International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V71I8P104

Volume 71 Issue 8, 21-23, August 2025

Original Article

A Study on Generalized Derivation Acting on Jordan Ideal in Prime Rings

V. K. Yadav

[1]*Department of Mathematics, D. S College, Aligarh, Affiliated to R.M.P.S. University, Aligarh, U.P., India.*

Corresponding Author : iitdvishalny@gmail.com

Received: 12 June 2025 Revised: 19 July 2025 Accepted: 05 August 2025 Published: 18 August 2025

Abstract *- Let R be a 2-torsion-free prime ring and J be a non-zero Jordan ideal of R. Suppose that F: R → R is a generalized derivation associated with a non-zero derivation d. If* $F(xy) - d(x)d(y) \in Z(R)$*, for all* $x, y \in J$*, then R is commutative.*

Keywords *- Prime ring, Generalized derivation, Jordan ideal.*

Mathematics Subject Classification (2000) - *16W 25, 16N 60, 16R50.*

1. Introduction

Throughout this paper, R denotes an associative ring with center $Z(R)$. A ring R is said to be a prime ring if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$. A ring R is said to be 2-torsion-free if $2x = 0$ implies $x = 0$. We denote operation o as a Jordan product, which is defined on R as $xoy = xy + yx$, for all $x, y \in R$ and $[x, y]$ denotes the Lie product of x, y, which is defined as $[x, y] = xy - yx$, for all $x, y \in R$. An additive subgroup J of R is called a Jordan ideal of R if $uor \in J$, for all $u \in J$ and $r \in R$. An additive mapping d from R to R is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is said to be a generalized derivation if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$.

Firstly, E.C. Posner [11] proved very striking results on derivation in prime rings and established a relation between additive mappings and the structure of a ring. Many authors have implemented Posner's theorems; further information can be found in [5],[9],[10]. In this line, Ram Awtar [1] proved some results on Jordan ideals and Lie ideals in a prime ring and also proved that, if J is a Jordan ideal of R, then $4j^2r, 4rj^2, 4jrj \in J$, where $j \in J, r \in R$. Further, Zaidi [12] et al. proved a result which states that if R is a ring and J is a non-zero Jordan ideal of R, then $2J[R, R] \subseteq J$ and $2[R, R]J \subseteq J$. In 1991, M. Bresar [3] introduced the concept of generalized derivation in rings, which is the generalization of derivation because every derivation is a generalized derivation, but not conversely. In this sequence, Ashraf [2] et al. proved that if R is a prime ring which is 2 torsion free and F is a generalized derivation associated with derivation d on R. If F satisfies any one of the following conditions: (i) $F(xy) - xy \in Z(R)$; (ii) $F(xy) - yx \in Z(R)$; (iii) $F(x)F(y) - xy \in Z(R)$; (iv) $F(x)F(y) - yx \in Z(R)$, for all $x, y \in I$, where I is an ideal of R, then R is commutative.

Recently, Oukhtite L. [7] et al. proved that, if R is a 2-torsion-free prime ring, J is a non-zero ideal of R, and F satisfies any one of the following conditions:
(i) $F(xy) - xy \in Z(R)$; (ii) $F(xy) - yx \in Z(R)$; (iii) $F(x)F(y) - xy \in Z(R)$;
(iv) $F(x)F(y) - yx \in Z(R)$ for all $x, y \in J$, then R is commutative.

Motivated by the results of Oukhtite L. [7], we continue this line of investigation. In this paper, I have studied generalized derivation acting on a Jordan ideal in a 2-torsion-free prime ring:

2. Preliminary Results

The following Lemma will be used in the proof of the main results;

Lemma 2.1. [[8], Lemma 2.6] If J is a non-zero Jordan ideal such that aJb = 0, then either a = 0 or b = 0.

Lemma 2.2. [[7], Fact 3] If R is a noncommutative ring such that $a[r, xy]b = 0$ for all $x, y \in J$, $r \in R$, then either $a = 0$ or $b = 0$.

Lemma 2.3. [[7], Fact 6] Let i be a positive integer and set $J_0 = J$, then $J_i = \{x \in J_{i-1} \mid d(x) \in J_i\}$ is a non-zero Jordan ideal; moreover, if $J \cap Z(R) /= 0$, then $J_i \cap Z(R) /= 0$.

Lemma 2.4. [[10], Lemma 2.2] If d is a derivation of R such that $d(x^2) = 0$ for all $x \in J$, then $d = 0$.

