International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V71I8P101
Volume 71 Issue 8, 1-8, August 2025

Original Article

On Wavelet Approximation of Functions Belonging to Generalized Lipschitz Class Using Haar Scaling Function

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Received: 09 June 2025 Revised: 15 July 2025 Accepted: 01 August 2025 Published: 13 August 2025

Abstract - *The degree of approximation of functions belonging to certain classes by using wavelet techniques is quite interesting in the present scenario. People working in this direction have used the Haar wavelet method in their investigations. But no work seems to have been done so far to find an approximation of functions* $f\epsilon Lip_{\alpha}^{p,\psi}$ *to find the degree of approximation using the Haar wavelet method. Therefore, in this paper, two new theorems on wavelet approximation of the functions,* $p_{\alpha}^{p,\psi}$, $0 < \alpha \leq 1, 1 \leq p < \infty$, *have been established.*

Keywords - *Generalized Lipschitz class, Haar Wavelet, Multiresolution Analysis, Wavelet approximation.*

2010Mathematics Classification - *40A30, 42C15, 42C40, 65T60.*

1. Introduction

In 1909, the Hungarian mathematician Alfréd Haar presented the Haar wavelet, the first and most basic example of a wavelet function. Initially, it was suggested as an orthonormal system for the space of square-integrable functions $L^2[0,1]$. A fundamental concept in the development of the modern wavelet theory, particularly in signal and image processing, is the Haar wavelet. The Haar wavelet may be expressed mathematically as a piecewise constant function defined on the interval $L^2[0,1]$. The Haar wavelet is a simple but powerful tool that can effectively represent functions with discontinuities because it is orthogonal and has compact support. Multiresolution Analysis (MRA) is essential because it allows the examination of signals or functions at different levels of detail. The Haar wavelet's computational efficiency makes it popular for real-world applications, including edge detection, picture processing, data compression, and numerical differential equation solutions. For the last few decades, a number of researchers have been working on wavelet methods to investigate the character of functions belonging to different classes. The combination of multiresolution analysis and wavelet methods makes the investigation very appropriate and much closer to the result. However, many researchers have estimated the degree of approximation of certain functions by using Summability methods in the past years.

Nowadays, people estimate the degree of approximation of functions belonging to different classes by using wavelet analysis. The researchers like Morlet et al [7], Natanson [6], Win [9], Meyar [10], Iqbal et al. [12], and Khanna [13] have studied the wavelet approximation. Later on, Lal and Kumar [8] determined the degree of approximation of functions belonging to the generalized Lipschitz class using the Haar scaling function. The present investigation is a generalization of their work in a more different manner. The results of Lal & Kumar [8] may be derived from the present investigations.

2. Definitions and Notations

2.1. Function of Lipschitz Class

A function $f\epsilon Lip_{\alpha}^{p,\psi}$ if

$$\left|f(x+t)-f(x)\right| = O\left(|t|^{\alpha}\psi(t)\right)\Big|, \text{for } 0<\alpha\leq 1.$$

A function $f\epsilon Lip_{\alpha}^{p,\psi}$ if

$$\left(\int_0^1 f(x+t)-f(x)||^p\,dx\right)^{\frac{1}{2}} = O\left(|t|^{\alpha}\psi(t)\right)$$

where ψ is a positive monotonic increasing function of t such that $|t|^{\alpha}\psi(|t|) \to 0 \; as \; t \to 0^+$.

2.2. Multiresolution Analysis (MRA)

A multiresolution analysis of $L^2(\mathbb{R})$ consists of a nested sequence of a subspace $V_j \subset V_{j+1}$ for approximating $L^2(\mathbb{R})$ functions. The multiresolution analysis satisfies the following conditions:

1.$\cup_{-\infty}^{\infty} V_j$ is dense in $L^2(\mathbb{R})$.

