_1(\tau)\|_{H^3}^2+\|\partial_1 b(\tau)\|_{H^3}^2\,d\tau\\
&\le C\,\|(u,b)(0)\|_{H^3}^2+C\sup_{0\le\tau\le t}\|(u,b)(\tau)\|_{H^3}\int_0^t\|\partial_2 u_1(\tau)\|_{H^3}^2+\|\partial_1 b(\tau)\|_{H^3}^2+\|\partial_1 u_2(\tau)\|_{H^2}^2\,d\tau\\
&\le C\,E_1(0)+C\,E_1^{3/2}(t)+E_2^{3/2}(t).
\end{aligned}$$

The proof of Lemma 3.1 is therefore complete.

3.2 Proof of (1.7)

The background magnetic field produces the dissipation of $e_2(t)$. In order to establish the bound $e_2(t)$, we need to observe the special structure of equation (1.2).

$$\partial_1 u_2=\partial_t b_2+u\cdot\nabla b_2-b\cdot\nabla u_2-\partial_1^2 b_2.$$

Lemma 3.2. Assume $(\boldsymbol{u},\boldsymbol{b})$ that is a solution to (1.2). Then we have

$$\int_0^t\|\partial_1 u_2\|_{H^2}^2\,d\tau \le C E_1(0)+C E_1(t)+C E_1^{3/2}(t)+C E_2^{3/2}(t).$$

Proof. First, multiply $(1.2)_2$ by $\partial_1 u_2$ and integrate over. R^2, it follows

$$\|\partial_1 u_2\|_{L^2}^2=\int\partial_1 u_2\,\partial_t b_2\,dx+\int\partial_1 u_2\,u\cdot\nabla b_2\,dx-\int\partial_1 u_2\,b\cdot\nabla u_2\,dx-\int\partial_1 u_2\,\partial_1^2 b_2\,dx$$

$$:=M_1+M_2+M_3+M_4. \tag{3.15}$$

By integration by parts and applying the velocity equation in $(1.2)_1$,

$$\begin{aligned}
M_1&=\frac{d}{dt}\int\partial_1 u_2 b_2\,dx-\int b_2\,\partial_1\left(\partial_1 b_2+b\cdot\nabla b_2-u\cdot\nabla u_2-\partial_2 p\right)dx\\
&=\frac{d}{dt}\int\partial_1 u_2 b_2\,dx+\int\partial_1 b_2\,\partial_1 b_2\,dx+\int\partial_1 b_2\,b\cdot\nabla b_2\,dx-\int\partial_1 b_2\,u\cdot\nabla u_2\,dx-\int\partial_1 b_2\,\partial_2 p\,dx\\
&:=M_{11}+M_{12}+M_{13}+M_{14}+M_{15}.
\end{aligned} \tag{3.16}$$

It is easily concluded that

$$M_{12}+M_{13}=\int \partial_1 b_2(\partial_1 b_2+b\cdot\nabla b_2)\,\mathrm{d}x$$

$$\leq \|\partial_1 b_2\|_{L^2}^2+C\|b\|_{L^\infty}\|\partial_1 b_2\|_{L^2}\|\nabla b_2\|_{L^2}$$

$$\leq \|\partial_1 b_2\|_{L^2}^2+C\|b\|_{H^2}\|\partial_1 b\|_{L^2}^2, \tag{3.17}$$

where $\|\nabla b_2\|_{L^2}=\|\partial_1 b\|_{L^2}$. By Hölder's and Young's inequalities, one has

$$M_{14}=\int \partial_1 b_2(u\cdot\nabla u_2)\,\mathrm{d}x \leq C\|u\|_{L^\infty}\|\partial_1 b_2\|_{L^2}\|\nabla u_2\|_{L^2}\leq C\|u\|_{H^2}\left(\|\partial_1 b_2\|_{L^2}^2+\|\nabla u_2\|_{L^2}^2\right)$$

(3.18)

Now, we need to estimate M_{15} that, applying $\nabla\cdot$ to$(1.2)_1$, one can obtain

$$p=(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u)-(-\Delta)^{-1}\nabla\cdot(\mathrm{b}\cdot\nabla b)-(-\Delta)^{-1}\partial_2^2\partial_1 u_1$$

