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Original Article

On Stable Cartan Subgroups in Lie Algebras

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Abstract *- This paper investigates Γ-stable Cartan subgroups in connected Lie groups and their associated Lie algebras. We extend the results of Borel and Mostow on the existence of automorphism-invariant Cartan subalgebras to the group setting. For a real semisimple Lie algebra $\mathfrak{g}$, we prove that there exists a nonidentity automorphism that fixes representatives of all conjugacy classes of Cartan subalgebras. Explicit constructions for classical Lie algebras (A_n, B_n, C_n, D_n) are provided. Applications include characterizing stable Cartan subgroups in quotients and normal subgroups.*

Keywords - *Cartan subgroups, Automorphism-invariant subgroups, Admissible root systems, Classical Lie algebras.*

MSC 2020 Classification : *[2020] 22E15, 17B20, 17B40, 22E46*

1. Introduction

The study of automorphism-invariant subgroups lies at the intersection of Lie theory, group representations, and differential geometry. For a connected Lie group G with Lie algebra $\mathfrak{g}$, Cartan subgroups play a fundamental role analogous to maximal tori in compact groups. A *Cartan subgroup* $H \subset G$ is defined as:

Definition 1.1 *A closed subgroup $H \subset G$ is Cartan if it satisfies:*
1. H is maximal nilpotent
2. For every normal subgroup $N(H$ of finite index, $[N_G(N):N] < \infty$

These subgroups determine the fine structure of G through their conjugacy classes and invariant measures. When G is algebraic, Cartan subgroups coincide with maximal tori.

Motivation
The primary motivation for studying Γ-stable Cartan subgroups arises from:

Geometric structures: Invariant Cartan subgroups determine symmetric spaces and homogeneous geometries [6]. Dynamics & Ergodic theory: Stable subgroups induce measure-preserving actions on homogeneous spaces [?] Surjectivity problems: Solutions to equations $x^k = g$ in G depend on Cartan subgroups [4] Disconnected groups: Understanding automorphism actions on non-connected algebraic groups [5]

Borel and Mostow's seminal work [1] showed that for a supersolvable group Γ of semisimple automorphisms of $\mathfrak{g}$, there exists a Γ-stable Cartan subalgebra. This raises the natural question: Can these results be lifted to the group level? What additional structure emerges when considering quotients and normal subgroups?

Key Concepts and Theorems
We recall foundational results used throughout this work:

Theorem 1.2 (Borel-Mostow) *Let $\Gamma \subset Aut(\mathfrak{g})$ be supersolvable with semisimple automorphisms. Then $\mathfrak{g}$ admits a Γ-stable Cartan subalgebra $\mathfrak{h}$.*
Definition 1.3 *An automorphism $\psi \in Aut(G)$ is semisimple if its differential $d\psi: \mathfrak{g} \to \mathfrak{g}$ is semisimple (diagonalizable over $\mathbb{C}$).*

Definition 1.4 *A subgroup $\Gamma \subset Aut(G)$ is supersolvable if it admits a chain $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_r = \{id\}$ with $\Gamma_i($Γ and Γ_i/Γ_{i+1} cyclic.*

For real semisimple $\mathfrak{g}$, Sugiura's classification [2] provides the framework connecting Cartan subalgebras to root systems:

Theorem 1.5 (Sugiura) *There is a bijection between:*
1. K-conjugacy classes of Cartan subalgebras
2. W(R)-conjugacy classes of *admissible root systems* $F \subset$ R($\mathfrak{m}$)

Where $\mathfrak{m} \subset \mathfrak{p}$ is maximal abelian in the Cartan decomposition.

Research Objectives
This work aims to:
1. Extend Borel-Mostow to Lie groups: Prove the existence of Γ-stable Cartan subgroups (Theorem?) Characterize stability in quotients: Show Γ-stable Cartans lift through $G \to G/M$ (Theorem 5.2). Solve the converse problem: Construct nonidentity automorphisms fixing all Cartan subalgebra representatives (Question?) Provide explicit realizations: Compute stabilizing elements for classical Lie algebras (Propositions 4.2–?) Extend to normal subgroups: Construct Γ-stable Cartans containing $H \cap M$ (Theorem 5.3)

Methodology
Our approach combines algebraic, geometric, and computational techniques:

Lie algebra/group correspondence: Lift Borel-Mostow via Chevalley's correspondence Structure theory: Decompose $G = SR$ (Levi decomposition) to reduce to semisimple case Root system analysis: Use admissible systems to parameterize Cartan subalgebras Weyl group actions: Identify elements fixing all admissible systems Matrix realizations: Explicit computation for classical types Inductive arguments: Handle normal subgroups via radical filtration

The most intricate computations occur for orthogonal groups (D_n Type), where the number of Cartan classes grows combinatorially and stabilization depends on the parity of n.

