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Original Article

Study of 2-D Unsteady In-Compressible Viscous Flow & Heat Transfer with Wall Slip Boundary Conditions: A Numerical Approach

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Abstract *- This study introduces a numerical method to address the phenomenon of fluid flow and heat transfer within a rectangular area under wall slip boundary conditions. The established marker and cell (MAC) technique [1] has been effectively used for discretizing the governing equations relevant to the study. The MAC method's solution algorithm has been employed to calculate the flow variables for high Reynolds numbers and a range of Prandtl numbers. The numerical computations were performed in accordance with the stability criteria established through the von Neumann analysis. An examination of the influence of high Reynolds numbers and Prandtl numbers on flow variables has been outlined.*

Keywords *- Flow variables, Heat transfer, Marker-And-Cell (MAC) method, Reynolds number, Prandtl numbers.*

1. Introduction

The proposed fluid flow issue has been a prominent field of theoretical, numerical, and experimental research in recent years. Fluid flow and heat transfer are important phenomena in nature, living organisms, and a variety of practical situations. The 2-D unsteady viscous flow equations are used for modeling various physical phenomena, which include pipe flow, weather and blood flow, flow around airfoils. Most of the flows we experienced in everyday life are turbulent, and many of these exhibit high Reynolds numbers. Heat transfer is an important phenomenon in various processes in a number of residential, industrial and commercial facilities. Unsteady heat transfer occurs in food process engineering, heating or cooling of solid bodies made from good thermal conductors, thermal and hydraulic power plants, heating and air-conditioning of buildings, air-heaters, design of electrical machinery and electronic circuits, weather prediction and environmental pollution, oil exploration, etc.

The MAC method, associated with free surface flows, is a finite difference numerical technique that addresses the velocity and pressure profiles necessary to calculate the behavior of incompressible fluid flow. Harlow and Welch [1] employed the Marker-And-Cell (MAC) technique for numerical computations of time-dependent viscous incompressible flow in a free surface. Various advancements concerning MAC methods can be observed in the literature [2-4]. Ghia et al. [6] utilized a multigrid method to achieve high-resolution results for the 2-D incompressible Navier-Stokes equations. Erturk et al. [13] presented numerical solutions for the 2-D incompressible steady flow at elevated Reynolds numbers. R.A. Nicolaides [8] suggested the analysis and convergence of the MAC scheme for linear problems, while R.A. Nicolaides and X. Wu [9] explored the analysis and convergence of the MAC scheme focused on the Navier-Stokes problem. The convergence of the Marker-and-Cell Scheme for the Incompressible flow utilizing Non-Uniform Grids was discussed by Gallouet Thierry et al. [19]. Research on the Navier-Stokes Equations for slip boundary situations has been examined in the literature [11-12] and [15]. The vast range of applications for unsteady compressible flow involving heat and mass transfer, as previously mentioned, serves as the impetus for this investigation. A review of the literature indicates that due to the complexities associated with the rectangular domain, there has been no numerical study on the flow variables for 2-D unsteady flow with heat transfer in a rectangular environment, including conditions for slip walls and temperature boundaries. Furthermore, to explore the significance of the previously listed applications, it is essential to derive numerical solutions for the unknown flow variables. To address this need, a numerical method for solving the aforementioned problem has been explored. The primary contribution of this study is the effective application of the MAC method to solve the problem of unsteady 2-D incompressible flow with heat transfer, including initial and boundary conditions in a rectangular domain. The MAC differencing technique has been utilized to discretize the governing equations. The Marker – and – Cell scheme has been used effectively for the computation of the numerical values of the unknown variables u, v, p, and T. The time-dependent variations of these flow variables, under specific parameters, within the rectangular domain, have been analyzed.

The proposed study is outlined as follows: Section 2 deals with the governing equations of the problem. Section 3 provides a comprehensive analysis of the MAC method, including the solution algorithm and the calculations of the results. The conclusions of the proposed numerical study are found in Section 4.