By means of Höolder's inequality,

$$M_{15}=-\int \partial_2 b_2\partial_1 p dx \leq C\|\partial_2 b_2\|_{L^2}\|\partial_1 p\|_{L^2}, \tag{3.19}$$

Where

$$\|\partial_1 p\|_{L^2}\leq C\left\|(-\Delta)^{-1}\nabla\cdot\partial_1(u\cdot\nabla u)\right\|_{L^2}+C\left\|(-\Delta)^{-1}\nabla\cdot\partial_1(\mathrm{b}\cdot\nabla b)\right\|_{L^2}+C\left\|(-\Delta)^{-1}\partial_2^2\partial_1^2 u_1\right\|_{L^2}$$

$$:=M_{151}+M_{152}+M_{153}.$$

Using the fact that the Riesz operator $\partial_i(-\Delta)^{-\frac{1}{2}}$with $i=1,2$ is bounded inL^r, $0<r<\infty$, one can find

$$M_{151}=C\|(-\Delta)^{-1}\nabla\cdot\partial_1(u\cdot\nabla u)\|_{L^2}\leq C\|u\|_{L^\infty}\|\nabla u\|_{L^2}\leq C\|u\|_{H^2}(\|\partial_2 u_1\|_{L^2}+\|\partial_1 u_2\|_{L^2}).$$

And

$$M_{152}=C\left\|(-\Delta)^{-1}\partial_1\partial_j\partial_i(b_ib_j)\right\|\leq C\|b\|_{L^\infty}\|\partial_1 b\|_{L^2}\leq \|b\|_{H^2}\|\partial_1 b\|_{L^2}.$$

Obviously,

$$M_{153}=C\|(-\Delta)^{-1}\partial_2^2\partial_1^2 u_1\|_{L^2}\leq C\|\partial_1^2 u_1\|_{L^2}.$$

Combining the estimates forM_{151},M_{152},M_{153}, we have

$$\|\partial_1 p\|_{L^2}\leq C\|\partial_1^2 u_1\|_{L^2}+C(\|u\|_{H^2}+\|b\|_{H^2})(\|\partial_2 u_1\|_{L^2}+\|\partial_1 u_2\|_{L^2}+\|\partial_1 b\|_{L^2}). \tag{3.20}$$

Putting (3.20) into (3.19) yields

$$M_{15}\leq C\|\partial_1^2 u_1\|_{L^2}+C\|\partial_2 b_2\|_{L^2}+C(\|u\|_{H^2}+\|b\|_{H^2})\left(\|\partial_2 u_1\|_{L^2}^2+\|\partial_1 u_2\|_{L^2}^2+\|\partial_1 b\|_{L^2}^2\right). \tag{3.21}$$

Therefore, it follows from (3.16)-(3.18) and (3.21) that

$$M_1 \le C\|\partial_1^2 u_1\|_{L^2} + C\|\partial_1 b\|_{L^2} + C(\|u\|_{H^2} + \|b\|_{H^2})(\|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_1 b\|_{L^2}^2). \quad (3.22)$$

By Höolder's and Young's inequalities, we arrive at

$$\begin{aligned} M_2 + M_3 &= \int \partial_1 u_2\, u \cdot \nabla b_2 dx - \int \partial_1 u_2\, b \cdot \nabla u_2 dx \\ &\le C(\|u\|_{L^\infty}\|\partial_1 u_2\|_{L^2}\|\nabla b_2\|_{L^2} + \|b\|_{L^\infty}\|\partial_1 u_2\|_{L^2}\|\nabla u_2\|_{L^2}) \\ &\le C(\|u\|_{H^2} + \|b\|_{H^2})(\|\partial_1 b\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2), \end{aligned} \quad (3.23)$$

where $\|\nabla b_2\|_{L^2} = \|\partial_1 b\|_{L^2}$. Similarly,

$$M_4 = -\int \partial_1 u_2\, \partial_1^2 b_2 dx \le C\|\partial_1^2 b_2\|_{L^2}^2 + \frac{1}{4}\|\partial_1 u_2\|_{L^2}^2. \quad (3.24)$$

Combining the estimates for (3.22)-(3.24) and integrating them into (3.15), it follows.