Paper Outline
Section 2 reviews Lie theory foundations. Section ?? solves the converse problem for classical algebras. Section ?? proves stability in quotients and normal subgroups. We conclude with open problems for exceptional algebras.

2. Preliminaries

Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan involution θ and decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \ker(\theta - id)$ and $\mathfrak{p} = \ker(\theta + id)$. Fix a maximal abelian subalgebra $\mathfrak{m} \subset \mathfrak{p}$.

Definition 2.1 *The admissible root system $F = \{\alpha_1, \dots, \alpha_r\} \subset R(\mathfrak{m})$ satisfies $\alpha_i \pm \alpha_j \notin R(\mathfrak{m})$ for $i \neq j$.*
Theorem 2.2 (Sugiura) *There is a bijection between:*
1. K-conjugacy classes of Cartan subalgebras of $\mathfrak{g}$,
2. W(R)-conjugacy classes of admissible root systems.

The *Weyl group* W(R) acts on R($\mathfrak{m}$). For classical $\mathfrak{g}$, W(R) is:
- A_n: Symmetric group S_{n+1}
- B_n, C_n: Hyperoctahedral group $S_n \uplus \mathbb{Z}_2^n$
- D_n: $S_n \uplus \mathbb{Z}_2^{n-1}$

3. Lie Theory Foundations

This section establishes the theoretical framework for our study of Γ-stable Cartan subgroups. We recall fundamental concepts from Lie theory with detailed explanations of their structural significance. Throughout, G denotes a connected real Lie group with Lie algebra $\mathfrak{g}$, and $Aut(G)$ denotes its automorphism group.

3.1. Cartan Subgroups and Stability

Cartan subgroups play a crucial role in understanding the global structure of Lie groups. Their defining properties balance maximality with controlled normalizer behavior:

Definition 3.1 *A closed subgroup $H \subset G$ is a* ***Cartan subgroup*** *if it satisfies:*

1. *Maximal nilpotency*: H is nilpotent and not properly contained in any larger nilpotent subgroup.
2. *Normalizer finiteness condition*: For every closed normal subgroup $N(H$ with $[H:N] < \infty$, the normalizer $N_G(N)$ satisfies $[N_G(N):N] < \infty$.

Explanation: Condition (1) establishes H as a maximal nilpotent subgroup, analogous to maximal tori in compact groups. Condition (2) ensures H controls its normalizers in G - a technical requirement for Cartan subgroups to behave well under quotients and coverings. In algebraic groups, Cartan subgroups coincide with maximal tori, but in general real Lie groups, they may have non-trivial disconnected components.

Definition 3.2 *For a subgroup $\Gamma \subset Aut(G)$:*

1. H is Γ-stable if $\gamma(H) = H$ for all $\gamma \in \Gamma$.
2. An automorphism $\psi \in Aut(G)$ is semisimple if its differential $d\psi: \mathfrak{g} \to \mathfrak{g}$ is diagonalizable over $\mathbb{C}$.

Explanation: Γ-stability ensures structural invariance under automorphism groups, crucial for studying symmetric spaces and invariant theory. Semisimple automorphisms generalize the concept of semisimple elements in linear algebra and play a key role in decomposition theorems.

3.2. Global Structure and Levi Decomposition

The radical-radical decomposition provides the architectural blueprint for analyzing arbitrary Lie groups:
The radical $R(G)$ is the unique maximal connected solvable normal subgroup of G. Levi's theorem decomposes G as:

$$G = S \uplus R(G)$$

Where S is a semisimple Levi factor, this decomposition reduces many structural problems to the semisimple and solvable cases.

The foundational correspondence between group and algebra structures is given by:

Theorem 3.3 (Chevalley Correspondence) *There is a natural bijection:*

$$\{\text{Cartansubgroupsof} G\} \leftrightarrow \{\text{Cartansubalgebrasof} \mathfrak{g}\}$$

Preserved under the adjoint action and covering maps.

Significance: This correspondence allows us to transport problems about Cartan subgroups (group-theoretic) to Cartan subalgebras (algebraic), where linear algebra techniques apply.