2. Mathematical Formulation

The governing equations in a rectangular domain using the Boussinesq approximation in the dimensionless form are as follows:

$$Continuity equation\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (1)$$

$$X-Momemtum\ \frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial P}{\partial x}-\left(\frac{1}{Re}\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0 (2)$$

$$Y-Momemtum\ \frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial P}{\partial y}-\left(\frac{1}{Re}\right)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0 (3)$$

$$Energy equation\ \frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}-\frac{1}{Pr}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=0 (4)$$

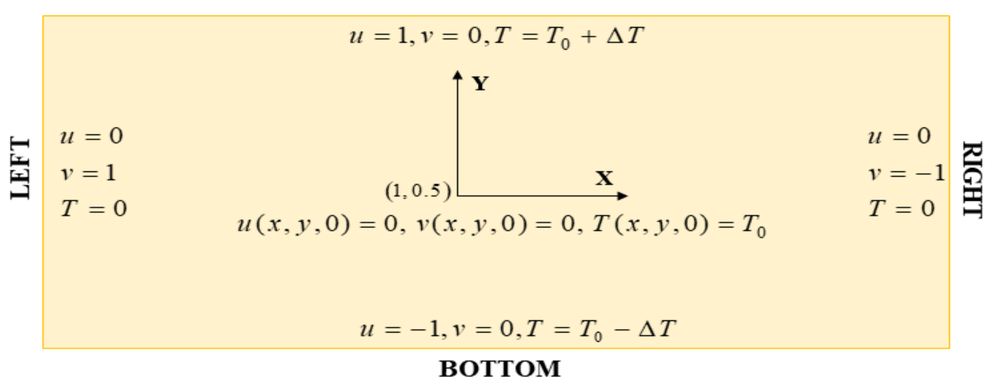

Fig. 1 The Computational Rectangular Domain

3. Numerical Methodology and Computation

Solve the above equations (1) – (4) suitably using the MAC method along with the initial and boundary conditions. Consider a MAC staggered grid for u, v, P and T nodes as shown in Figure 2. Using the method, the various derivatives appearing in equation (2) are calculated as follows:

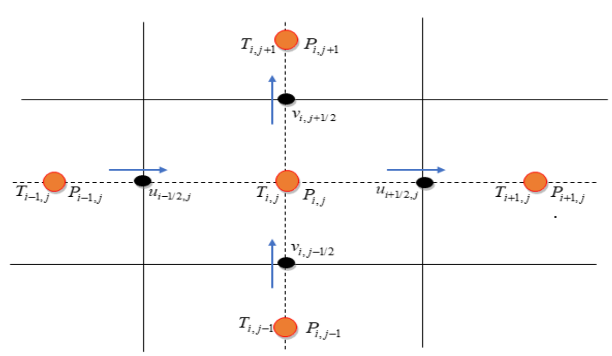

Fig. 2 MAC Staggered Grid System

The time-splitting method, also known as the fractional time-step method [7], which is fully explicit in nature, has been implemented. Applying it to the equation. (2) & (3) on the staggered grid from t to $\hat{t}$:

$$\frac{\hat{u}_{i+1/2,j} - u^n_{i+1/2,j}}{\Delta t} = \frac{(u^n_{i,j})^2 - (u^n_{i+1,j})^2}{\Delta x} + \frac{(uv)^n_{i+1/2,j-1/2} - (uv)^n_{i+1/2,j+1/2}}{\Delta y} + \frac{u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j}}{\mathrm{Re}\Delta x^2} + \frac{u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{\mathrm{Re}\Delta y^2} \tag{5}$$

$$\frac{\hat{v}_{i,j+1/2} - v^n_{i,j+1/2}}{\Delta t} = \frac{(uv)^n_{i-1/2,j+1/2} - (uv)^n_{i+1/2,j+1/2}}{\Delta x} + \frac{(v^n_{i,j})^2 - (v^n_{i,j+1})^2}{\Delta y} + \frac{v^n_{i+1,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i-1,j+1/2}}{\mathrm{Re}\Delta x^2} + \frac{v^n_{i,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i,j-1/2}}{\mathrm{Re}\Delta y^2} \tag{6}$$

Now advancing from t^n to $\hat{t}$, and then $\hat{t}$ to t^{n+1}, the elliptical pressure equation is obtained

$$\nabla^2 P^{n+1} = \frac{\nabla . \hat{U}}{\Delta t} \tag{7}$$

with boundary condition $\frac{\partial P^{n+1}}{\partial n} = 0$.