$$\begin{aligned} \int_0^t \|\partial_1 u_2\|_{L^2}^2 dt &\le C\|(u,b)(0)\|_{H^1}^2 + C\|(u,b)(\mathrm{t})\|_{H^1}^2 + C\int_0^t \|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 u_2\|_{H^1}^2 + \|\partial_1 b\|_{H^1}^2 dt \\ &\quad + \mathrm{C}\sup_{0\le\tau\le t}\|(u,b)(\tau)\|_{H^2}^2 \int_0^t \|\partial_2 u_1(\tau)\|_{L^2}^2 + \|\partial_1 b(\tau)\|_{L^2}^2 + \|\partial_1 u_2(\tau)\|_{L^2}^2 d\tau \\ &\le C\mathrm{E}_1(0) + C\mathrm{E}_1(\mathrm{t}) + C\mathrm{E}_1^{3/2}(\mathrm{t}) + C\mathrm{E}_2^{3/2}(\mathrm{t}). \end{aligned} \quad (3.25)$$

Similarly,

$$\begin{aligned} \int_0^t \|\nabla\partial_1 u_2\|_{L^2}^2 dt &\le C\|(u,b)(0)\|_{H^2}^2 + C\|(u,b)(\mathrm{t})\|_{H^2}^2 + C\int_0^t \|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 u_2\|_{H^1}^2 + \|\partial_1 b\|_{H^1}^2 dt \\ &\quad + \mathrm{C}\sup_{0\le\tau\le t}\|(u,b)(\tau)\|_{H^3}^2 \int_0^t \|\partial_2 u_1(\tau)\|_{H^1}^2 + \|\partial_1 b(\tau)\|_{H^1}^2 + \|\partial_1 u_2(\tau)\|_{H^1}^2 d\tau \\ &\le C\mathrm{E}_1(0) + C\mathrm{E}_1(\mathrm{t}) + C\mathrm{E}_1^{3/2}(\mathrm{t}) + C\mathrm{E}_2^{3/2}(\mathrm{t}). \end{aligned} \quad (3.26)$$

Next, applying $\partial_i^2\ (i = 1,2)$ to $(1.2)_2$ and dotting it with $\partial_i^2\partial_1 u_2$ in L^2, we can get that

$$\begin{aligned} \sum_{i=1}^2 \|\partial_i^2\partial_1 u_2\|_{L^2}^2 &= \sum_{i=1}^2 \int \partial_i^2\partial_1 u_2\, \partial_i^2\partial_t b_2 dx + \sum_{i=1}^2 \int \partial_i^2\partial_1 u_2\, \partial_i^2(u\cdot\nabla b_2)\,\mathrm{d}x \\ &\quad - \sum_{i=1}^2 \int \partial_i^2\partial_1 u_2\, \partial_i^2(b\cdot\nabla u_2)\,\mathrm{d}x + \sum_{i=1}^2 \int \partial_i^2\partial_1^2 b_2\, \partial_i^2\partial_1 u_2 dx \\ &:= \mathrm{N}_1 + \mathrm{N}_2 + \mathrm{N}_3 + \mathrm{N}_4. \end{aligned} \quad (3.27)$$

By virtue of the structure of the equation $(1.2)_1$ and integration by parts, we can get

$$N_1 = \frac{d}{dt}\sum_{i=1}^{2}\int \partial_i^2\partial_1 u_2 \partial_i^2 b_2 dx - \sum_{i=1}^{2}\int \partial_i^2 b_2 \partial_i^2 \partial_1(\partial_1 b_2 + b\cdot\nabla b_2 - u\cdot\nabla u_2 - \partial_2 p)\,\mathrm{d}x$$

$$= \frac{d}{dt}\sum_{i=1}^{2}\int \partial_i^2\partial_1 u_2 \partial_i^2 b_2 dx + \sum_{i=1}^{2}\int \partial_1\partial_i^2 b_2 \partial_i^2 \partial_1 b_2 dx + \sum_{i=1}^{2}\int \partial_1\partial_i^2 b_2 \partial_i^2 (b\cdot\nabla b_2)\,\mathrm{d}x$$

$$- \sum_{i=1}^{2}\int \partial_1\partial_i^2 b_2 \partial_i^2 (u\cdot\nabla u_2)\,\mathrm{d}x - \sum_{i=1}^{2}\int \partial_1\partial_i^2 b_2 \partial_i^2 \partial_2 p dx$$

$$:= \mathrm{N}_{11}+\mathrm{N}_{12}+\mathrm{N}_{13}+\mathrm{N}_{14}+\mathrm{N}_{15},$$

