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Original Article

Domination in Cubic Circulant Graphs

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Abstract *- A Cayley graph is a graph constructed out of a group Γ and its generating set A. In this paper, the domination number of Cay(Z_n, A) = Cir(n, A), for the generating set A with cardinality three is determined.*

Keywords *- Cayley graphs, Domination number, Dominating set, Gamma set.*

2000 Mathematics Subject Classification: 05C.

1. Introduction

Cayley graphs are useful in routing problems of parallel computing. Circulant graphs are Cayley graphs. Tamizh Chelvam and Rani[2,3,4] obtained the domination number, independent domination number, total and connected domination numbers for circulant graphs with respect to a class of generating sets of Z_n.

1.1. Foundational Surveys and Classical Bounds

The research on domination in graphs has its roots in early work on combinatorial optimization and network theory. Plummer's survey (2006)[6] specifies upper bounds on the domination number of cubic graphs and deals with related invariants such as toughness and matchings. Reed (1996)[7] showed that $\gamma(G) \leq \frac{3n}{8}$for each connected cubic graph G of order n, and conjectured $\gamma(G) \leq ceil\left(\frac{n}{3}\right)$. This was later disproved by Kostochka & Stodolsky[8], who produced infinite families of cubic graphs with $\gamma(G) \sim \left(\frac{1}{3}+\frac{1}{69}\right)n$.

1.2. Bounds and Conjectures under Girth and Bipartiteness Constraints

In order to make a breakthrough towards the $\frac{n}{3}$ barrier, further structural assumptions have been made. Löwenstein & Rautenbach (2008)[9] showed that cubic graphs of at least girth 83 satisfy $\gamma(G) \leq \frac{n}{3}$. The two best-known "one-third conjectures" ensued: Verstraete's conjecture claims the same bound for girth ≥6, and Kostochka's conjecture asserts $\gamma(G) \leq \frac{n}{3}$ for every cubic bipartite graph. Recent developments by Dorbec & Henning (2024)[10] confirm Verstraete's conjecture in the absence of 7- and 8-cycles and sets Kostochka's bound for bipartite cubic graphs without 4- or 8-cycles.

1.3. Connections with Total Domination and Irredundance

Variants of domination have been considered in parallel. Henning & Yeo (2017)[11], showed that $\frac{\gamma_t(G)}{\gamma(G)} \leq 2$ for every cubic graph and also stated the conditions for equality.

1.4. Advanced Variants and Recent Structural Results

Sheng & Lu (2020)[12] settled the Goddard–Henning conjecture by finding paired-domination number for all cubic graphs.

For large-girth graphs, Joos et al.[13] achieved the bound of $\gamma(G) \leq 0.29987n + O\left(\frac{n}{g}\right)$.

Most recently, new decomposition theorems (2025)[14]capture the internal structure of minimum dominating sets in any cubic graph, opening the door to further structural insight.

1.5. Open Problems and Future Directions

Even with these advances, key questions remain open:

1. General one-third conjecture: Is $\gamma(G) \leq \frac{n}{3}$ true for every cubic graph?
2. Extremal characterizations: Which structures have equality in bounds for total and paired domination?
3. General classes: Can the new structural theorems be generalized to irregular or directed cubic networks?

Future work will likely combine combinatorial constructions, probabilistic methods, and computation experiments to bridge these gaps and improve domination bounds in cubic graphs.

For the concept of domination and Cayley graph one may refer[1,5]. Throughout this paper, the generating set A of Z_n taken as A $=\left\{a, \frac{n}{2}, n-a\right\}$ where $1 \leq a \leq n-1$. Since the elements of A are arbitrary, A need not be a generating set of Z_n. It is sometimes referred to as the Cayley subset of Z_n. In view of this, the circulant graph under consideration need not be connected. Addition is performed modulo n.

Theorem 1.1. [1]For any graph G with n vertices, $ceil\left(\frac{n}{1+\Delta(G)}\right) \leq \gamma(G) \leq n-\Delta(G)$ where $\Delta(G)$ is the maximum degree of G.

2. Main Results

In this section, the domination number and upper bound of the domination number of a class of circulant graphs Cir(n, A) is obtained.

Theorem 2.1. Let n, k and a be positive integers such that $n \neq 8k$, n is even and with $1 \leq a \leq floor\left(\frac{n-1}{2}\right)$ be the generating set with $\gcd\left(a, \frac{n}{2}, n-a\right) = 1$ and G=Cir(n, A). Then $\gamma(G) = ceil\left(\frac{n}{4}\right)$.

Proof. Let $A = \left\{a, \frac{n}{2}, n-a\right\}$ and D=$\{0, s, 2s, \cdots (\ell-1)s\}$ where $\ell = ceil\left(\frac{n}{4}\right)$ and $s = \frac{n}{2} + 2a$. Note that some of the elements of D exceed n and hence addition modulo n is operated.