Applying the central difference numerical scheme:

$$\frac{P^{n+1}_{i+1,j} - 2P^{n+1}_{i,j} + P^{n+1}_{i-1,j}}{\Delta x^2} + \frac{P^{n+1}_{i,j+1} - 2P^{n+1}_{i,j} + P^{n+1}_{i,j-1}}{\Delta y^2} = \frac{1}{\Delta t}\left[\frac{\hat{u}_{i+1/2,j} - \hat{u}_{i-1/2,j}}{\Delta x} + \frac{\hat{v}_{i,j+1/2} - \hat{v}_{i,j-1/2}}{\Delta y}\right] \tag{8}$$

Now, the velocity field is being calculated at time level t^{n+1}:

$$u^{n+1}_{i+1/2,j} = \hat{u}_{i+1/2,j} - \frac{\Delta t}{\Delta x}\left(P^{n+1}_{i+1,j} - P^{n+1}_{i,j}\right), v^{n+1}_{i,j+1/2} = \hat{v}_{i,j+1/2} - \frac{\Delta t}{\Delta y}\left(P^{n+1}_{i,j+1} - P^{n+1}_{i,j}\right). \tag{9}$$

Similarly, equation (4) has been discretized suitably:

$$\frac{T^{n+1}_{i,j} - T^n_{i,j}}{\Delta t} = u^n_{i,j}\left(\frac{T^n_{i,j} - T^n_{i+1,j}}{\Delta x}\right) + v^n_{i,j}\left(\frac{T^n_{i,j} - T^n_{i,j+1}}{\Delta y}\right) + \left(\frac{1}{Pr}\right)\left(\frac{T^n_{i+1,j} - 2T^n_{i,j} + T^n_{i-1,j}}{\Delta x^2} + \frac{T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j-1}}{\Delta y^2}\right) \tag{10}$$

Here $u^n_{i,j} = \frac{1}{2}\left(u^n_{i+1/2,j} + u^n_{i-1/2,j}\right), v^n_{i,j} = \frac{1}{2}\left(v^n_{i,j+1/2} + v^n_{i,j-1/2}\right)$

The von Neumann convergence analysis has been used suitably to find the practical stability conditions:

$$\max_{i,j}\left[\left(\frac{|u_{i,j}|}{\Delta x} + \frac{|v_{i,j}|}{\Delta y}\right)\Delta t\right] < \frac{1}{2}, \max\left[\frac{\Delta t}{\mathrm{Re}}\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right)\right] < \frac{1}{10}$$

$$\Delta t \max_{i,j}\left[\left(\frac{u_{i,j}}{\Delta x} + \frac{v_{i,j}}{\Delta y}\right)\frac{1}{Re} + \frac{2}{Pr}\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right)\right] \leq 1 \tag{11}$$

3.1. Solution Algorithm

3.1.1. Prediction Step

- Using (5) and (6), the results for $\hat{u}$ &$\hat{v}$ have been obtained at respective grid points.
- The initial and boundary conditions are applied.
- The time advancement is explicit, and hence the equations are solved algebraically.
- The stability condition (11) must be satisfied.
- Divergence calculation of the velocity profiles at every time advancement step:
-

$$\nabla . u = \frac{u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j}}{\Delta x} + \frac{v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2}}{\Delta y}$$

At all grid points, the sum of the divergence magnitude must have a machine-zero value; otherwise, the calculation must be performed with a smaller time step.

3.1.2. Pressure Calculation

- Pressure-Poisson equation given by eqn. (8) to be numerically solved for pressure calculation using the homogeneous Neumann boundary conditions.

3.1.3. Velocity Correction

- The numerical computation for the velocity profiles is calculated t^{n+1} using the equation. (9).

3.1.4. Temperature Calculation

- Calculate the temperature from the equation. (10).

