International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V71I9P102

Volume 71 Issue 9, 9-16, September 2025

Original Article

Fibonacci and Lucas Numbers and their Bi-Complex Extension

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Received: 13 July 2025 Revised: 15 August 2025 Accepted: 03 September 2025 Published: 18 September 2025

Abstract *- This paper aims to introduce a new type of Lucas and Fibonacci numbers, which are said to be bi-complex Lucas and bi-complex Fibonacci numbers. Also, we prove the D'ocagne identity. Further, we give the relation between the identities of negabicomplex Lucas numbers, negabicomplex Fibonacci numbers, and the Binet formula.*

Keywords *- Bi-complex numbers, Binet formula, Fibonacci numbers, Lucas numbers.*

1. Introduction and Motivation

The concept of bi-complex numbers was explored in 1892 by Corrado Segre, an Italian mathematician [15]. Among the most influential studies, the comprehensive treatment provided by G.B. Price [13] stands out as a key reference in the field. bicomplex numbers form a special class within the framework of complex Clifford algebras and can be considered a natural extension of the traditional complex number system.

The Fibonacci sequence traces its origin to the Italian mathematician Leonardo Fibonacci, who introduced it in his 1202 book *Liber Abaci*. Among several mathematical problems discussed in the text, one involved modeling the growth of a rabbit population, a scenario that gave rise to the now-famous Fibonacci sequence. This sequence of numbers is obtained by adding two previous numbers, beginning with 0 and 1, resulting in the series: 0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987, and so on.

Mathematically, this sequence is denoted by F_t and given by:

$$F_t = F_{t-1} + F_{t-2}\ , where\ F_0 = 0, F_1 = F_2 = 1\ and\ t \geq 2.$$

In contrast, the Lucas sequence, named after the French mathematician Édouard Lucas, who extensively studied and popularized these numbers, follows the same recurrence relation as the Fibonacci sequence. However, it differs in its initial terms. The Lucas sequence is defined as,
2,1,3,4,7,11,18,29,47,76,123,199,322,521,843,1364,..

Lucas is often recognized as a pioneer in integer sequences for his significant contributions to the Fibonacci and Lucas number systems.

The general recurrence formula is:

$$L_{r+1} = L_r + L_{r-1}\ \ where\ L_0 = 2, L_1 = 1, and\ r \geq 1.$$

There are many works on Fibonacci and Lucas numbers in the literature. The properties, relations, and results between Fibonacci and Lucas numbers can be found in Dunlap [2], Güven and Nurkan [3], Kome [9], Koshy [10], Parajuli et al. [12], Vajda [17], Verner and Hoggatt [18]. Also, Fibonacci and Lucas quaternions were described by Horadam in [5], then many studies related to these quaternions were done in Akyiğit [1], Halici [4], Iyer [6], Nurkan [11], Swamy [16].

2. Preliminaries

This section discusses the detailed preliminaries of bi-complex Fibonacci and Lucas numbers.

Definition 2.1

The classical complex numbers, denoted by $\mathbb{C}$, are defined as:

$$\mathbb{C} = \{\boldsymbol{a} + \boldsymbol{bi} : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R} \ \boldsymbol{and} \ \boldsymbol{i^2} = -\mathbf{1}\}$$

Definition 2.2

The set of bi-complex numbers denoted by $\mathbb{C}_2$ is defined by

$$\mathbb{C}_2 = \{u + vj : u,v \in \mathbb{C} \ and \ j^2 = -1\}$$

Since u and v are complex numbers, writing $u = a+bi$ and $v = c+di$ gives us a way to represent the bi-complex numbers as;

$$u + vj = a_1 + b_1 i + c_1 j + d_1 ij \text{ , where } a_1, b_1, c_1, d_1 \in \mathbb{R}.$$

Let the bi-complex numbers be

$$x = a_1 + b_1 i + c_1 j + d_1 ij \ \ and \ \ y = a_2 + b_2 i + c_2 j + d_2 ij.$$