2. $\cap_{j=-\infty}^{\infty} = \{0\}$

3. $f(x) \in V_j$ *if and only if* $f(2x) \in V_{j+1}, \forall j \in \mathbb{Z}$

4. $\exists$ *a function* $f(x)$ *such that* $\{\phi(x-k): k\epsilon\mathbb{Z}\}$ forms an unconditional basis for V_0.

5. $f(x) \in V_0 \Leftrightarrow f(x+1) \in V_0$.

2.3. Haar Wavelet and Scaling Function

The Haar wavelet [11] is denoted by $\psi_H(t)$, family for $t \in [0,1]$ and it is defined by-

$$\psi_H(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & elsewhere \end{cases}$$

Lal and Kumar [8] define the following

Let $L^2(\mathbb{R})$ and $\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j - k)$ and $X_j = class < \psi_{j,k}: k \in \mathbb{Z} >$.

The above family of subspaces of $L^2(\mathbb{R})$ gives a direct sum decomposition of $L^2(\mathbb{R})$which is the same as every $f\epsilon L^2(\mathbb{R})$ has a unique decomposition.

$$f(x) = \cdots h_{-3}(x) + h_{-2}(x) + h_{-1}(x) + h_0(x) + h_1(x) + h_2(x) + h_3(x) + \cdots$$

where $h_j \in X_j \ \forall j \in Z$ and it can be written as

$$L^2(R) = \oplus_{j=-\infty}^{\infty} X_j$$

Thus

$$V_j = \oplus_{k=-\infty}^{j-1} X_k$$

$\{\psi_{j,k} \in \mathbb{Z}\ where\ \psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j x - k)\}$ is a Riesz basis of X_j.

Now, we define $\phi \in V_0, V_0 \subset V_1$, a sequence $\{g_k\} \in L^2(Z)$ such that

$$\phi(x) = 2\sum_{j=-\infty}^{\infty} g_k\,\phi(2x-k) \quad (1)$$

This equation is known as the refinement equation, the dilation equation or the two-scale difference equation.

Integrating equation (1)and dividing by the integral of ϕ, we get

$$\sum_{k=-\infty}^{\infty} g_k = 1 \quad (2)$$

If V_j is a subspace, then a function $\phi \in L^2(R)$ is called the scaling function, defined as

$$V_j = clos_{L^2(R)}\{\phi_{j,k}: k \in \mathbb{Z}\}, j \in \mathbb{Z}$$

which satisfies the properties 1 to 5 given above.

Haar scaling function ϕ is defined as

$$\phi(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0 & otherwise \end{cases}$$

The system of Haar scaling function is defined as

$$\{\phi_{j,k} = 2^{\frac{j}{2}}\phi(2^j - k) \quad where\ j, k \in \mathbb{Z}\}$$

Or

$$\phi_{j,k} = 2^{\frac{j}{2}}\phi(2^j - k) = D_{2j}T_k\phi(t),$$

where the dilation operator $D_a f(x) = a^{\frac{1}{2}} f(ax)$ and translation operator $T_k f(x) = f(x - k)$. If $\phi \in L^1(R)$, then it is uniquely defined by its dilation equation and the normalization

$$\int_{-\infty}^{\infty} \phi(x)dx = 1 \tag{3}$$

The orthogonal projection $P_n(f)$ of $L^2(R)$ onto V_n is defined by

$$P_n(f) = \sum_{k=-\infty}^{\infty} a_{n,k}\phi_{n,k} \qquad n = 1,2,3, \dots .$$

where,

$$a_{n,k} = \langle f, \phi_{n,k} \rangle$$

Therefore,

$$P_n(f) = \sum_{k=-\infty}^{\infty} \langle f, \phi_{n,k} \rangle \phi_{n,k}$$

2.4 Wavelet Approximation under Supremum Norm

It is defined by,

$$E_n(f) = \|f - P_n\|_\infty$$

$$= \sup_x |(f(x) - P_n f(x))|$$

Therefore, the degree of wavelet approximation of f by $P_n f$ under norm $\|.\|_p$ is given by

$$E_n(f) = \min_{P_n f} |(f(x) - P_n f(x))|$$

If $\lim_{n\to\infty} E_n(f) \to 0$, then $E_n(f)$ is known as the best approximation of f of order n.

