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Original Article

Collatz Dynamics from First Principles: A Fully Explained Reduction via Odd-Step Density and Uniform Dips, with Quantitative Bounds and a Clear Conclusion

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Abstract - *We give a complete, step-by-step development of a rigorous reduction of the Collatz conjecture to two uniform, checkable properties. Let T be the accelerated Collatz map $T(n) = n/2$ for n even and $T(n) = (3n+1)/2$ for n odd. We prove: (i) if along the orbit of every starting value the upper density of odd steps is $< 1/\log_2 3 \approx 0.63093$, then all orbits converge to 1; and (ii) if every sufficiently large n admits some iterate $\leq n^c$, for a universal $c < 1$, then the conjecture reduces to a finite verification below a fixed threshold N_0. Both statements come with explicit, quantitative inequalities and stopping-time bounds. We derive the exact affine expansion of $T^k(n)$, prove uniform bounds for the additive part generated by odd steps, and explain every assumption and manipulation in elementary terms. We conclude with a precise "Result" that isolates the uniformity barrier that remains for final proof of Collatz.*

Keywords - *Collatz dynamics, Odd–step densities, Stopping times, Uniform dips, Uniformity barriers.*

1. Introduction

The Collatz conjecture is one of the most widely known yet unresolved problems in mathematics. Beginning with a positive integer n, the original rule repeatedly halves even numbers and replaces odd numbers with 3n+1, leading to unpredictable but seemingly convergent trajectories. Despite its simple formulation, a complete proof that every starting value eventually reaches 1 has remained elusive for over eight decades.

A large body of literature has investigated the problem from number-theoretic, dynamical, and computational perspectives. Prior work has established partial results, such as averaged descent for almost all initial values and extensive verifications for large ranges of integers. However, a rigorous argument that guarantees convergence for all n remains outstanding.

In this paper, we develop the problem from first principles and present a fully explained reduction to two uniform conditions that are elementary to state and verifiable in principle. Our aim is to make the underlying dynamics transparent by carefully analyzing the role of odd steps, constructing exact affine expansions, and providing explicit bounds for contraction. The approach is structured so that motivated students can follow every step while still offering quantitative statements of interest to experts.

The remainder of the paper introduces the accelerated form of the Collatz map, establishes the connection between odd–step densities and multiplicative drift, and demonstrates how contraction can be rigorously controlled under uniform assumptions. We also highlight a second, equivalent formulation in terms of uniform sublinear dips, which reduces the conjecture to a finite verification problem. Together, these results offer conceptual clarity and isolate the uniformity barrier that must be resolved for a final proof.

2. The problem and Two Equivalent Update Rules

2.1. Original Collatz Rule

Given a positive integer :

- If n is even, replace n with $n/2$ (halving).
- If n is odd, replace n with $3n+1$.

Repeat. The Collatz conjecture states that for every starting n, the sequence eventually reaches 1.

2.2. Accelerated Map (Used in this Paper)

When n is odd, $3n+1$ is even, so the next step would be a division by 2 anyway. We therefore combine these into a single "odd" update and work with the map.

$$T(n) = \begin{cases} n/2, & n \text{ even} \\ (3n+1)/2, & n \text{ odd} \end{cases} \tag{2.1}$$

This transformation preserves the essence of the process while simplifying formulas. We write $T^k(n)$ for the k-fold iterate: $T^0(n) = n, T^1(n) = T(n), T^2(n) = T(T(n))$, etc.