We leave the proofs of the following easy Lemma to the readers.

Lemma 2.5. Let J be a non-zero Jordan ideal of R. Suppose that d is a derivation on R such that $d(x) = x$ for all $x \in J$, then $d = 0$.

3. Main Results

Theorem 3.1. Let R be a 2-torsion free prime ring and J be a non-zero Jordan ideal of R. Suppose that $F: R \to R$ is a generalized derivation, associated with non-zero derivations d, such that $F(xy) - d(x)d(y) \in Z(R)$, for all $x, y \in J$, then R is commutative.

Proof. First of all, we show that $J \cap Z(R) /= 0$. On the contrary, if $J \cap Z(R) = 0$. We have, $F(xy) - d(x)d(y) \in Z(R)$ (1)
for all $x, y \in J$. Replacing y by $4[r, uv]y$ in (1), where $u, v \in J, r \in R,$ we get

$$4(F(x[r, uv]) - d(x)d([r, uv])y + 4x[r, uv]d(y) - 4d(x)[r, uv]d(y) \in Z(R) \quad (2)$$

for all $u, v, x, y \in J$. As $4x[r, uv]d(y) = 2x[r, uv]od(y) + 2[x[r, uv], d(y)] \in J$ and also $4d(x)[r, uv]d(y) \in J$ for all $x, y, u, v \in J_1$.

So $4(F(x[r, uv])-d(x)d([r, uv])y+4x[r, uv]d(y) - 4d(x)[r, uv]d(y) \in J.$
But $J \cap Z(R) = 0$, hence we get,

$$(F(x[r, uv]) - d(x)d([r, uv])y + x[r, uv]d(y) - d(x)[r, uv]d(y) = 0 \quad (3)$$

for all $x, y, u, v \in J_1, r \in R$. Replacing y by $4yz^2$ in (3), where $z \in J_1$. we get
$(x - d(x))[r, uv]yd(z^2)=0$ (4)

for all $x, y, u, v \in J_1, r \in R.$

Using lemma 2.2 we obtain either $x - d(x) = 0$ or $yd(z^2) = 0$. If $yd(z^2) = 0$, this implies $d(z^2) = 0 \ \forall z \in J_1$. Then by Lemma 2.4, $d = 0$, a contradiction. If $x\text{-}d(x) = 0 \ \forall x \in J_1$, then in application of Lemma 2.5, again we get $d = 0$, a contradiction. Therefore $J \cap Z(R) /= 0$.

Replacing y by $4yu^2$ in (1), where $u \in J$, we get
$4(F(xy) - d(x)d(y))u^2 + 4xyd(u^2) - 4d(x)yd(u^2) \in Z(R)$ (5)

for all $x, y, u \in J$. Since $F(xy) - d(x)d(y) \in Z(R)$, we get
$[xyd(u^2), u^2] - [d(x)yd(u^2), u^2] = 0$ (6) f

or all $x, y, u \in J$. Replacing x by $4xu^2$ in (6), we get
$[xu^2yd(u^2), u^2] - [d(x)u^2yd(u^2), u^2] - [xd(u^2)yd(u^2), u^2] = 0$ (7)

for all $x, y, u \in J$. Replacing y by $4u^2y$ in (6) and subtracting from (7), we obtain

$$[xd(u^2)yd(u^2), u^2] = 0 \quad (8)$$

for all $x, y, u \in J$.
As $4d(u^2)yd(u^2)x = 4(d(u)ou)yd(u^2)x = 2(d(u)ou)o(yd(u^2)x)$

$+2[(d(u)ou), yd(u^2)x] \in J$
for all $x, y, u, \in J_1$, then replacing x by $4d(u^2)yd(u^2)x$ in (8) we obtain
$[d(u^2)yd(u^2), u^2]xd(u^2)yd(u^2) = 0$ (9)

for all $x, y, u \in J_1$. Replacing x by $4xu^2$ in (9), we get
$[d(u^2)yd(u^2), u^2]xu^2d(u^2)yd(u^2) = 0$(10)