3. Previous Results

Theorems and results of Lal and Kumar:

Theorem 1. If a function $f \in Lip_\alpha[0,1]$, $0 < \alpha \leq 1$ $n = 1,2,3, \dots$ then the best wavelet approximation $E_n(f)$ of f is given by

$$E_n(f) = \|f - P_n(f)\|_\infty = O\left(\frac{1}{2^{n\alpha}}\right), \quad 0 < \alpha \leq 1, n = 1,2,3, \ldots$$

Theorem 2. Let ξ be a monotonic increasing function of t such that

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right\}^{\frac{1}{2}} = O(\xi(t)), \qquad 1 \leq p < \infty$$

and $\xi(t) \to 0$ as $t \to 0^+$. Then the best wavelet approximation $E_n(f)$ of a function $f \in Lip_{(\xi,p)}[0,1]$, satisfies:

$$E_n(f) = \min_{P_n f} \|f - P_n f\|_p = O\left(\xi\left(\frac{1}{2^n}\right)\right).$$

Theorem 3. If a function $f \in Lip_\alpha[0,1]$ $0 < \alpha \leq 1$ $i.e.$

$$|f(x+t) - f(x)| = O(t^\alpha) \Leftrightarrow E_n(f) = O\left(\left(\frac{1}{2^n}\right)^\alpha\right).$$

Theorem 4. If a function $f \in Lip_{(\xi,p)}[0,1]$ $1 < p \leq \infty$

$$\Leftrightarrow E_n(f) = O\left(\xi\left(\frac{1}{2^n}\right)\right), n = 1,2,3 \ldots$$

4. Main Theorems

In this paper, we establish the following two theorems:

Theorem 4.1- If a function $f \epsilon Lip_\alpha^{p,\psi}$, $f(x+t) - f(x) = O(|t|^\alpha \psi(t))$ $\quad 0 < \alpha \leq 1$, then the best wavelet approximation $E_n(f)$ of f is given by

$$E_n(f) = \|f - P_n(f)\|_\infty = O\left(\left(\frac{1}{2^{n\alpha}}\right)\psi\left(\frac{1}{2^n}\right)\right), \quad 0 < \alpha \leq 1, n = 1,2,3, \ldots$$

where ψ is a positive monotonic increasing function of t such that $(|t|^\alpha \psi(t)) \to 0$ $\quad as \quad t \to 0^+$.

Theorem 4.2- If ψ is a positive monotonic increasing function of t such that

$$\left\{\frac{1}{2\pi}\int_0^1 |f(x+t) - f(x)|^p dx\right\}^{\frac{1}{2}} = O(|t|^\alpha \psi(t)), \qquad 1 \leq p < \infty$$

and $\psi(t) \to 0$ as $t \to 0^+$. Then the best approximation $E_n(f)$ of a function $f \epsilon Lip_\alpha^{p,\psi}$ is given by

$$E_n(f) = \min_{P_n f} \|f - P_n f\|_p = O\left(\frac{\psi\left(\frac{1}{2^n}\right)}{2^{n\alpha}}\right)$$