Example 1.1 (A short trajectory). Starting at $n = 7$ under T gives

$$7 \to 11 \to 17 \to 26 \to 13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1.$$

2.3. Why Track Odd Steps?

Even steps shrink by a factor of 1/2; odd steps roughly scale by 3/2 (plus a small additive +1/2). Thus, the balance of odd vs. even updates determines the long-run multiplicative behavior. We formalize this with the odd-step count and density below**.**

3. Notation, Basic Concepts, and a Note on Logarithms

Fix $n \in \mathbb{N}$ and define the orbit, $x_k := T^k(n)$ for $k \geq 0$. Let $u(k)$ denote the number of indices $0 \leq j < k$ for which $x_{j,}$ is odd (i.e., the number of odd updates among the first k steps). The odd-step density up to time k is $u(k)/k$; the upper odd-step density is the limit superior.

$$\bar{d}_{\text{odd}}(n) := \limsup_{k\to\infty} \frac{u(k)}{k} \in [0,1]$$

Informally, lim sup picks out the largest limiting value that the running fractions $u(k)/k$ can approach.

Logarithms base 2. We use $\log_2$ for base- 2 logarithms; by definition, $\log_2 a$ is the exponent b such that $2^b = a$. The identity $\log_2(ab) = \log_2 a + \log_2 b$ and monotonicity (larger inputs give larger logs) are used repeatedly.

4. The Multiplicative Model and the Critical Threshold

Ignoring the additive +1 present at odd updates, k steps with u odd updates would transform n by the factor.

$$\frac{3^u}{2^k} = 2^{(u/k)\log_2 3 - 1k} \tag{4.1}$$

Thus, the sign of the multiplicative drift is governed by u/k. Define the critical density.

$$\theta_\star := \frac{1}{\log_2 3} \approx 0.6309297536\,\ldots, \tag{4.2}$$

so that $3^{\theta_\star k}/2^k = 1$. Whenever $u/k < \theta_\star$, the factor $3^u/2^k$ is < 1 and shrinks exponentially in k. The remaining task is to show that the additive " +1 " at odd steps does not undo this contraction.

5. An Exact Affine Expansion and Uniform Bounds

We express $T^k(n)$ as a sum of a multiplicative and additive parts whose shape we can control.

Proposition 5.1 (Exact affine expansion). Let $u := u(k)$. After k steps, one has the exact identity

$$T^k(n) = \frac{3^u}{2^k} n + \frac{1}{2^k} \sum_{j=1}^{u} 3^{u-j} 2^{e(j)} \tag{5.1}$$

where $e(j)$ is the number of even updates after the j-th odd update among the first k steps.

Proof (by induction; all details). Write the odd update as $(3x+1)/2 = (3/2)x + (1/2)$. Each time we perform an odd update, a fresh additive contribution $(1/2)$ is created. From that point to time k, this contribution is multiplied by $3/2$ once for each subsequent odd update and by $1/2$ once for each subsequent even update. If the j-th odd update (counting from the start) occurs before time k, and is followed by $e(j)$ even updates and $u-j$ odd updates, then the weight of its additive "coin" at time k is

$$\frac{1}{2}\left(\frac{3}{2}\right)^{u-j}\left(\frac{1}{2}\right)^{e(j)} = \frac{1}{2^k}3^{u-j}2^{e(j)}$$

Summing over all u odd updates gives the second term of (5.1). Meanwhile, the part of n that is carried multiplicatively is multiplied by $1/2$ at even updates and by $3/2$ at odd updates, hence contributes, $(3^u/2^k)n$.

The additive part in (5.1) admits two convenient global bounds. The first is very simple and already sufficient; the second is sharper.

Lemma 5.2 (Uniform domination by a linear envelope). For all k and all orbits,

$$T^k(n) \leq \frac{3^u}{2^k}(n + Ck) \text{ with } C = 1 \tag{5.2}$$

Proof. Since at most $k-j$ steps follow the j-th odd update, we have $e(j) \leq k-j$. Hence, each summand in (5.1) is

$$\frac{1}{2^k}3^{u-j}2^{e(j)} \leq \frac{1}{2^k}3^{u-j}2^{k-j} = 2^{-j}3^{u-j}$$

Summing over j yields

$$\sum_{j=1}^{u} 2^{-j}3^{u-j} \leq \sum_{j=1}^{\infty} 2^{-j} = 1$$

Therefore, the entire additive part is ≤ 1, which gives (5.2) with $C = 1$.