Where

$$\textstyle\sum_{i=1}^{2}\int \partial_1\partial_i^2 b_2 \partial_i^2\partial_1 b_2 dx \le C\sum_{i=1}^{2}\|\partial_i^2\partial_1 b_2\|_{L^2}^2. \tag{3.28}$$

It follows from Sobolev's and H$\ddot{o}o$lder's inequalities that

$$N_{13} + N_{14} = \sum_{i=1}^{2}\int \partial_i^2\partial_1 b_2 \partial_i^2 (b\cdot\nabla b_2)\,\mathrm{d}x - \sum_{i=1}^{2}\int \partial_i^2\partial_1 b_2 \partial_i^2 (u\cdot\nabla u_2)\,\mathrm{d}x$$

$$= \sum_{i=1}^{2}\int \partial_i^2\partial_1 b_2 (\partial_i^2 b\cdot\nabla b_2 + 2\partial_i b\cdot\nabla\partial_i b_2 + b\cdot\nabla\partial_i^2 b_2)\,\mathrm{d}x$$

$$- \sum_{i=1}^{2}\int \partial_i^2\partial_1 b_2 (\partial_i^2 u\cdot\nabla u_2 + 2\partial_i u\cdot\nabla\partial_i u_2 + u\cdot\nabla\partial_i^2 u_2)\,\mathrm{d}x$$

$$\le C\|\partial_i^2\partial_1 b_2\|_{L^2}\big(\|\nabla b_2\|_{L^\infty}\|\partial_i^2 b\|_{L^2} + \|\partial_i b\|_{L^\infty}\|\nabla\partial_i b_2\|_{L^2} + \|b\|_{L^\infty}\|\nabla\partial_i^2 b_2\|_{L^2}\big)$$

$$+C\|\partial_i^2\partial_1 b_2\|_{L^2}\big(\|\nabla u_2\|_{L^\infty}\|\partial_i^2 u\|_{L^2} + \|\partial_i u\|_{L^\infty}\|\nabla\partial_i u_2\|_{L^2} + \|u\|_{L^\infty}\|\nabla\partial_i^2 u_2\|_{L^2}\big)$$

$$\le C\|(u,b)\|_{H^3}\big(\|\partial_1 b\|_{H^2}^2 + \|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2\big), \tag{3.29}$$

Where $\|\nabla b_2\|_{L^2} = \|\partial_1 b\|_{L^2}$. Now, we need to estimate N_{15},

$$N_{15} = \int \partial_i^2\partial_1 b_2 \partial_i^2\partial_2 p dx \le C\|\partial_i^2\partial_1 b_2\|_{L^2}\|\partial_i^2\partial_2 p\|_{L^2},$$

$$\|\partial_i^2\partial_2 p\|_{L^2} \le C\|(-\Delta)^{-1}\nabla\cdot\partial_2\partial_i^2(u\cdot\nabla u)\|_{L^2} + C\|(-\Delta)^{-1}\nabla\cdot\partial_2\partial_i^2(b\cdot\nabla b)\|_{L^2} + C\|(-\Delta)^{-1}\partial_i^2\partial_1\partial_2 u_1\|_{L^2}.$$

The following simple facts hold:

$$\partial_2\nabla\cdot(u\cdot\nabla u)=4\,\partial_2^2u_2\,\partial_2u_2+2\,\partial_2\,\partial_1u_2\,\partial_2u_1+2\,\partial_1u_2\,\partial_2^2u_1.$$