Multiplying (9) by u^2 from the right side, and subtracting from (10), we obtain
$[d(u^2)yd(u^2), u^2]J[d(u^2)yd(u^2), u^2] = 0$ (11)

for all $x, y, u \in J_1$. In light of Lemma 2.1, we get $[d(u^2)yd(u^2), u^2] = 0$, for all $x, y, u \in J_1$. Therefore $d(u^2)yd(u^2)u^2 - u^2d(u^2)yd(u^2) = 0$. As $4yd(u^2)z = 2yd(u^2)oz + 2[yd(u^2), z] \in J$ then replacing y by $2yd(u^2)z$ we get,
$d(u^2)y[d(u^2), u^2]zd(u^2) = 0$ (12)

for all $x, y, z, u \in J_1$. Again, by application of Lemma 2.1, we get either $d(u^2) = 0$ or $[d(u^2), u^2] = 0$. If $d(u^2) = 0$ for all $u \in J_1$, by Lemma 2.4, $d = 0$ a contradiction, therefore we get
$[d(u^2), u^2] = 0$ (13)

for all $u \in J_1$. Let $0 /= t \in J_1 \cap Z(R)$ and replacing u by $2rt$, where $r \in R$, we obtain
$[d(r^2), r^2] = 0$ (14)

for all $r \in R$. Therefore, in application of Theorem 3 of [4], we find that $[R, R]d(R) = 0$, hence R is commutative. This completes the proof.

References

[1] Ram Awtar, "Lie and Jordan Structure in Prime Rings with Derivations," *Proceedings of the American Mathematical Society*, vol. 41, pp. 67-74, 1973. [CrossRef] [Google Scholar] [Publisher Link]

[2] Mohammad Ashraf, Asma Ali, and Shakir Ali, "Some Commutativity Theorems for Rings with Generalized Derivations," *Southeast Asian Bulletin of Mathematics*, vol. 31, pp. 415-421, 2007. [Google Scholar]

[3] Matej Bresar, "On the Distance of the Composition of Two Derivations to the Generalized Derivations," *Glasgow Mathematical Journal*, vol. 33, no. 1, pp. 89-93, 1991. [CrossRef] [Google Scholar] [Publisher Link]

[4] Tsiu-Kwen Lee, "Semiprime Rings with Hypercentral Derivation," *Canadian Mathematical* Bulletin, vol. 38, no. 4, pp. 445-449, 1995. [CrossRef] [Google Scholar] [Publisher Link]

[5] Joseph H. Mayne, "Centralizing Automorphisms of Lie Ideals in Prime Rings," *Canadian Mathematical Bulletin*, vol. 35, no. 4, pp. 510-514, 1992. [CrossRef] [Google Scholar] [Publisher Link]

[6] A. Mamouni, L. Oukhtite, and M. Samman, "Commutativity Theorems for ∗-Prime Rings with Differential Identites on Jordan Ideal," *ISRN Algebra*, vol. 2012, no. 1, pp. 1-11, 2012. [CrossRef] [Google Scholar] [Publisher Link]

[7] L. Oukhtite, and A. mamouni, "Commutativity Theorems for Prime Rings with Generalized Derivations on Jordan Ideals," *Journal of Taibah University for Science*, vol. 9, no. 3, pp. 314-319, 2015. [CrossRef] [Google Scholar] [Publisher Link]

[8] L. Oukhtite, and A. mamouni, "Derivations Satisfying Certain Algebraic Identities on Jordan Ideals," *Arabian Journal of* Mathematics, vol. 1, pp. 341-346, 2012. [CrossRef] [Google Scholar] [Publisher Link]

[9] Lahcen Oukhtite, and Abdellah Mamouni, "Generalized Derivations Centralizing on Jordan Ideals of Rings with Involution," *Turkish Journal of Mathematics*, vol. 38, no. 2, pp. 233-239, 2014. [CrossRef] [Google Scholar] [Publisher Link]

[10] Lahcen Oukhtite, Abdellah Mamouni, and Charef Beddani, "Derivations on Jordan Ideals in Prime Rings," *Journal of Taibah University for Science*, vol. 8, no. 4, pp. 364-369, 2014. [CrossRef] [Google Scholar] [Publisher Link]

[11] Edward C. Posner, "Derivations in Prime Rings," *Proceedings of American Mathimatical Society*, vol. 8, no. 6, pp. 1093-1100, 1957. [CrossRef] [Google Scholar] [Publisher Link]

[12] S.M.A. Zaidi, Mohammad Ashraf, and Shakir Ali, "On Jordan Ideals and Left (θ, θ) - Derivations in Prime Rings," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 37, pp. 1957-1964, 2004. [CrossRef] [Google Scholar] [Publisher Link]