Claim D is a dominating set. To prove that $V = N[D]$. As it is always true that $N[D]N[D] \subseteq V$, it is enough to prove that $V \subseteq N[D]$. Let $v \in V$. Any element $v \in V$ could be written as $v = \left(\frac{n}{2} + 2a\right) i + j$ where $0 \leq i \leq (\ell-1)$ and j=0,a,$\frac{n}{2}$,n-a. When $v = \left(\frac{n}{2} + 2a\right) i + j$ where $0 \leq i \leq (\ell-1)$ and j=a,$\frac{n}{2}$,n-a then $v \in N(D)$ and when j=0, $v \in D$.

Hence $V = N[D]$ and so D is a dominating set. Also $|D| = \ell = ceil\left(\frac{n}{4}\right)$. Hence $\gamma(G) \leq ceil\left(\frac{n}{4}\right)$.

By Theorem 1.1 [1], we have $\gamma(G) \geq ceil\left(\frac{n}{4}\right)$. Therefore $\gamma(G) = ceil\left(\frac{n}{4}\right)$.

Example 2.2. Suppose n = 42 and A = {5, 21, 37}, then $\ell = ceil\left(\frac{42}{4}\right)$=11
$s = \frac{n}{2} + 2a = 31$ and γ–set, D = {0, 31, 20, 9, 40, 29, 18, 7, 38, 27, 16}.

Theorem 2.3. Let n, k, a be positive integers such that n = 8k and A $=\left\{a, \frac{n}{2}, n-a\right\}$ with $1 \leq a \leq floor\left(\frac{n-1}{2}\right)$be the generating set with g.c.d$\left(a, \frac{n}{2}, n-a\right) = 1$.

Proof. Let A $=\left\{a, \frac{n}{2}, n-a\right\}$ and D = $\{0, s, 2s \cdots \ell s\}$ where $\ell = ceil\left(\frac{n}{4}\right)$ and $s = \frac{n}{2} + 2a$

Claim D is a dominating set

Consider $D_1 = \{0, s, 2s \cdots (\ell-1)s\}$. Then the element $\left(\frac{n}{2} - 1\right) a$ is not dominated by D_1 . Let D $=D_1 \cup \{\ell s\}$. Since $\ell s = 2k(4k + 2a) = k(8k + 4a) = n + 4ak = 4ak = \frac{n}{2} a$, we have $\left(\frac{n}{2} - 1\right) a$ is dominated by ℓs . Therefore D is a dominating set. Also $|D| = \ell + 1 = ceil\left(\frac{n}{4}\right) + 1$ Hence $\gamma(G) \leq ceil\left(\frac{n}{4}\right) + 1$.

Example 2.4. Suppose n = 56 and A = {9, 28, 47},then $\ell = ceil\left(\frac{56}{4}\right) = 14$ and γ–set, D = {0, 46, 36, 26, 16, 6, 52, 42, 32, 22, 12, 2, 48, 38, 28}.

Theorem 2.5. Let n, k and a be positive integers such that $n \neq 8k$, n is even with $\frac{n}{2}$ is not a prime number and A = $\left\{a, \frac{n}{2}, n-a\right\}$ with $1 \leq a \leq floor\left(\frac{n-1}{2}\right)$ and g.c.d $\left(a, \frac{n}{2}, n-a\right) = g \neq 1$. Suppose G=Cir(n,A) then $\gamma(G) \leq g\left(ceil\left(\frac{n}{4g}\right)\right)$.

Proof. Suppose A = $\left\{a, \frac{n}{2}, n-a\right\}$with $1 \leq a \leq floor\left(\frac{n-1}{2}\right)$ and g.c.d.(a,$\frac{n}{2}$, n − a) = g $\neq$ 1. Then G is a disconnected graph with g components. Let the components are $G_0, G_1, \cdots, G_{g-1}$. The component G_0 has vertex set $V_0 = N[A \cup \{0\}]$ and is connected. Similar to the proof of Theorem 2.1, one can prove that $D_0 = \{0, s, 2s, \cdots (\ell-1)s\}$ where $\ell = ceil\left(\frac{n}{4g}\right)$and $s = \frac{n}{2} + 2a$ is a dominating set with $|D| = \ell$. Hence $\gamma(G_0) \leq \ell$. Also G_i, $1 \leq i \leq g-1$ is a connected component of G with vertex set V$V_i = N[V_0 + i] = N[(A+i) \cup \{i\}]$ and $D_i = D_0 + i = \{i, s+i, 2s+i, \cdots, (\ell-1)s+i\}$ as a dominating set with $|D_i| = \ell$ and hence $\gamma(G_i) \leq \ell$. Therefore $\gamma(G) = \gamma(G_0) + \gamma(G_1) + \cdots + \gamma(G_{g-1}) \leq g\ell = g\left(ceil\left(\frac{n}{4g}\right)\right)$.