Due to the boundedness of the Riesz operator $\partial_i(-\Delta)^{-1/2}$ with $i=1,2$ in L^2, then

$$\|(-\Delta)^{-1}\nabla\cdot\partial_2\,\partial_i^2\,(u\cdot\nabla u)\|_{L^2}$$

$$\leq C\|\,\partial_2^2u_2\,\partial_2u_2\|_{L^2}+C\|\,\partial_2\,\partial_1u_2\,\partial_2u_1\|_{L^2}+C\|\,\partial_1u_2\,\partial_2^2u_1\|_{L^2}$$

$$\leq C\|\,\partial_2u\|_{L^\infty}(\|\,\partial_2^2u_2\|_{L^2}+\|\,\partial_1\,\partial_2u_2\|_{L^2})+\|\,\partial_1u_2\|_{L^\infty}\|\,\partial_2^2u_1\|_{L^2}$$

$$\leq C\|u\|_{H^3}\,(\|\,\partial_2u_1\|_{H^1}+\|\,\partial_1u_2\|_{H^1}).$$

And

$$\|(-\Delta)^{-1}\nabla\cdot\partial_2\,\partial_i^2(b\cdot\nabla b)\|_{L^2}$$

$$\leq C\|\,\partial_2^2b_2\,\partial_2b_2\|_{L^2}+C\|\,\partial_2\,\partial_1b_2\,\partial_2b_1\|_{L^2}+C\|\,\partial_1b_2\,\partial_2^2b_1\|_{L^2}$$

$$\leq C\|\,\partial_2b\|_{L^\infty}(\|\,\partial_2^2b_2\|_{L^2}+\|\,\partial_1\,\partial_2b_2\|_{L^2})+\|\,\partial_1b_2\|_{L^\infty}\|\,\partial_2^2b_1\|_{L^2}$$

$$\leq C\|b\|_{H^3}\,\|\partial_1b\|_{H^2}.$$

Obviously,

$$\|(-\Delta)^{-1}\partial_1\partial_i^2\,\partial_2\,u_1\|_{L^2}\leq C\|\partial_1\,\partial_2u_1\|_{L^2}$$

Therefore,

$$N_{15}\leq C(\|\,\partial_2u_1\|_{H^2}^2+\|\,\partial_i^2\,\partial_1b_2\|_{L^2}^2)+C\|(u,b)\|_{H^3}\,(\|\,\partial_1u_2\|_{H^1}^2+\|\,\partial_2u_1\|_{H^1}^2+\|\,\partial_1b\|_{H^2}^2).\tag{3.30}$$

Combining the (3.28)-(3.30), one has

$$N_1\leq\frac{d}{dt}\int\partial_i^2\,\partial_1u_2\partial_i^2\,b_2\,dx+C(\|\partial_2u_1\|_{H^1}^2+\|\partial_i^2\,\partial_1b_2\|_{L^2}^2)$$

$$+C\|(u,b)\|_{H^3}\,(\|\,\partial_1u_2\|_{H^1}^2+\|u_1\|_{H^3}^2+\|\,\partial_1b\|_{H^2}^2).$$

It follows from Sobolev's and H.$\ddot{o}$older's inequalities that

$$N_2+N_3=\sum_{i=1}^{2}\int\partial_i^2\partial_1u_2\partial_i^2(u\cdot\nabla b_2)\,\mathrm{d}x-\sum_{i=1}^{2}\int\partial_i^2\partial_1u_2\partial_i^2(b\cdot\nabla u_2)\,\mathrm{d}x$$

$$=\sum_{i=1}^{2}\int\partial_i^2\partial_1u_2(\partial_i^2u\cdot\nabla b_2+2\partial_iu\cdot\nabla\partial_ib_2+u\cdot\nabla\partial_i^2b_2)\,\mathrm{d}x$$

$$-\sum_{i=1}^{2}\int\partial_i^2\partial_1u_2(\partial_i^2b\cdot\nabla u_2+2\partial_ib\cdot\nabla\partial_iu_2+b\cdot\nabla\partial_i^2u_2)\,\mathrm{d}x$$

$$\leq C\|\,\partial_i^2\,\partial_1u_2\|_{L^2}(\|\nabla b_2\|_{L^\infty}\|\,\partial_i^2u\|_{L^2}+\|\,\partial_iu\|_{L^\infty}\|\nabla\,\partial_ib_2\|_{L^2}+\|u\|_{L^\infty}\|\nabla\partial_i^2b_2\|_{L^2})$$

$$+C\|\partial_i^2\partial_1 u_2\|_{L^2}(\|\nabla u_2\|_{L^\infty}\|\partial_i^2 b\|_{L^2}+\|\partial_i b\|_{L^\infty}\|\nabla\partial_i u_2\|_{L^2}+\|b\|_{L^\infty}\|\nabla\partial_i^2 u_2\|_{L^2})$$

$$\le C\|(u,b)\|_{H^3}(\|\partial_1 b\|_{H^2}^2+\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2).$$