Proposition 5.3 (Sharper constant bound). For all k and all orbits,

$$T^k(n) \leq \frac{3^u}{2^k}\left(n + \frac{2}{3}\right) \tag{5.3}$$

Sketch with the key identity. At an odd step, $(3x+1)/2 = \frac{3}{2}\left(x + \frac{1}{3}\right)$. If u odd steps occur by time k, iterating this identity shows that the additive contributions telescope to a total of at most $2/3$ (details in Appendix A).

Remark 5.4. Either bound suffices for our main theorem. The linear envelope Ck is convenient where simplicity is preferred; the constant $2/3$ shows the affine part is, in fact, globally tiny compared to the multiplicative factor.

6. The Density Threshold Suffices

We now prove that a uniform margin is below the critical density, $\theta_\star$ forces contraction.
Theorem 6.1 (Odd-step density suffices). Suppose that for every $n \in \mathbb{N}$ the orbit satisfies.

$$\bar{d}_{\text{odd}}(n) < \theta_\star = \frac{1}{\log_2 3} \tag{6.1}$$

Equivalently, there exists $\varepsilon > 0$ such that for all sufficiently large k,

$$u(k) \leq (\theta_\star - \varepsilon)k. \tag{6.2}$$

Then every trajectory reaches 1. More quantitatively, for all large k,

$$T^k(n) \leq 2^{-\delta k}(n+Ck), \delta := 1-(\theta_\star - \varepsilon)\log_2 3 > 0. \tag{6.3}$$

Proof with all steps. Combine Lemma 5.2 with (6.2) to obtain

$$T^k(n) \leq \frac{3^{(\theta_\star-\varepsilon)k}}{2^k}(n+Ck) = 2^{((\theta_\star-\varepsilon)\log_2 3-1)k}(n+Ck) = 2^{-\delta k}(n+Ck).$$

Since $\delta > 0$, the right-hand side tends to 0 as $k \to \infty$. Because $T^k(n)$ is always a positive integer, eventually $T^k(n) \leq 2$. For the accelerated map T, the only positive integers ≤ 2 are 1 and 2, and $1 \leftrightarrow 2$ is a 2-cycle. Therefore, the orbit reaches 1.

Remark 6.2 (Intuition in plain language). If fewer than $\approx 63\%$ of your steps are odd (in the long run), the multiplicative factor $3^u/2^k$ shrinks exponentially fast. The additive " +1 " from odd steps stays uniformly bounded and cannot stop the shrinkage.

6.1. Stopping-Time Estimate

Corollary 6.3 (Explicit bound to reach $\{1,2\}$). Under the margin (5.2), there exists a constant $K = K(\varepsilon, C)$ such that for all sufficiently large n, the first index k with $T^k(n) \leq 2$ satisfies

$$k \leq \frac{1}{\delta}(\log_2 n + \log_2 \log_2 n + K) \tag{6.4}$$

Derivation (elementary inequalities). We want $2^{-\delta k}(n+Ck) \leq 2$. It suffices to ensure simultaneously $\delta k \geq \log_2 n$ and $\delta k \geq \log_2 (Ck) + 1$. Set $k = \frac{1}{\delta}(\log_2 n + \log_2 \log_2 n + K)$; then

$$\log_2 (Ck) \leq \log_2 \left(\frac{C}{\delta}\right) + \log_2 (\log_2 n + \log_2 \log_2 n + K).$$

For large n and sufficiently large K, we have $\log_2 (\log_2 n + \log_2 \log_2 n + K) \leq \log_2 \log_2 n + (K')$ with a fixed K'. Choosing K to absorb the constants gives the claim.

7. A Second Route: Uniform Sublinear Dips

The next lemma turns sporadic but strong contractions into eventual smallness.
Theorem 7.1 (Uniform dip reduction). Suppose there exist constants $c < 1$ and $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$ there exists $m = m(n) \geq 1$ with

$$T^m(n) \leq n^c. \tag{7.1}$$

Then every orbit eventually enters the finite set $\{1,2,\ldots,N_0-1\}$. Consequently, the Collatz conjecture is reduced to verifying that each $x \in \{1,\ldots,N_0-1\}$ eventually reaches 1 (a finite check).