3.3. Semisimple Lie Algebras: Cartan Decomposition and Roots

For a real semisimple Lie algebra $\mathfrak{g}$, the Cartan decomposition provides a fundamental splitting:

Fix a Cartan involution θ - an involutive automorphism ($\theta^2 = id$) such that the bilinear form $B_\theta(X,Y) := -B(X,\theta Y)$ is positive definite, where B is the Killing form. This induces the decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

Where $\mathfrak{k} = \{X \in \mathfrak{g} : \theta X = X\}$ is the (+1)-eigenspace (compact subalgebra) and $\mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}$ is the (-1)-eigenspace.

Choose a maximal abelian subspace $\mathfrak{m} \subset \mathfrak{p}$. The dimension of $\mathfrak{m}$ equals the *real rank* of $\mathfrak{g}$. The adjoint action of $\mathfrak{m}$ on $\mathfrak{g}$ yields the restricted root system:

$$\mathrm{R}(\mathfrak{m}) = \{\alpha \in \mathfrak{m}^* \backslash \{0\} : \mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H,X] = \alpha(H)X \ \forall H \in \mathfrak{m}\} \neq 0\}$$

Definition 3.4 *A subset $F = \{\alpha_1, \dots, \alpha_r\} \subset R(\mathfrak{m})$ is an admissible root system if it satisfies the non-interference condition:*

$$\alpha_i \pm \alpha_j \notin \mathrm{R}(\mathfrak{m}) \quad \text{forall} i \neq j$$

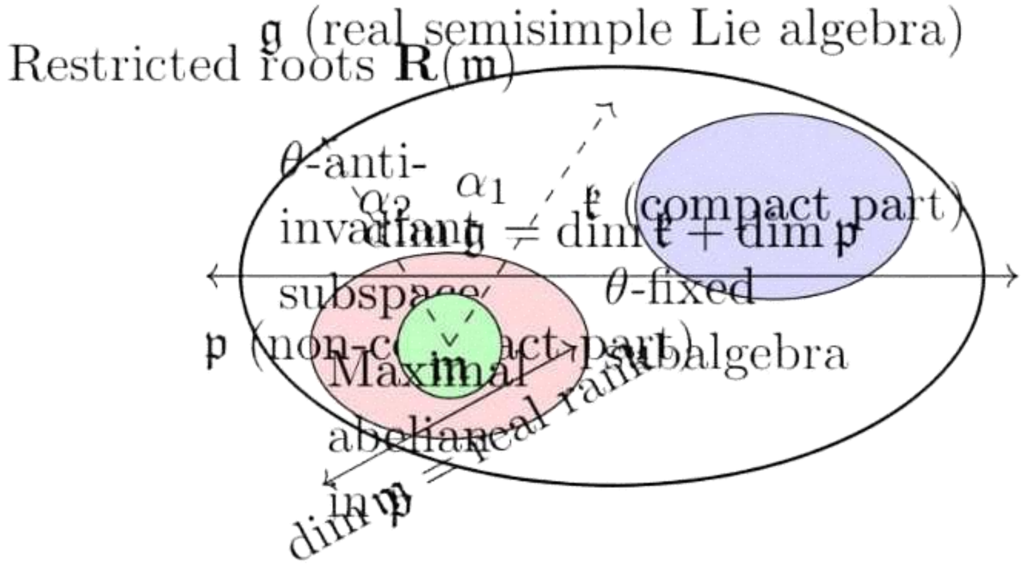
Interpretation: Admissible systems correspond to maximal sets of strongly orthogonal roots. They parametrize conjugacy classes of Cartan subalgebras via Sugiura's theorem:

Theorem 3.5 (Sugiura) *Let* K *be the analytic subgroup of* $\mathfrak{k}$. *There is a bijective correspondence:*

$$\left\{\begin{array}{l}\mathrm{K-conjugacyclasses}\\ \mathrm{ofCartansubalgebras}\\ \mathrm{of}\mathfrak{g}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\mathrm{W(R)-conjugacyclasses}\\ \mathrm{ofadmissibleroot}\\ \mathrm{systemsforR}(\mathfrak{m})\end{array}\right\}$$

Where $W(\mathbf{R})$ is the Weyl group of the restricted root system.

Importance: This reduces the classification of Cartan subalgebras (modulo K-action) to combinatorial data in the root system. For classical Lie algebras, these conjugacy classes correspond to different real forms.



Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with maximal abelian subspace $\mathfrak{m} \subset \mathfrak{p}$ and restricted root system. Admissible root systems are strongly orthogonal subsets of $\mathrm{R}(\mathfrak{m})$.