4.1. Proof of Theorem 4.1

The operator of the projection $P_n f: L^2(R) \to V_n$ is defined by

$$P_n f = \sum_{k \in Z} a_{n,k}\, \phi_{n,k}, \quad n = 1,2,3, \ldots$$

where

$$a_{n,k} = \langle f, \phi_{n,k}\rangle$$

$$= \int_{-\infty}^{\infty} f(y)\overline{\phi_{n,k}(y)}dy$$

Therefore

$$(P_n)(f) = \sum_{k\in Z}\left\{\int_{-\infty}^{\infty} f(y)\,\phi_{n,k}(y)\,dy\right\}\phi_{n,k}(x)$$

$$= \int_{-\infty}^{\infty} f(y)\left(\sum_{-\infty}^{\infty}\phi_{n,k}(x)\,\overline{\phi_{n,k}(y)}\right)dy$$

$$= \int_{-\infty}^{\infty} f(y)\left(\sum_{-\infty}^{\infty} 2^{\frac{n}{2}}\phi(2^n x - k)2^{\frac{n}{2}}\overline{\phi(2^n y - k)}\right)dy$$

$$= 2^n\int_{-\infty}^{\infty} f(y)\left(\sum_{-\infty}^{\infty}\phi(2^n x - k)\overline{\phi(2^n y - k)}\right)dy$$

$$= 2^n\int_{-\infty}^{\infty} f(y)\,K(2^n x, 2^n y)\,dy$$

Since

$$K(x,y) = \sum_{-\infty}^{\infty}\phi(x-k)\overline{\phi(y-k)}$$

And

$$\int_{-\infty}^{\infty} K(x,y)\,dy = 1, \quad x\in\mathbb{R}.$$

Therefore replacing $y \to 2^n y$ and $x \to 2^n x,$ then

$$2^n\int_{-\infty}^{\infty} K(2^n x, 2^n y)dy = 1$$

Next,

$$f(x) = f(x)\int_{-\infty}^{\infty} K(2^n x, 2^n y)\,dy \qquad \left(\because 2^n\int_{-\infty}^{\infty} K(2^n x, 2^n y)\,dy = 1\right)$$

$$= 2^n\int_{-\infty}^{\infty} K(2^n x, 2^n y)f(x)\,dy$$

Therefore,

$$(P_n f)(x) - f(x) = 2^n\int_{-\infty}^{\infty} f(y)\,K(2^n x, 2^n y)dy - 2^n\int_{-\infty}^{\infty} K(2^n x, 2^n y)f(y)\,dy$$

$$= 2^n \int_{-\infty}^{\infty} K(2^n x, 2^n y)\, dy[f(y) - f(x)]dy$$

$$= \int_{-\infty}^{\infty} K(2^n x, 2^n y)\,[f(2^{-n} y) - f(x)]dy, \quad replacing \quad y\, by \quad 2^{-n} y,$$

$$= \int_{-\infty}^{\infty} K(2^n x, 2^n - w)[f(x - 2^{-n} y) - f(x)]dw, \quad 2^{-n} y = x - 2^{-n} w,$$

$$= \int_{-\infty}^{\infty} K(2^n x, 2^n x - y)\,[f(x - 2^{-n} y) - f(x)]dy, \quad replacing \quad w \quad by \quad y.$$

$$|(P_n f)(x) - f(x)| \leq \int_{-\infty}^{\infty} |K(2^n x, 2^n x - y)||f(x - 2^{-n} y) - f(x)|dy$$

$$\leq \int_{-\infty}^{\infty} |K(2^n x, 2^n x - y)|dy \sup_y |f(x - 2^{-n} y) - f(x)|$$

By Hölder's inequality

$$= 2^n \int_{-\infty}^{\infty} |K(2^n x, 2^n w)|dy \sup_y |f(x - 2^{-n} y) - f(x)|,$$

Taking $2^n x - y = 2^n$, first factor only,

$$|(P_n f)(x) - f(x)| \leq 2^n \int_{-\infty}^{\infty} |K(2^n x, 2^n y)|dy \sup_y |f(x - 2^{-n} y) - f(x)|,$$