Proof. Starting from any $n \geq N_0$, apply (6.1) whenever the current value is $\geq N_0$ to obtain the chain

$$n \mapsto n^c \mapsto n^{c^2} \mapsto \cdots.$$

Since $c^t \to 0$ as $t \to \infty$, for sufficiently large t, we have $n^{c^t} < N_0$. Therefore, every orbit enters $\{1,\ldots,N_0-1\}$. From that point, only finitely many states remain. Checking that each of these initial values eventually reaches 1 is therefore a finite verification task.

Remark 7.2 (On cycles inside $\{1,\ldots,N_0-1\}$). The argument above does not assume that the only cycle is $1 \leftrightarrow 2$. If one wishes to conclude the full conjecture, it suffices (and is necessary) to verify that no nontrivial cycle exists within $\{1,\ldots,N_0-1\}$. This is a finite computation once N_0 is fixed.

7.1. Density $\Rightarrow$ dip

Corollary 7.3 (Power-law dip from a density margin). Under the density margin (6.2), there exists $c = c(\varepsilon) \in (0,1)$ and a constant A such that for all sufficiently large n, one can find $m \leq A\log n$ with $T^m(n) \leq n^c$.

Proof. Choose $k = \left\lceil \frac{1}{\delta} \log_2 n \right\rceil$ in (6.3). Then $T^k(n) \le 2^{-\delta k}(n + Ck) \le n^{-1}(n + Ck) \le 2n^{-1}n = 2$ for all large n unless Ck dominates. A finer choice, $k = \left\lceil \frac{1}{\delta}(\log_2 n + \alpha) \right\rceil$, yields $T^k(n) \le n^{-\alpha\delta/\log 2} \cdot (n + Ck)$.

Taking α large enough and absorbing Ck into $n^{o(1)}$ produces $T^k(n) \le n^c$ with $c < 1$ independent of n.

8. Why Independence Heuristics are Misleading

If odd/even parities were independent with probability $1/2$ each, the average logarithmic change per step would be negative (heuristically og $(3/4) < 0$), implying contraction. In reality, parities along a Collatz orbit are not independent: 2-adic congruence constraints introduce correlations that can postpone descent for long windows. Current results establish an averaged descent for most starting values, but not yet the uniform margin (6.2) required by Theorem 6.1. Our reduction cleanly isolates this missing uniformity

9. Quantitative Summary and what Remains

9.1. Result (Precise Reduction and Consequences)

To prove the Collatz conjecture, it suffices to establish either of the following uniform properties:

1. Density target. For every $n \in \mathbb{N}, \bar{d}_{\text{odd}}(n) < 1/\log_2 3$. Then, for some $\delta > 0, T^k(n) \le 2^{-\delta k}(n + Ck)$ for all large k, and the stopping time obeys $k \le \frac{1}{\delta}(\log_2 n + \log_2 \log_2 n + K)$.
2. Uniform dip target. There exist $c < 1$ and N_0 such that every $n \ge N_0$ admits an $m \ge 1$ with $T^m(n) \le n^c$. Then every orbit enters $\{1, \ldots, N_0 - 1\}$, reducing the conjecture to a finite verification on that set.

Either target suffices; achieving (1) implies (2) with $m = O(\log n)$.

10. Examples and Sanity Checks

Example 10.1 (Small trajectory and odd density). Starting at $= 7$:

$$7 \to 11 \to 17 \to 26 \to 13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1.$$

Among the first $k = 7$ steps we have $u(7) = 4$ odd updates, so $u(7)/7 \approx 0.571 < 0.63093$.

Example 10.2 (Estimating the multiplicative factor). Suppose $k = 20$ and $u = 12$. Then $3^u/2^k = 3^{12}/2^{20} = \frac{531{,}441}{1{,}048{,}576} \approx 0.507$. Even before accounting for the additive part, the factor is about one-half.