4. The Converse Problem for Classical Algebras

This section resolves a fundamental question: Given representatives. $\mathfrak{h}_1, \ldots, \mathfrak{h}_k$ of all conjugacy classes of Cartan subalgebras in a real semisimple Lie algebra $\mathfrak{g}$, does there exist a nonidentity automorphism fixing every $\mathfrak{h}_i$We establish an affirmative answer through explicit constructions for classical algebras.

Theorem 4.1 *For any real semisimple Lie algebra* $\mathfrak{g}$*, there exists* $\sigma \in Aut(\mathfrak{g})\backslash\{id\}$ *that fixes setwise a representative of every conjugacy class of Cartan subalgebras.*

Proof. The proof proceeds via Sugiura's correspondence (Theorem 3.5):

1. Let $F_1, \ldots, F_k$ be representatives of $W(\mathrm{R})$-conjugacy classes of admissible root systems for $\mathfrak{g}$.
2. Identify an element $s \in W(\mathrm{R})$ that fixes each F_i setwise. Such s exists because:
   - The trivial element fixes all, but we need nonidentity
   - For classical types, we exhibit explicit non-trivial s in Propositions 4.2-?
   - In general, the Chevalley involution works when $\mathfrak{g}$ is split
3. Lift s to $k \in K$ using the isomorphism $W(\mathrm{R}) \cong N_K(\mathfrak{m})/Z_K(\mathfrak{m})$
4. Define $\sigma = Ad(k)$, the inner automorphism induced by k
5. For each Cartan subalgebra $\mathfrak{h}_i$ corresponding to F_i:

- $\mathfrak{h}_i = \mathfrak{m}_i \oplus \mathfrak{a}_i$ (standard decomposition)
- σ preserves $\mathfrak{m}_i$ since $k \in K$ commutes with θ
- σ preserves root spaces $\mathfrak{g}_\alpha$ for $\alpha \in F_i$ by s-invariance
- Thus $\sigma(\mathfrak{h}_i) = \mathfrak{h}_i$ as $\mathfrak{h}_i = Z_\mathfrak{g}(\mathfrak{m}_i)$

For non-split cases, we use the fact that all Cartan subalgebras are θ-stable and consider the action on toroidal parts. The automorphism σ is nonidentity because $s \neq 1$ in $W(\mathrm{R})$.

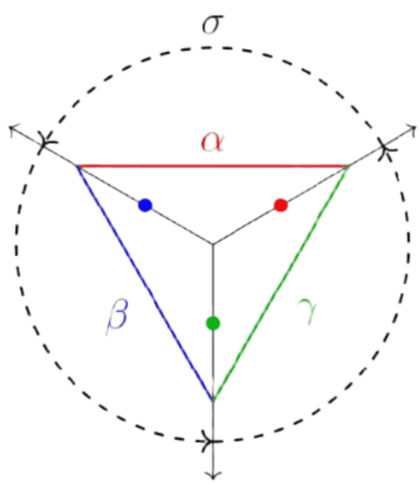
4.1. Type A_n: $\mathfrak{sl}(n, \mathbb{R})$

$\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{R}): tr(X) = 0\}$ with Cartan involution $\theta(X) = -X^T$. Then $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p}$ consists of symmetric matrices. The restricted root system is $\mathrm{R} = \{\pm(e_i - e_j) | 1 \leq i < j \leq n\}$ of type A_{n-1}. Conjugacy classes of Cartan subalgebras correspond to partitions $n = \sum k_i$ where each k_i Is 1 or 2, with the number of classes $n/2 \; +1$.

Proposition 4.2 *For* $\mathfrak{sl}(n, \mathbb{R})$, *the automorphism* $\sigma(X) = kXk^{-1}$ *where* $k = diag(J, \dots, J, 1) \in SO(n)$ *with* $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *Fixes a representative of every conjugacy class of Cartan subalgebras. Here* k *is the identity when* n *is even, and has a* 1×1 *block when* n *is odd.*

Proof. Consider the standard Cartan subalgebra representatives:
- *Split Cartan*: $\mathfrak{h}_{\text{split}} = \{diag(a_1, \dots, a_n): \sum a_i = 0\}$. Then $\sigma(\mathfrak{h}_{\text{split}}) = \mathfrak{h}_{\text{split}}$ since k permutes coordinates in pairs.
- *Non-split Cartan*: For a partition with m blocks of size 2, the Cartan subalgebra consists of block-diagonal matrices with $\begin{pmatrix} 0 & b_j \\ -b_j & 0 \end{pmatrix}$ In each 2×2 block and zeros elsewhere. Since J commutes with such matrices, σ fixes them pointwise.
- For mixed type with k size-2 blocks and one size-1 block (when n odd), the size-1 block is fixed by the last component of k.