The solutions for the flow variables have been derived using a computational rectangular staggered grid. At the initial time $t = 0$, the velocity field is set to zero, and the temperature within the rectangular domain is 0.50, with a temperature gradient of 0.50 at both the left and right walls. Numerical computations have been performed for Reynolds numbers $Re = 2500, 5000, 7500, 10000$ and Prandtl numbers $Pr = 1.0, 3.0, 5.0, 7.0$ with a time step of $\Delta t = 0.001$ sec. The numerical solutions for the u-velocity have been calculated, and the results are displayed in Figure 3. Likewise, the numerical solutions for the v-velocity have been computed, with the results illustrated in Figure 4. The numerical computations have been performed for Reynolds values of 2500, 5000, 7500, and 10000.

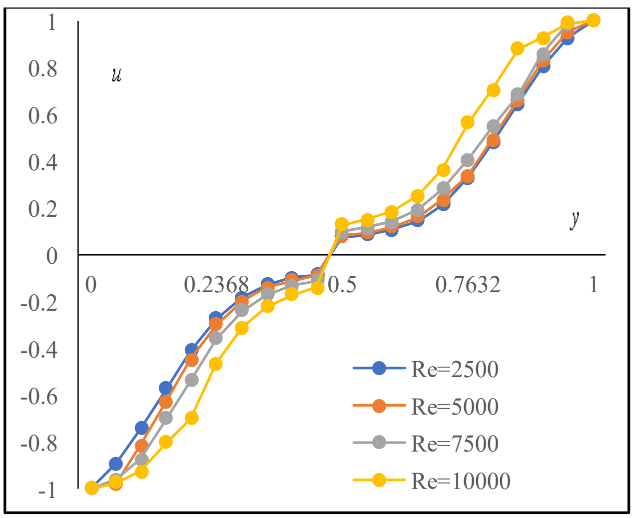