Example 2.6. Suppose n = 20 and A = {6, 10, 14}, then g.c.d.(6, 10, 14) = g = 2 and $\ell = ceil\left(\frac{20}{8}\right) = 3$. Hence there are two components with one component having V_0 = {0, 2, 4, 6, 8, 10, 12, 14, 16, 18} with $s = \frac{n}{2} + 12 = 2$ and $\gamma -$ set, D_0= {0, 2, 4} and the other component having V_1 ={1, 3, 5, 7, 9, 11, 13, 15, 17, 19} with $\gamma -$ set, D_1 = {1, 3, 5}. Therefore the $\gamma -$ set of G is $D = D_0 \cup D_1 = \{0,1,2,3,4,5\}$ with $\gamma(G) = g\ell = g\left(ceil\left(\frac{n}{4g}\right)\right) = 6$.

Similarly one can prove the following theorem.

Theorem 2.7. Let n, k and a be positive integers such that n = 8k and A = $\left\{a, \frac{n}{2}, n-a\right\}$with $1 \leq a \leq floor\left(\frac{n-1}{2}\right)$and g.c.d.(a,$\frac{n}{2}$, n − a) = g $\neq$1.

Remark 2.8. In Theorem 2.7, one can find that for a fixed value of n, depending on the set A, the value of $\gamma(G) = g\left(ceil\left(\frac{n}{4g}\right)\right)$ or $g\left(ceil\left(\frac{n}{4g}\right) + 1\right)$, which is clear from the following example.

Example 2.9. Suppose n = 24 and A = {3, 12, 21}, then g.c.d.(3, 12, 21) = g = 3 and $\ell = ceil\left(\frac{24}{12}\right) = 2$. Hence there are three components with the first component having V_0 = {0, 3, 6, 9, 12, 15, 18, 21} with $s = \frac{n}{2} + 6 = 18$ and $\gamma -$ set, D_0= {0, 18, 12}; the second component having V_1 ={1, 4, 7, 10, 13, 16, 19, 22} with $\gamma -$ set, D_1 = {1, 19, 13}; the third component having V_2= {2, 5, 8, 11, 14, 17, 20, 23} with $\gamma -$ set, D_2= {2, 20, 14}. Therefore the $\gamma -$ set of G is$D = D_0 \cup D_1 \cup D_2 = \{0,1,2,12,13,14,18,19,20\}$ with $\gamma(G) = g(\ell+1) = g\left(ceil\left(\frac{n}{4g}\right) + 1\right) = 9$.

Suppose n = 24 and A = {4, 12, 20}, then g.c.d.(4, 12, 20) = 4 and $\ell = ceil\left(\frac{24}{16}\right) = 2$.

Hence there are four components with the first component having V_0= {0, 4, 8, 12, 16, 20} with $s = \frac{n}{2} + 8 = 20$ and $\gamma -$ set, D_0= {0, 20};
the second component having V_1= {1, 5, 9, 13, 17, 21} with $\gamma -$ set, D_1 = {1, 21}; the third component having V_2 = {2, 6, 10, 14, 18, 22} with $\gamma -$ set, D_2 = {2, 22} and the fourth component having V_3= {3, 7, 11, 15, 19, 23} with $\gamma -$ set, D_3 = {3, 23}. Therefore the $\gamma -$ set of G is $D = D_0 \cup D_1 \cup D_2 \cup D_3 = \{0,1,2,3,20,21,22,23\}$ with $\gamma(G) = g\ell = g\left(ceil\left(\frac{n}{4g}\right)\right) = 8$.

Remark 2.10. Suppose n = 4k, for some positive integer k and A =$\left\{\frac{n}{4}, \frac{n}{2}, \frac{3n}{4}\right\}$, then g.c.d.$\left(\frac{n}{4}, \frac{n}{2}, \frac{3n}{4}\right) = \frac{n}{4}$. The graph has $\frac{n}{4}$ components and $\gamma -$ set for each component will be {0}, {1}, . . . , {$\frac{n}{4} - 1$} respectively. Hence $\gamma -$ set of G = {0} ∪ {1} ∪ . . . {$\frac{n}{4} - 1$}. Hence $\gamma(G) = \frac{n}{4}$.

Remark 2.11. Whenever D is a $\gamma -$ set of G, D + m, where $1 \leq m \leq n-1$ is also a $\gamma -$ set of G.

Remark 2.12. Let n, k be positive integers, n$\neq$ 8k and $\frac{n}{2}$ is a prime number, A = $\{1, \frac{n}{2}, n-1\}$ be the generating set for G = Cir(n, A) and D be a $\gamma -$ set of G. Then for any positive integer a which is relatively prime to n, aA is the generating set for G_a = Cir(n, aA) and aD is a $\gamma -$ set of Ga.

Remark 2.13. Let n be a positive integer, A = $\{1, n, n-1\}$ be the generating set for G = Cir(n, A) and D be a $\gamma -$ set of G. Then for any positive integer a, such that g.c.d.(a, n , n − a) = 1, aA is the generating set for G_a = Cir(n, aA) and aD is a $\gamma -$ set of G_a.

3. Conclusion

In this paper the domination number for the cubic circulant graphs is found. Work has been initiated to find the domination number of circulant graphs for higher odd order generating sets.

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