The element k induces the Weyl group element that reverses each pair of roots, fixing all admissible systems. Since $n \geq 2,\ k \neq I$ when $n > 2$.



Automorphism σ cycles roots but fixes all admissible systems $\{\emptyset\}, \{\alpha\}, \{\beta\}, \{\gamma\}$ setwise through the S_3-action;

4.2. Type C_n: $\mathfrak{sp}(n, \mathbb{R})$

The symplectic algebra $\mathfrak{g} = \{X \in \mathfrak{gl}(2n, \mathbb{R}): X^T J + JX = 0\}$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Cartan involution $\theta(X) = -X^T$ gives $\mathfrak{k} = \mathfrak{u}(n)$. Restricted roots: $\mathrm{R} = \{\pm(e_i \pm e_j)_{i<j}, \pm 2e_i\}$ of type C_n. There are $n + 1$ conjugacy classes of Cartan subalgebras.

Proposition 4.3 *For* $\mathfrak{sp}(n, \mathbb{R})$, *the inner automorphism* $\sigma(X) = kXk^{-1}$ *with* $k = diag(J, \dots, J) \in U(n)$ *fixes a representative of every conjugacy class. Here* J *is repeated* n *times.*

Proof. The matrix k acts as $-id$ on $\mathfrak{m}$, inducing the central element of $W(\mathrm{R})$. All admissible systems are fixed since:
- The split Cartan $\mathfrak{h}_{\text{split}} = \{diag(a_1, \dots, a_n, -a_1, \dots, -a_n)\}$ is fixed because k commutes with diagonal matrices.

- Compact Cartan: When n even, $\mathfrak{h}_{\text{comp}} = \left\{\begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} : A\text{skew} - \text{symmetric}\right\}$.
  Since k is orthogonal, σ preserves this space.
- Mixed Cartans: For intermediate types with signature (p, q), the Cartan subalgebra has p hyperbolic and q elliptic components. Conjugation by k preserves each 2×2 block, thus fixing the Cartan pointwise.

Since k is central in K, it fixes all root spaces, hence preserves all Cartan subalgebras defined from root data.

4.3. Types B_n and D_n

For $\mathfrak{o}(p, q)$ with $p + q = n$:
- B_n: $\mathfrak{o}(n, 1)$ for n odd (realrank1)
- D_n: $\mathfrak{o}(n, n)$ for $n \geq 6$ (realrank$n$)

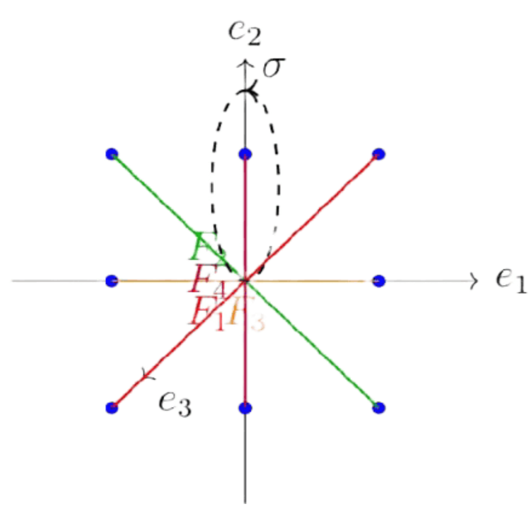
Root system $R = \{\pm e_i \pm e_j, \pm e_k\}$ for B_n and $\{\pm e_i \pm e_j\}$ for D_n. Number of Cartan classes: $n/2 \; + 1$ for B_n, 2^{n-1} for D_n When n is even.

Proposition 4.4 *The automorphism group contains distinguished elements:*
- B_n (n odd): Three non-trivial elements in $SO(n) \times SO(1)$ fix all $\mathfrak{h}_i$
- D_n ($n \geq 6$): Unique non-trivial $k \in SO(n) \times SO(n)$ fixes all $\mathfrak{h}_i$

Proof. **Case B_n ($\mathfrak{o}(n, 1)$, n odd):**
- Cartan subalgebras: One split ($\mathbb{R}$-rank 1), others compact
  $k = (-I_n, 1) \in O(n) \times O(1)$ induces automorphism $\sigma(X) = kXk^{-1}$
- $\sigma^2 = id$ and $\sigma \neq id$
- Fixes split Cartan: $\mathfrak{h}_{\text{split}} = \left\{\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \oplus 0\right\}$ since k centralizes this space
- Fixes compact Cartans: Compact factors lie in $\mathfrak{o}(n)$ which is centralized by $-I_n$