Then the addition and multiplication of x and y are given by

$$\boldsymbol{x} + \boldsymbol{y} = (\boldsymbol{a_1} + \boldsymbol{a_2}) + (\boldsymbol{b_1} + \boldsymbol{b_2})\boldsymbol{i} + (\boldsymbol{c_1} + \boldsymbol{c_2})\boldsymbol{j} + (\boldsymbol{d_1} + \boldsymbol{d_2})\boldsymbol{ij} \tag{1}$$

And

$$x \times y = (a_1 a_2 - b_1 b_2 - c_1 c_2 + d_1 d_2) + (a_1 b_2 + b_1 a_2 - c_1 d_2 - d_1 c_2)\, i + (a_1 c_2 + c_1 a_2 - b_1 d_2 - d_1 b_2) j + (a_1 d_2 + d_1 a_2 + b_1 c_2 + c_1 b_2) ij. \tag{2}$$

The multiplication by a real scalar p is given by

$$px = pa_1 + pb_1 i + pc_1 j + pd_1 ij$$

The set $\mathbb{C}_2$ is a commutative ring, and it is a real vector space with addition and scalar multiplication [13].

Definition 2.3

The complex conjugate of $w = p + qi$ *is* $\overline{w} = p - qi$ and there are different conjugations for a bi-complex number $x = (p + qi) + (r + si)\, j$, with i, j, and k (=i j) given by Rochan and Shapiro [14] are as follows:

$$\begin{cases} x^i = [(p + qi) + (r + si)j]^i = (p - qi) + (r - si)j \\ x^j = [(p + qi) + (r + si)j]^j = (p + qi) - (r + si)j \\ x^k = [(p + qi) + (r + si)j]^k = (p - qi) - (r - si)j \end{cases} \tag{3}$$

Then the product of bicomplex with their different conjugates is given as [7];

$$\begin{cases} x \times x^i = (p^2 + q^2 - r^2 - s^2) + 2(pr + qs)j \\ x \times x^j = (p^2 - q^2 + r^2 - s^2) + 2(pq + rs)i \\ x \times x^k = (p^2 + q^2 + r^2 + s^2) + 2(ps - qr)\ ij. \end{cases} \tag{4}$$

Definition 2.4

The modulus of a bi-complex number x can be characterized in several ways, depending on the type of conjugation applied. These different moduli are defined as follows [14]:

$$|x|_i = \sqrt{x \times x^i}$$
$$|x|_j = \sqrt{x \times x^j}$$

$$|x|_k = \sqrt{x \times x^k}$$
$$|x| = \sqrt{Re(x \times x^k)}$$

Now we proceed to define the bi-complex extensions of Fibonacci and Lucas numbers.

Definition 2.5

The bi-complex versions of the Fibonacci number are denoted by BF_r and is given by

$$BF_r = F_r + F_{r+1}i + F_{r+2}j + F_{r+3}k \tag{5}$$

And the bi-complex versions of Lucas numbers are denoted by BL_r, and is given by

$$BL_r = L_r + L_{r+1}i + L_{r+2}j + L_{r+3} \tag{6}$$

Where $F_r and\ L_r$ represent the r^{th} terms of the Fibonacci and Lucas sequences, respectively.

Fundamental Operations on bi-complex Fibonacci numbers:

Let us define two bi-complex Fibonacci numbers as follows:

$$BF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

and

$$BF_m = F_m + F_{m+1}i + F_{m+2}j + F_{m+3}k$$

Definition 2.6

The sum or difference of two bi-complex Fibonacci numbers is given component-wise:

$$BF_n \pm BF_m = (F_n \pm F_m) + (F_{n+1} \pm F_{m+1})i + (F_{n+2} \pm F_{m+2})j + (F_{n+3} \pm F_{m+3})k$$