Fig. 3 *u*-velocity profiles

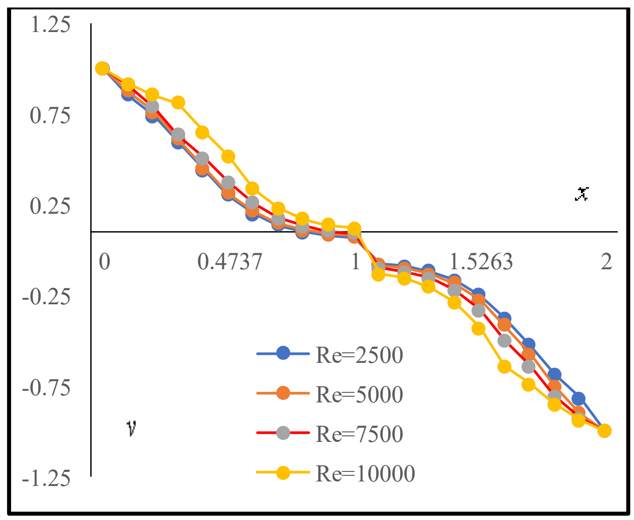

Fig. 4 *v*-velocity profiles

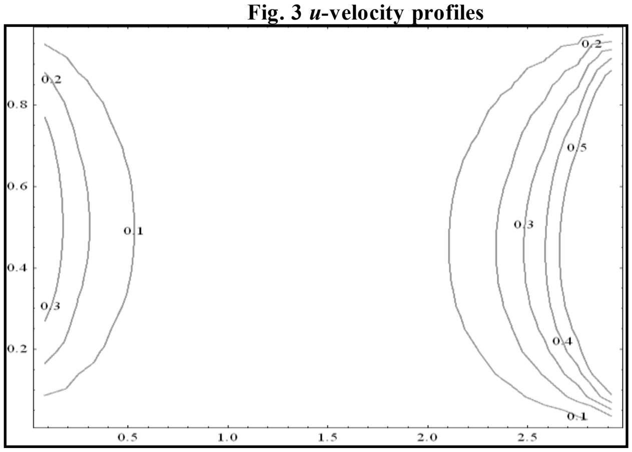

Fig. 5 Temperature variation for Re =2500

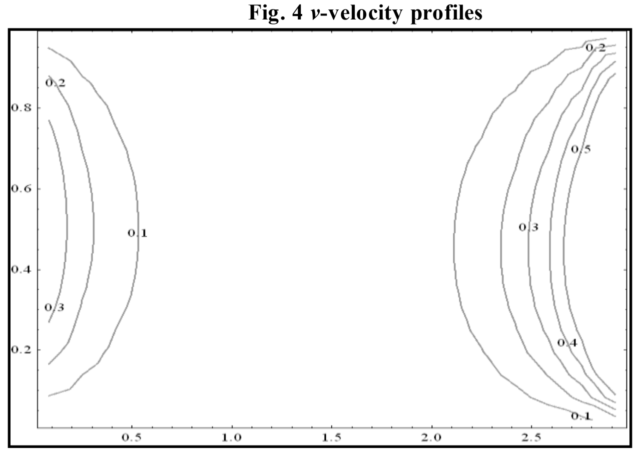

Fig. 6 Temperature variation for Re = 5000

The temperature flow behavior for various Reynolds numbers ($Re = 2500, 5000, 7500, 10000$) is depicted in Figures 5 through 8. These numerical findings were determined for Prandtl numbers ($Pr = 1.0, 3.0, 5.0, 7.0, 9.0, 11.0$). The numerical computations for temperature at the midpoint of the rectangular area have been performed, and the computational results for various Prandtl numbers are presented in Figure 9.

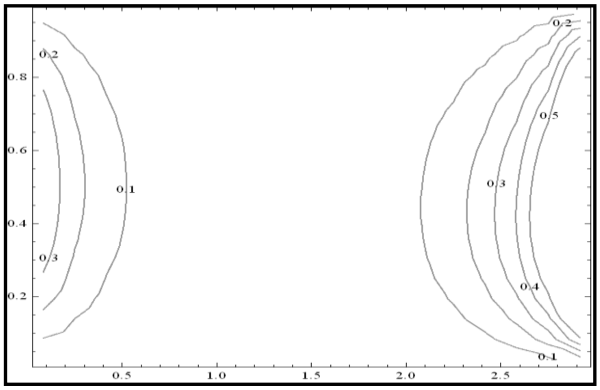

Fig. 7 Temperature variation for Re = 7500.

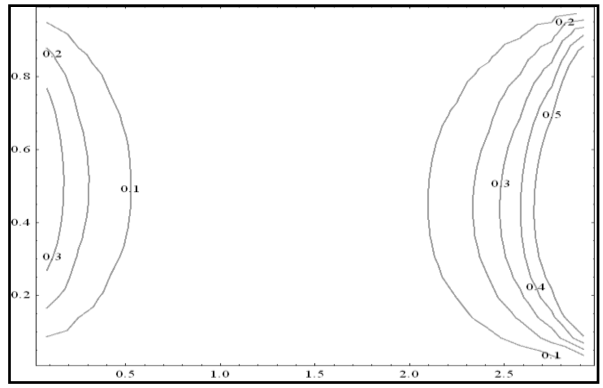

Fig. 8 Temperature variation for Re = 10000.