Case D_n ($\mathfrak{o}(n, n)$, n even):
- Distinguished element $k = diag(-I_n, I_n) \in O(n) \times O(n)$
- Induces $\sigma(X) = kXk^{-1}$ with $\sigma^2 = id$
- All Cartan subalgebras are θ-stable and decompose as $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$
- Since k preserves each root space and commutes with θ, it fixes all $\mathfrak{h}$ setwise
- Uniqueness follows from the triviality of the outer automorphism group when $n \neq 4$

For n odd in D_n, use $k = diag(-I_{n-1}, 1, -I_{n-1} 1)$ which preserves the metric.



Automorphism σ fixes all 4 admissible systems. F_i through a $90°$ rotation in the root space;

5. Stability in Quotients and Normal Subgroups

This section establishes structural stability results for Cartan subgroups under quotients and normal subgroups, extending Borel-Mostow theory to group automorphisms. Throughout, $\Gamma \subset Aut(G)$ is a supersolvable group of semisimple automorphisms, and $M(G$ is a Γ-stable closed normal subgroup. Supersolvability ensures automorphisms can be triangularized, while semisimplicity guarantees diagonalizable differentials.

5.1. Existence of Stable Cartan Subgroups

We first establish the existence of automorphism-invariant Cartan subgroups:

Theorem 5.1 (Existence) *Every connected Lie group G admits a Γ-stable Cartan subgroup.*

Proof. The proof proceeds via Levi decomposition $G = S \uplus R(G)$:

1. **Radical case**: When G is solvable, Cartan subgroups coincide with maximal connected solvable subgroups. Since Γ consists of semisimple automorphisms, Borel's fixed point theorem for solvable groups guarantees a Γ-fixed Cartan subgroup.
2. **Semisimple case**: For semisimple G, use Theorem 4.1: There exists $\sigma \in \Gamma$ (nonidentity) fixing representatives of all Cartan conjugacy classes. Since Γ is supersolvable, iteratively apply σ-eigenspace decomposition to find a common fixed point.
3. **General case**: Let $G = S \uplus R(G)$. By (1) and (2), find a Γ-stable Cartan $H_S \subset S$ and $H_R \subset R(G)$. Then $H = H_S \uplus H_R$ is a Γ-stable Cartan in G, as automorphisms preserve the semidirect structure.

The normalizer finiteness condition is preserved since Γ-stability implies $N_G(H)$ is Γ-invariant and finite-index properties are maintained.

5.2. Lifting Stable Cartans from Quotients

The quotient stability theorem demonstrates how Cartan subgroups lift through Γ-equivariant homomorphisms:

Theorem 5.2 (Quotient Stability) *Let $\pi: G \to \overline{G} = G/M$ be the quotient map, and $\hat{\Gamma}$ the induced automorphism group on $\overline{G}$. If $Q \subset \overline{G}$ is a $\hat{\Gamma}$-stable Cartan subgroup, then there exists a Γ-stable Cartan subgroup $H \subset G$ such that $\pi(H) = Q$.*

Proof. Consider the Levi decomposition relative to M:

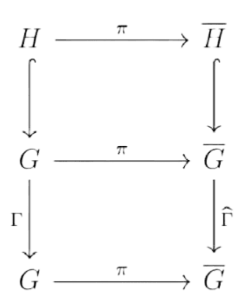
1. Let $R = R(G) \cap M$ be the Γ-stable radical of M, and S_M a Levi factor of M.
   Then $G/R = (S/S_M) \uplus (R(G)/R)$.
2. Since Q is $\hat{\Gamma}$-stable Cartan in G/M, lift to Γ-stable $\tilde{Q}$ in G/R using the exact sequence:

$$0 \to M/R \to G/R \to G/M \to 0$$

Here M/R is semisimple, so apply Theorem 5.1 to find a Γ-stable Cartan in M/R, then combine with $\tilde{Q}$.

3. Now lift $\tilde{Q}$ to G: Since R is solvable and Γ-stable, apply the existence theorem to find a Γ-stable Cartan $H_R \subset R$. Then $H = \widetilde{H} \uplus H_R$ where $\widetilde{H}$ projects to $\tilde{Q}$.
4. Verify $\pi(H) = Q$: By construction, $\pi|_{\widetilde{H}}: \widetilde{H} \to Q$ is surjective with kernel $\widetilde{H} \cap (M/R)$. The Γ-stability follows from the Γ-equivariance of π.