Definition 2.7

The product of $BF_n and\ BF_m$ Can be expanded as:

$$\begin{aligned} BF_n . BF_m = & (F_n F_m - F_{n+1}F_{m+1} - F_{n+2}F_{m+2} + F_{n+3}F_{m+3}) \\ & + (F_n F_{m+1} + F_{n+1}F_m - F_{n+2}F_{m+3} - F_{n+3}F_{m+2})i \\ & + (F_n F_{m+2} + F_{n+2}F_m - F_{n+1}F_{m+3} - F_{n+3}F_{m+1})j \\ & + (F_n F_{m+3} + F_{n+3}F_m + F_{n+1}F_{m+2} + F_{n+2}F_{m+1})k. \end{aligned}$$

3. Main Results

This section presents and proves a few theorems involving bicomplex Fibonacci and Lucas numbers. Besides this, we also provide some results about the Negabicomplex and Negalucas numbers.

Theorem 3.1. $For\ n \geq 0$, the bi-complex Fibonacci numbers BF_n Satisfy the following relation.

$$BF_n^2 + BF_{n+1}^2 = BF_{2n+1} + F_{2n+2} + F_{2n+5} - 3iF_{2n+5} - jF_{2n+6} + 3kF_{2n+4}.$$

Proof:

According to the defined form of bi-complex Fibonacci number, we have

$$BF_n = F_n + F_{n+1}i + F_{n+2}\mathrm{j} + F_{n+3}k$$

Using classical Fibonacci identities from (Vajda [17])

We use the relation, which is a key identity related to the square of two consecutive Fibonacci terms:

$$F_n^2 + F_{n+1}^2 = F_{2n+1}$$

Another notable relation involves the difference of squares,

$$F_{n+1}^2 - F_{n-1}^2 = F_{2n}$$

A general identity connecting the product of two pairs of Fibonacci numbers to a single term is given by:

$$F_m + F_{n+1}F_{m+1} = F_{n+m+1}$$

We proceed to derive, from (Vajda [17])

The sum of squares of two consecutive bicomplex Fibonacci quaternions, denoted by

$$BF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k,$$

and

$$BF_{n+1} = F_{n+1} + F_{n+2}i + F_{n+3}j + F_{n+4}k$$

Then the sum of the squares of two consecutive bicomplex Fibonacci numbers is expressed as:

$$BF_n^2 + BF_{n+1}^2 = (F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k)^2 + (F_{n+1} + F_{n+2}i + F_{n+3}j + F_{n+4}k)^2$$

which expands to:

$$= [\,(F_{2n+3} + 2(F_nF_{n+1} - F_{n+2}F_{n+3})i + 2(F_nF_{n+2} - F_{n+1}F_{n+3})j + 2(F_nF_{n+3} + F_{n+1}F_{n+2})k] + [F_{2n+5} + 2(F_{n+1}F_{n+2} - F_{n+3}F_{n+4})i + 2(F_{n+1}F_{n+3} - F_{n+2}F_{n+4})j + 2(F_{n+1}F_{n+4} + F_{n+2}F_{n+3})k]$$

Combining the terms, we obtain:

$$= F_{2n+1} + F_{2n+2}i + F_{2n+3}j + F_{2n+4}k + F_{2n+2} + F_{2n+5} - 3F_{2n+5}i - F_{2n+6}j + 3F_{2n+4}k$$

Which implies:

$$BF_{2n+1} + F_{2n+2} + F_{2n+5} - 3iF_{2n+5} - jF_{2n+6} + 3kF_{2n+4}.$$

□

Theorem 3.2. *For* $r \geq 0$ The bi-complex Lucas numbers BL_r Satisfy the following relation.

$$(BL_r - BL_{r+1}i + BL_{r+2}j - BL_{r+3}k) = -5L_{r+3} + 2(L_{r+2} + L_{r+4})j.$$

Proof:

By using the definition of the bi-complex Lucas number, we have

$$BL_r = L_r + L_{r+1}i + L_{r+2}j + L_{r+3}k$$

We proceed to derive:

$$(BL_r - BL_{r+1}i + BL_{r+2}j - BL_{r+3}k)$$

Expanding and simplifying this expression, we get,

$$= [(L_r + L_{r+1}i + L_{r+2}j + L_{r+3}k) - (L_{r+1} + L_{r+2}i + L_{r+3}j + L_{r+4}k)i + (L_{r+2} + L_{r+3}i + L_{r+4}j + L_{r+5}k)j - (L_{r+3} + L_{r+4}i + L_{r+5}j + L_{r+6}k)k]$$

Arranging and simplifying the scalar and i,j and k components, we isolate the scalar and j components,

$$= L_r + L_{r+2} - L_{r+4} - L_{r+6} + 2(L_{r+2} + L_{r+4})j$$

Using the Lucas recurrence relation,

$$L_{n+2} = L_{n+1} + L_n$$

Now putting these into expression, we get

$$= L_r + L_{r+2} - (L_{r+3} + L_{r+2}) - (L_{r+5} + L_{r+4}) + 2(L_{r+2} + L_{r+4})j$$

Which implies that:

$$= (L_r - 3L_{r+3} - L_{r+4} - L_{r+2}) + 2(L_{r+2} + L_{r+4})j$$

By simplifying, we get,

$$= -5L_{r+3} + 2(L_{r+2} + L_{r+4})j.$$

□

Theorem 3. 3. For non–negative integers *r* and *s*, the D'Ocagne type relation for bicomplex Fibonacci numbers is expressed as:

$$BF_s \cdot BF_{r+1} - BF_{s+1} \cdot BF_r = (-1)^r BF_{s-r} + (-1)^{r+1}[F_{s-r} + F_{s-r+1}i - (F_{s-r-2} + 2F_{s-r+2})j + 2F_{s-r-1}k].$$

Proof:

We begin by applying the classical D'ocagne identity for standard Fibonacci numbers (Weisstein [19]):

$$F_sF_{r+1} - F_{s+1}F_r = (-1)^r F_{s-r}$$

Using the definition of the bi-complex Fibonacci number

$$BF_r = F_r + F_{r+1}i + F_{r+2}j + F_{r+3}k,$$

$$BF_s = F_s + F_{s+1}i + F_{s+2}j + F_{s+3}k,$$

$$BF_{r+1} = F_{r+1} + F_{r+2}i + F_{r+3}j + F_{r+4}k,$$
$$BF_{s+1} = F_{s+1} + F_{s+2}i + F_{s+3}j + F_{s+4}k,$$

We compute the expression.

$$BF_s \cdot BF_{r+1} - BF_{s+1} \cdot BF_r = (F_s + F_{s+1}i + F_{s+2}j + F_{s+3}k) \cdot (F_{r+1} + F_{r+2}i + F_{r+3}j + F_{r+4}k) - (F_{s+1} + F_{s+2}i + F_{s+3}j + F_{s+4}k) \cdot (F_r + F_{r+1}i + F_{r+2}j + F_{r+3}k)$$

By applying the properties from Fibonacci algebra along with well-known Fibonacci identities, the expression can be rewritten and simplified as;

$$= [F_{r+2}(F_s + F_{s+4}) - F_{s+2}(F_r + F_{r+4})]\, i + [2(-1)^r(F_{s-r-2} + F_{s-r+2})]\, j + [(-1)^r \cdot (F_{s-r-2} + F_{s-r+2})]\, k + (-1)^r \cdot BF_{s-r} - (-1)^r BF_{s-r}$$

Which implies that:

$$= (-1)^r \cdot BF_{s-r} + (-1)^{r+1} \cdot [F_{s-r} + F_{s-r+1}i - (F_{s-r-2} + 2F_{s-r+2})j + 2F_{s-r-1}k].$$

□

In the following theorem, we prove the bicomplex Fibonacci and Lucas numbers. BF_r and BL_r for their negative indices with $r \geq 0$, denoted by BF_{-r} and BL_{-r}, which are called Negabicomplex and Negalucas numbers, respectively.