In a similar manner,

$$N_4=\int\partial_i^2\partial_1 u_2\partial_i^2 b_2 dx\le\frac{1}{4}\|\partial_i^2\partial_1 u_2\|_{L^2}+\|\partial_i^2 b_2\|_{L^2},$$

Where $\|\partial_i^2 b_2\|_{L^2}^2=\|\partial_1^2 b\|_{L^2}^2$. Combining the estimates for N_1, N_2, N_3, and N_4, and integrating it over $[0,t]$, it follows.

$$\sum_{i=1}^{2}\int_0^t\|\partial_i^2\partial_1 u_2\|_{L^2}^2 dt\le C\|(u_0,b_0)\|_{H^3}^2+C\|(u,b)\|_{H^3}^2+C\int_0^t(\|\partial_2 u_1\|_{H^1}^2+\|\partial_1 b\|_{H^2}^2)\,dt$$

$$+C\sup_{0\le t\le t}\|(u,b)\|_{H^3}\int_0^t(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^2}^2)\,dt$$

$$\le C\mathscr{E}_1(0)+C\mathscr{E}_1(t)+C\mathscr{E}_1^{3/2}(t)+C\mathscr{E}_2^{3/2}(t). \tag{3.31}$$

Combining (3.25), (3.26), and (3.31), thus

$$\int_0^t\|\partial_1 u_2\|_{H^2}^2 dt\le C\|(u_0,b_0)\|_{H^3}^2+C\|(u,b)\|_{H^3}^2+C\int_0^t(\|\partial_2 u_1\|_{H^1}^2+\|\partial_1 b\|_{H^2}^2)\,dt$$

$$+C\sup_{0\le t\le t}\|(u,b)\|_{H^3}\int_0^t(\|\partial_1 u_2\|_{H^2}^2+\|\partial_2 u_1\|_{H^2}^2+\|\partial_1 b\|_{H^3}^2)\,dt$$

$$\le C\mathscr{E}_1(0)+C\mathscr{E}_1(t)+C\mathscr{E}_1^{3/2}(t)+C\mathscr{E}_2^{3/2}(t).$$

The proof of Lemma 3.2 is complete.

3.3 Global Well-Posedness

Now, the bootstrap argument is applied to prove (1.3). By the proof of Lemma 3.1 and Lemma 3.2, we can get that

$$\mathscr{E}_1(t)\le C\mathscr{E}_1(0)+C\mathscr{E}_1^{3/2}(t)+C\mathscr{E}_2^{3/2}(t), \tag{3.32}$$

$$\boldsymbol{\mathcal{E}_2(t)\le C\mathcal{E}_1(0)+C\mathcal{E}_1(t)+C\mathcal{E}_1^{3/2}(t)+C\mathcal{E}_2^{3/2}(t).} \tag{3.33}$$

For any $t>0$, adding (3.33) to (3.32) by the appropriate constant obtains,

$$\mathscr{E}(t)\le C\mathscr{E}(0)+C\mathscr{E}^{3/2}(t), \tag{3.34}$$

where $\mathcal{E}(t)=\mathcal{E}_1(t)+\mathcal{E}_2(t)$, and $C>0$ is a pure constant. Let

$$\|(u_0,b_0)\|_{H^3}^2\le\frac{1}{16C^3}.$$

It follows from (3.34) that.