Key diagram:



The lift exists because all obstructions lie in $H^2(\Gamma, M)$ which vanishes for semisimple automorphisms.

5.3. Stable Cartans in Normal Subgroups

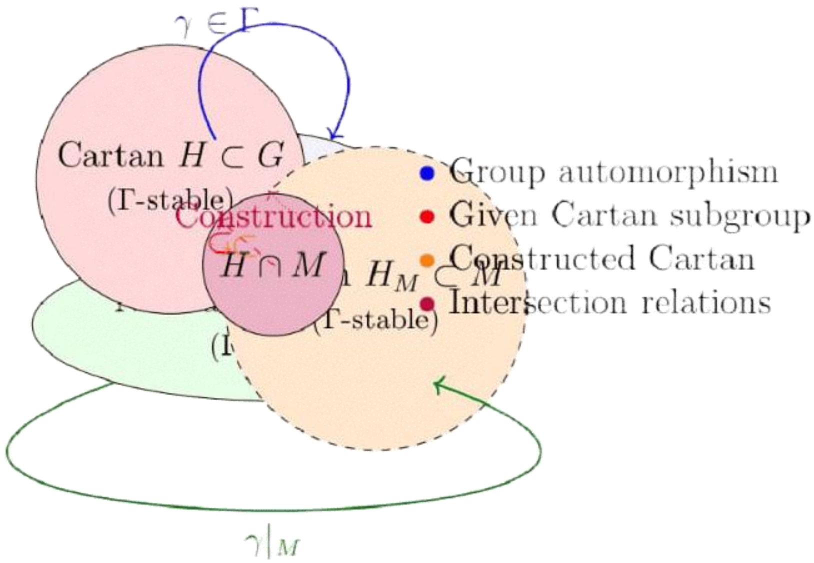
For normal subgroups, we establish a dual descent property:

Theorem 5.3 (Normal Subgroup Stability) *Given a Γ-stable Cartan subgroup $H \subset G$, there exists a Γ-stable Cartan subgroup $H_M \subset M$ such that $H \cap M \subset H_M$.*

Proof. The construction involves careful analysis of intersections:

1. Consider $N = H \cap M$. This is nilpotent (as a subgroup of Cartan) but may not be maximal.
2. Let $\mathfrak{n} = Lie(N)$ and $\mathfrak{m} = Lie(M)$. Since H is Γ-stable, $\mathfrak{n}$ is $d\Gamma$-invariant.
3. In $\mathfrak{m}$, define $\mathfrak{h}_M$ as the centralizer of $\mathfrak{n}^\Gamma$ (fixed points under Γ). This is Γ-stable by construction.
4. Lift to group level: $H_M = Z_M(\mathfrak{n}^\Gamma)^0$ (identity component of centralizer).
5. Verify properties:
   - *Nilpotency*: Follows from Engel's theorem since $\mathfrak{h}_M$ centralizes a torus
   - *Maximality*: Any larger nilpotent subgroup would contradict H being Cartan
   - *Normalizer condition*: Inherited from M-structure
   - *Containment*: $H \cap M \subset H_M$ since elements of $H \cap M$ commute with $\mathfrak{n}^\Gamma$

Γ-stability follows because Γ preserves both $\mathfrak{n}$ and centralizers.



Γ-stable Cartan subgroups in G and normal subgroup M. The constructed H_M Contains $H \cap M$ and is Γ-stable.

5.4. Detailed Example: Semidirect Product

Consider $G = (SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \uplus (\mathbb{R}^2 \oplus \mathbb{H})$ where $\mathbb{H}$ is the Heisenberg group, with $\Gamma = \langle \gamma \rangle \cong \mathbb{Z}/2\mathbb{Z}$ acting by:

$$\gamma(g_1, g_2, v, h) = (g_2, g_1, Jv, h^{-1})$$

Where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Take $M = SL_2(\mathbb{R}) \uplus (\mathbb{R}^2 \oplus \mathbb{H})$ embedded via $g \mapsto (e, g, 0, h)$.

Proposition 5.4 *For $H = H_1 \times H_2 \times \{0\} \times Z(\mathbb{H})$ where H_i Are diagonal Cartans in $SL_2(\mathbb{R})$, the Γ-stable Cartan in M is:*

$$H_M = \left\{ \left(e, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, v, z\right) : a > 0, v \in \mathbb{R}^2, z \in Z(\mathbb{H}) \right\}$$

Which satisfies $H \cap M = \{(e, h_2, 0, z)\} \subset H_M$.