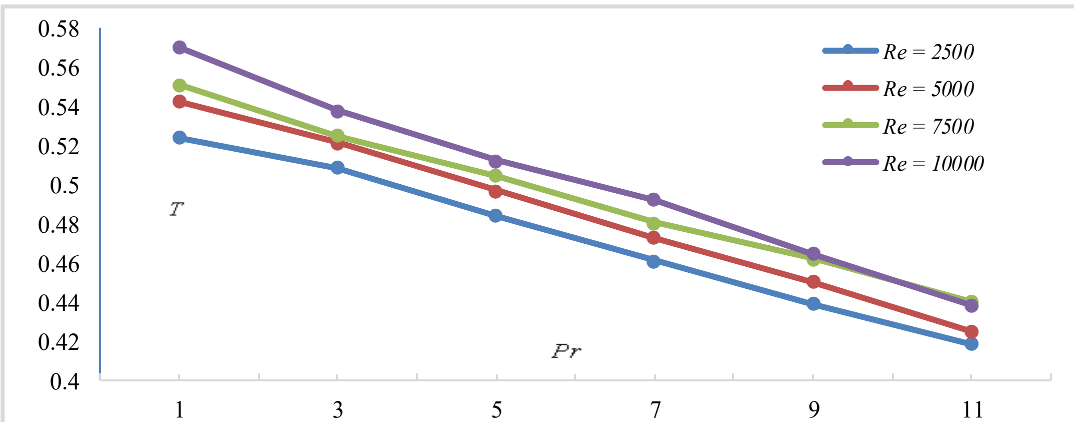

Fig. 9 Temperature variation at the midpoint

The pressure profiles for various grid points (0.0526,0.5), (1.0,0.0263), (1.0,0.5), (1.0,0.9737) have been depicted in Figure 10 for Reynolds numbers of 2500, 5000, 7500, and 10000.

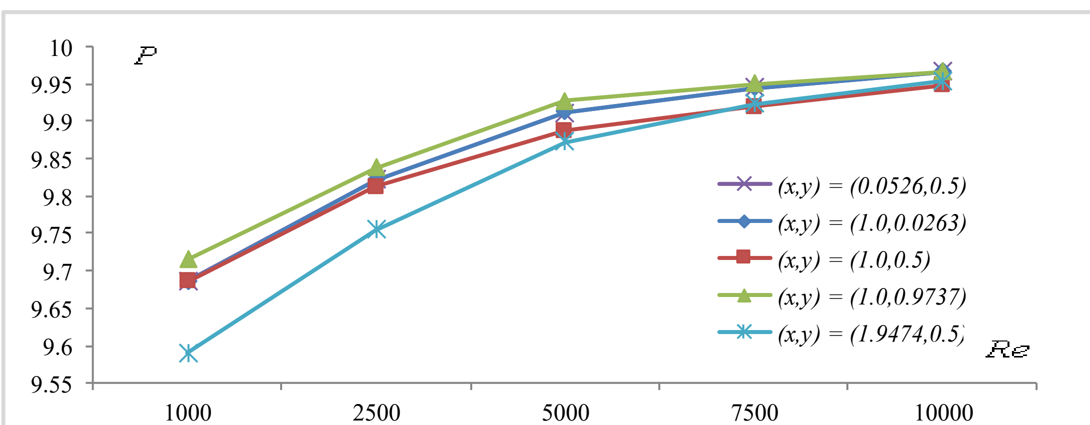

Fig. 10 Pressure variation at specific grids for different Reynolds numbers

4. Conclusion

In the present study, a numerical analysis of the incompressible flow of viscous fluids in a 2-D unsteady scenario, along with heat transfer through a coupled energy equation, has been presented. The wall slip boundary conditions within a rectangular domain have been applied. Numerical computations have been performed to compute unknown flow variables such as velocity profiles, pressure profiles, and temperature profiles for specific Reynolds numbers and Prandtl numbers. Using numerical computations, the behavior of the x-component of the velocity profile, referred to as u-velocity profiles, along a vertical line that passes through the geometric center of the rectangular domain has been observed. The boundary conditions at the left and right walls are set at $u = 0$, while the bottom and top wall boundaries are assigned $u = -1$ and $u = 1$, respectively. The results reflected that the absolute values of the u-velocity profiles diminish as the Reynolds number increases in the ranges of Re = 2500, 5000, 7500, and 10000. Similarly, the behavior of the y-component of the velocity profile, also known as v-velocity profiles, along a vertical line through the center of the rectangular domain has been examined. The boundary conditions at the bottom and top walls are maintained at $v = 0$, whereas the left and right walls are set at $v = 1$ and $u = -1$, respectively. The investigation reveals that the absolute values of the v-velocity profiles decline with an increase in the Reynolds number across the specified range of Re = 2500, 5000, 7500, and 10000. The numerical analysis demonstrated that the pressure profiles increase with the rising Reynolds number within the examined range of Re = 2500, 5000, 7500, and 10000, as observed at particular grid points. A numerical computation of temperature values has been performed with boundary conditions at the left and right walls set to $T = 0$, while the bottom and top wall boundaries have specified temperatures. The temperature variations revealed that, for a fixed value of Re, the temperature profiles decrease as Prandtl numbers increase, as summarized at the midpoint of the rectangular domain.

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