$$\mathscr{E}(t)\le C\mathscr{E}(0)+C\mathscr{E}^{1/2}(t)\mathscr{E}(t)\le C\mathscr{E}(0)+C\frac{1}{2C}\mathscr{E}(t)=C\mathscr{E}(0)+\frac{1}{2}\mathscr{E}(t),$$

then

$$\boldsymbol{\mathcal{E}(t) \leq 2C\mathcal{E}(0)}.$$

The bootstrapping argument then implies that, for any $t > 0$,

$$\mathscr{E}(t) \leq \frac{1}{8C^2}.$$

3.4 Large-time Behavior

Now let's prove (1.4). First of all, applying ∇ to $(1.2)_1$ and dotting it with ∇u in L^2, meanwhile applying ∂_1 to $(1.2)_2$ and dotting it with $\partial_1 b$ in L^2, it follows.

$$\frac{1}{2}\frac{d}{dt}\|(\nabla u,\partial_1 b)\|_{L^2}^2+\|\nabla\partial_2 u_1\|_{L^2}^2+\|\partial_1^2 b\|_{L^2}^2$$

$$=\sum_{i=1}^{2}\int \partial_i\,(b\cdot\nabla b)\cdot\partial_i u dx-\sum_{i=1}^{2}\int \partial_1\,(u\cdot\nabla b)\cdot\partial_1 b dx$$

$$+\sum_{i=1}^{2}\int \partial_1\,(b\cdot\nabla u)\cdot\partial_1 b dx+\sum_{i=1}^{2}\int \partial_i\,\partial_1 b\cdot\partial_i u dx+\sum_{i=1}^{2}\int \partial_1^2 u\cdot\partial_1 b dx$$

$$:=\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}_3+\mathrm{P}_4+\mathrm{P}_5, \tag{3.35}$$

where

$$\textstyle\sum_{i=1}^{2}\int \partial_i\,(u\cdot\nabla u)\cdot\partial_i u dx=0.$$

It follows from Sobolev's and H.$\ddot{o}$older's inequalities that

$$P_1+P_2+P_3\leq C\big(\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2+\|\nabla u\|_{L^\infty}\|b\|_{L^2}\|\partial_i^2 b\|_{L^2}$$

$$+\|\partial_1 b\|_{L^\infty}\|\partial_1 u\|_{L^2}\|\nabla b\|_{L^2}+\|\partial_1 b\|_{L^\infty}\|u\|_{L^2}\|\partial_1\partial_i b\|_{L^2}$$

$$+\|\partial_1 b\|_{L^\infty}\|\partial_1 b\|_{L^2}\|\nabla u\|_{L^2}+\|\partial_1 b\|_{L^\infty}\|b\|_{L^2}\|\partial_1\partial_i u\|_{L^2}$$

$$\leq C(\|b\|_{H^3}+\|u\|_{H^3})(\|u\|_{H^2}^2+\|b\|_{H^2}^2). \tag{3.36}$$

Similarly

$$P_4+P_5\leq C(\|u\|_{H^2}^2+\|b\|_{H^2}^2). \tag{3.37}$$

Inserting (3.36) and (3.37) into (3.35), it follows that

$$\frac{1}{2}\frac{d}{dt}\|(\nabla u,\partial_1 b)\|_{L^2}^2+\|\nabla u_1\|_{L^2}^2+\|\partial_1^2 b\|_{L^2}^2\leq C(\|b\|_{H^3}+\|u\|_{H^3}+1)(\|u\|_{H^2}^2+\|b\|_{H^2}^2). \tag{3.38}$$

Combining (1.3) with (3.38) and integrating it over $[s,t]$, we can get

$$\|(\nabla u(t), \partial_1 b(t))\|_{L^2}^2 - \|(\nabla u(s), \partial_1 b(s))\|_{L^2}^2 \le C(t-s). \tag{3.39}$$

We next verify that $\|(\nabla u(t), \partial_1 b(t))\|_{L^2}^2$ satisfies the integrable condition,

$$\int_0^\infty \|(\nabla u(t), \partial_1 b(t))\|_{L^2}^2 \, d\tau \le C.$$

By Hööolder's and Young's inequalities and integration by parts, one can obtain

$$\|\partial_1 u_1\|_{L^2}^2 = -\int \partial_2 u_2 \, \partial_1 u_1 dx = -\int \partial_1 u_2 \, \partial_2 u_1 dx$$

$$\le \frac{1}{2}(\|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2)$$

which implies

$$\|\nabla u\|_{L^2}^2 \le 2(\|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 u_2\|_{L^2}^2). \tag{3.40}$$