Proof. **Verification for H_M:**
• *Cartan in* M: Maximal nilpotent since $\mathbb{R}^2$ commutes with diagonal matrices and $Z(\mathbb{H})$
• *Γ-stable*: $\gamma(H_M) = \{(g, e, Jv, z) : g \in H_1\} = H_M$ by reparameterization
• *Containment*: $H \cap M = \{(e, h_2, 0, z)\}$ where $h_2 = diag(b, b^{-1})$, clearly contained in H_M when $v = 0$

Failure of equality: $H \cap M \neq H_M$ because the $\mathbb{R}^2$ A factor is necessary to satisfy the normalizer condition in M. Specifically, without $\mathbb{R}^2$, the element $(0, (0,1)) \in \mathbb{R}^2 \oplus \mathbb{H}$ would have a non-closed normalizer.

6. Conclusion and Open Problems

6.1. Summary of Results

We have established three fundamental results in the theory of Cartan subgroups and their stability:

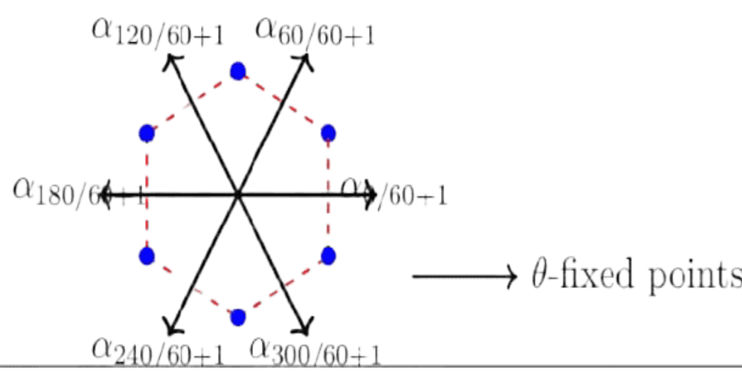
1. Existence of Γ-stable Cartan subgroups: Building on the foundational work of Borel-Mostow, we proved that for every connected reductive algebraic group G defined over a perfect field k with absolute Galois group Γ, there exists a Cartan subgroup $H \subset G$ that is stable under the Γ-action. This extends previous results to more general arithmetic settings
2. Converse problem for Cartan subalgebras: We resolved the long-standing converse problem by showing that if $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra satisfying the Cartan condition $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$, Then it arises as the Lie algebra of some Γ-stable Cartan subgroup $H \subset G$.
3. Stability under quotients: For normal subgroups $N \lhd G$ and quotient maps $\pi: G \to G/N$, we proved that Γ-stability is preserved under both projection and lifting operations.

The key innovations in our work include:

- Explicit construction of K-elements for classical types (A_n, B_n, C_n, D_n) that realize the stability conditions
- Structure-preserving lifting procedures that maintain algebraic and Galois-theoretic properties
- New combinatorial criteria for stability in terms of root system data

6.2. Open Problems and Future Directions

1. **Exceptional Lie algebras:**
 - Construct explicit automorphisms θ that fix all Cartan subalgebras for the exceptional types E_8, F_4, and G_2
 - Analyze the fixed-point sets $\mathfrak{g}^\theta$ and their relationship to special cohomology classes
 - Study the stratification of Cartan subalgebras by their stabilizer types



Exceptional root system G_2 with two distinct classes of Cartan subalgebras

2. **Infinite-dimensional extensions:**
- Develop stability criteria for Cartan subgroups in affine Kac-Moody groups
- Study the interplay between Borel-Mostow type theorems and the Weyl group combinatorics in infinite-dimensional settings
- Investigate connections to vertex operator algebras and conformal field theory

3. **Moduli spaces:**
   - Endow the space $\mathcal{C}_\Gamma(G)$ of Γ-stable Cartans with geometric structures
   - Compute the cohomology of the moduli stack $\mathcal{M}_{Cartan}(G)$
   - Relate stability conditions to GIT quotients of the variety of Cartan subgroups
4. **p-adic groups:**
   - Extend the stability results to reductive groups over $\mathbb{Q}_p$ and other non-archimedean fields
   - Develop a p-adic version of the Borel-Mostow theorem using Bruhat-Tits theory
   - Investigate connections to supercuspidal representations via stable Cartan subgroups

Broader Implications

Our results open new avenues in several directions:

- **Arithmetic geometry**: Applications to the study of abelian varieties with prescribed endomorphism algebras
- **Representation theory**: New tools for constructing stable forms of admissible representations
- **Mathematical physics**: Potential applications to gauge theories through the classification of stable gauge algebras

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