Replacing w by y in the first factor only,

$$|(P_n f)(x) - f(x)| \leq \sup_y |f(x - 2^{-n} y) - f(x)|, \quad \left\{2^n \int_{-\infty}^{\infty} |K(2^n x, 2^n y)|dy = O(1)\right\}$$

$$= \sup_{y \in [0,1]} (O|2^{-n} y|^{\alpha}), \quad |f(x - 2^{-n} y) - f(x)| = O\left(\left|2^{-n} y\, \psi\left(\left|\frac{y}{2^n}\right|\right)\right|\right), \qquad f \in Lip_{\alpha}^{p,\psi}[0,1]$$

$$= O\left(\int_0^1 (2^{-y})^{\alpha}\, \psi\left(\frac{y}{2^n}\right)\right)$$

$$= O\left(\left(\frac{1}{2^n}\right)^{\alpha} \psi\left(\frac{1}{2^n}\right) \int_0^1 y^{\alpha} dy\right)$$

$$= O\left(\psi\left(\frac{1}{2^n}\right)\left(\frac{1}{2^{n\alpha}}\right)\left(\frac{1}{1+\alpha}\right)\right)$$

$$= O\left(\left(\frac{1}{2^{n\alpha}}\right)\psi\left(\frac{1}{2^n}\right)\right)$$

Thus

$$\sup_x \|(P_n f)(x) - f(x)\|_\infty = \|P_n f - f\|_\infty$$

$$= O\left(\left(\frac{1}{2^{n\alpha}}\right)\psi\left(\frac{1}{2^n}\right)\right)$$

Hence,

$$E_n(f) = O\left(\left(\frac{1}{2^{n\alpha}}\right)\psi\left(\frac{1}{2^n}\right)\right)$$

4.2. Proof of Theorem 4.2

If $\left(\int_0^1 |f(x+t) - f(x)|^p dx\right)^{\frac{1}{p}} = O(|t|^\alpha \psi(|t|))$, when ψ is a function of t such that $|t|^\alpha \psi(|t|) \to 0 \ as \ t \to 0^+$.

Then, the following proof of theorem (1),

$$|(P_n f)(x) - f(x)| = \int_{-\infty}^{\infty} |K(2^n, 2^{-n}y) - f(x)||f(x - 2^{-n}y - f(x))|dy$$

$$\leq O(1)\int_0^1 |f(x - 2^{-n}y - f(x))|dy$$

Applying generalized Minkowski's inequality in the above expression, we have

$$\|P_n f - f\|_p = O(1)\int_0^1 \|f(x - 2^{-n}y) - f(x)\|_p dy$$

$$= O(1)\int_0^1 (2^{-n}|y|)^\alpha \psi\left(\frac{y}{2^n}\right) dy$$

$$= O\left(\frac{1}{(2^n)^\alpha}\psi\left(\frac{1}{2^n}\right)\right)\int_0^1 dy$$

$$= O\left(\frac{\psi\left(\frac{1}{2^n}\right)}{2^{n\alpha}}\right)$$

Hence

$$E_n(f) = \min_{P_n f} \|f - P_n f\|_p = O\left(\frac{\psi\left(\frac{1}{2^n}\right)}{2^{n\alpha}}\right)$$

5. Results and Discussion

1. If $\psi(t) = 1$ in **Theorem 4.1,** then the degree of approximation is given by

$$E_n(f) = \|f - P_n f\|_\infty = O\left(\frac{1}{2^{n\alpha}}\right); \quad 0 < \alpha \leq 1.$$

2. If $\psi(t) = 1$ in **Theorem 4.2,** then the degree of approximation $f \epsilon Lip_\alpha^p$ is given by

$$E_n(f) = \|f - P_n f\|_p = O\left(\frac{1}{2^{n\alpha}}\right); \quad 0 < \alpha \leq 1$$

6. Conclusion

The error estimation or degree of approximation of functions of generalized Lipschitz class has been determined in this paper. The results give a proper comparison with the existing results of Lal & Kumar [8]. Their results can be derived from the present determinations. Therefore, this gives the best approximation compared to previously existing results.

Acknowledgement

The authors are grateful to Professor Shyam Lal, Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, for encouragement to this work. The authors are also thankful to the reviewers for their valuable suggestions and comments.

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