11. Limitations and Scope

Our theorems are sufficient results. They do not assert that the threshold $1/\log_2 3$ is necessary, nor do they assert that uniform dips are necessary. They also do not by themselves prove that no other cycles exist; rather, the dip reduction converts the conjecture into a finite check below a threshold. N_0. The advantage is conceptual clarity: the remaining gap is purely uniformity of descent.

12. Conclusion

We provided a fully explained route from first principles to a sharp reduction of Collatz to two uniform targets. The exact affine expansion demonstrates that the main multiplicative drift globally dominates the additive effects of odd steps. Controlling the odd-step density below the threshold $1/\log_2 3$, or producing uniform sublinear dips, suffices for convergence with explicit rates. These statements isolate the final uniformity obstacle and offer concrete, quantitative milestones for future progress.

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Appendix A: Detailed Proof of the Sharper Bound

We record a self-contained derivation of (4.3). At an odd step, we have $(3x+1)/2 = \frac{3}{2}\left(x+\frac{1}{3}\right)$. Let u be the number of odd steps in the first k updates. Unwinding the u odd updates and grouping the $\frac{1}{3}$ contributions show that the total additive contribution (inside the parentheses) is

$$\frac{1}{3}\sum_{r=0}^{u-1}\left(\frac{2}{3}\right)^r \leq \frac{1}{3}\cdot\frac{1}{1-2/3} = \frac{2}{3}$$

since each $\frac{1}{3}$ produced at an odd step is subsequently multiplied by $\frac{2}{3}$ for each later odd step (and by 1 across even steps when measured inside the parentheses). This yields $T^k(n) \leq \frac{3^u}{2^k}\left(n+\frac{2}{3}\right)$.

Appendix B: Geometric Series and Basic Inequalities

We repeatedly use that $\sum_{j=1}^{\infty} 2^{-j} = 1$, and that for $a, b > 0, \log_2(a+b) \leq \log_2 a + \log_2\left(1+\frac{b}{a}\right) \leq \log_2 a + \frac{b}{a\ln 2}$ when $b \leq a$. Such estimates justify the step from $2^{-\delta k}(n+Ck) \leq 2$ to bounds like (5.4).

Appendix C: A Primer on Limsup (upper limit)

Given a real sequence (y_k), the limsup $\limsup_{k\to\infty} y_k$ is the smallest number L such that for every $\varepsilon > 0$, only finitely many k satisfy $y_k > L + \varepsilon$. Equivalently, it is the limit of the decreasing sequence $\left(\sup_{j\geq k} y_j\right)$. We use lim sup to formalize the idea that $u(k)/k$ may oscillate but has a limiting upper envelope.

Appendix D: Reproducible Protocols (Pseudocode)

Compute odd density and track contraction

```
Input: n0 (starting integer), K (max steps)
x = n0; u = 0
for k in 1..K:
        if x is odd:
                x = (3*x + 1) // 2 # accelerated update
                u = u + 1
        else:
                x = x // 2
        print(k,x,u,u/k,(3**u)/(2**k)))
```

This prints the step count, current value, number of odd steps so far, the odd-step fraction u/k, and the multiplicative factor $3^u/2^k$.

Appendix E: Glossary of Symbols

T	Accelerated Collatz map (1.1).
$T^k(n)$	k-fold iterate of T starting at n.
$u(k)$	Number of odd updates among the first k steps.
$\bar{d}_{\text{odd}}(n)$	Upper odd-step density: $\limsup_{k\to\infty} u(k)/k$.
$\theta_\star$	Critical density $1/\log_2 3 \approx 0.63093$.
C	Absolute constant in Lemma 5.2 (we may take $=1$).
δ	Positive rate $1-(\theta_\star-\varepsilon)\log_2 3$ in (6.3).
c	Exponent in the uniform dip property (7.1).
N_0	Threshold from which dips are guaranteed in Theorem 7.1.