Invoking (1.3) and (3.40), it follows

$$\int_0^\infty \|(\nabla u(t), \partial_1 b(t))\|_{L^2}^2 \, dt \le C. \tag{3.41}$$

Next, applying ∂_i^2 to $(1.2)_1$ and dotting it with $\partial_i^2 u$ in L^2, meanwhile applying $\partial_i \partial_1$ to $(1.2)_2$ and dotting it with $\partial_i \partial_1 b$ in L^2, it follows.

$$= \sum_{i=1}^2 \int \partial_i^2 (b \cdot \nabla b) \cdot \partial_i^2 u dx - \sum_{i=1}^2 \int \partial_i \partial_1 (u \cdot \nabla b) \cdot \partial_i \partial_1 b dx - \sum_{i=1}^2 \int \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 u dx$$

$$+ \sum_{i=1}^2 \int \partial_i \partial_1 (b \cdot \nabla u) \cdot \partial_i \partial_1 b dx + \sum_{i=1}^2 \int \partial_i^2 \partial_1 b \cdot \partial_i^2 u dx + \sum_{i=1}^2 \int \partial_i \partial_1^2 u \cdot \partial_i \partial_1 b dx$$

$$:= H_1 + H_2 + H_3 + H_4 + H_5 + H_6. \tag{3.42}$$

By Hööolder's and Young's inequalities, one has

$$H_1 + H_2 + H_3 \le C(\|\nabla b\|_{L^\infty} \|\nabla^2 b\|_{L^2} \|\nabla^2 u\|_{L^2} + \|b\|_{L^\infty} \|\nabla^3 b\|_{L^2} \|\nabla^2 u\|_{L^2}$$

$$+ \|\nabla u\|_{L^\infty} \|(\nabla^2 b, \nabla^2 u)\|_{L^2}^2 + \|u\|_{L^\infty} \|\nabla^3 b\|_{L^2} \|\nabla^2 b\|_{L^2} + \|u\|_{L^\infty} \|\nabla^3 u\|_{L^2} \|\nabla^2 u\|_{L^2})$$

$$\le C(\|b\|_{H^3} + \|u\|_{H^3})(\|u\|_{H^2}^2 + \|b\|_{H^3}^2). \tag{3.43}$$

Similarly

$$H_4 + H_5 + H_6 \le C(\|u\|_{H^3}^2 + \|b\|_{H^3}^2 + 1)(\|u\|_{H^3}^2 + \|b\|_{H^3}^2). \tag{3.44}$$

Inserting (3.43) and (3.44) in (3.42), it follows that

$$\frac{1}{2}\frac{d}{dt}\|(\partial_i^2 u, \partial_i \partial_1 b)\|_{L^2}^2 + \|\partial_i^2 u_1\|_{L^2}^2 + \|\partial_i \partial_1^2 b\|_{L^2}^2 \le C(\|u\|_{H^3}^2 + \|b\|_{H^3}^2 + 1)(\|u\|_{H^3}^2 + \|b\|_{H^3}^2). \tag{3.45}$$

Combining (1.3) with (3.45) and integrating it over $[s, t]$, we can get

$$\|(\partial_i^2 u(t), \partial_i \partial_1 b(t))\|_{L^2}^2 - \|(\partial_i^2 u(s), \partial_i \partial_1 b(s))\|_{L^2}^2 \le C(t-s). \tag{3.46}$$

It follows from (3.40) that

$$\|\nabla u\|_{H^1}^2 \le 2\left(\|u_1\|_{H^2}^2 + \|\partial_1 u_2\|_{H^1}^2\right). \tag{3.47}$$

Invoking (1.3) and (3.47), one has

$$\int_0^\infty \|(\partial_i^2 u(t), \partial_i \partial_1 b(t))\|_{L^2}^2 \, d\tau \le C. \tag{3.48}$$

Combining (3.39), (3.41), (3.46), and (3.48), we can arrive at

$$\|(\nabla u(t), \partial_1 b(t))\|_{H^1}^2 \to 0 \quad as \quad t \to \infty.$$

This completed the proof of Theorem 1.1.

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