Theorem 3.4. Negabicomplex and negalucas numbers BF_{-r} and BL_{-r}, satisfy the following relations.

$$BF_{-r} = (-1)^{r+1} BF_r + (-1)^r L_r(i + j + 2k).$$

and

$$BL_{-r} = (-1)^r BL_r + (-1)^{r+1} 5F_r(i + j + 2k).$$

Proof: Here, we use identity for the negative-indexed Fibonacci numbers (Knuth [8], Dunlap [2]):

$$F_{-r} = (-1)^{r+1} F_r.$$

Using this, the negabicomplex Fibonacci number becomes

$$BF_{-r} = F_{-r} + F_{-(r-1)}i + F_{-(r-2)}j + F_{-(r-3)}k$$

Using the identity of negative Fibonacci terms:

$$= (-1)^{r+1} F_r + (-1)^r F_{r-1}\, i + (-1)^{r-1} F_{r-2}j + (-1)^{r-2} F_{r-3}\, k$$

Arranging the terms:

$$BF_{-r} = (-1)^{r+1}(F_r + F_{r+1}\, i + F_{r+2}\, j + F_{r+3}k) - (-1)^{r+1} F_{r+1}\, i - (-1)^{r+1} F_{r+2}\, j - (-1)^{r+1} F_{r+3}\, k + (-1)^r F_{r-1}\, i\, (-1)^{r+1} F_{r-2}\, j + (-1)^r F_{r-3}\, k$$

Rearranging the terms:

$$= (-1)^{r+1} BF_r + (-1)^r(F_{r+1} + F_{r-1})i + (-1)^r(F_{r+2} - F_{r-2})j + (-1)^r(F_{r+3} + F_{r-3})k$$

In view of Vajda [17], we have

(a) $F_{r+1} + F_{r-1} = L_r$

(b) $F_{r+2} - F_{r-2} = L_r$

(c) $F_{r+3} + F_{r-3} = 2L_r$

Thus, we have

$$BF_{-r} = (-1)^{r+1} BF_r + (-1)^r L_r\, i + (-1)^r L_r\, j + (-1)^r 2L_r$$

After simplifying

$$= (-1)^{r+1} BF_r + (-1)^r L_r\, (i + j + 2k).$$

From the identity of the Negalucas number and the bicomplex Lucas number

$$BL_r = L_r + L_{r+1}i + L_{r+2}j + L_{r+3}\, k$$

The relation for Lucas numbers with negative indices is

$$L_{-r} = (-1)^r L_r \quad \text{(Knuth [8], Dunlap [2])}$$

We obtain

$$BL_{-r} = L_{-r} + L_{-(r-1)}i + L_{-(r-2)}j + L_{-(r-3)}k$$
$$= (-1)^r L_r + (-1)^{r-1} L_{r-1}i + (-1)^r L_{r-2}j + (-1)^{r-1} L_{r-3}k$$

After simplification, we get
$= (-1)^r (L_r + L_{r+1}i + L_{r+2}j + L_{r+3}k)$

$$-(-1)^r L_{r+1}i-(-1)^r L_{r+2}j-(-1)^r L_{r+3}k+(-1)^{r-1} L_{r-1}i+(-1)^r L_{r-2}j+(-1)^{r-1} L_{r-3}k$$
$$= (-1)^r BL_r+(-1)^{r+1}[(L_{r+1} + L_{r-1})i + (L_{r+2} - L_{r-2})j + (L_{r+3} + L_{r-3})k].$$

Here, we use $L_{s+r} + L_{s-r} = \begin{cases} 5F_sF_r & if\ r\ is\ odd \\ L_sL_r, & otherwise \end{cases}$

(Koshy [10]). The sum of Lucas numbers one step ahead and behind the index r is 5 times the r^{th} Fibonacci number:
$$L_{r-1} + L_{r+1} = 5\ F_r,$$

The difference between Lucas numbers two steps ahead and behind index r is also 5 times the r^{th} Fibonacci number:
$$L_{r+2} - L_{r-2} = 5F_r,$$

The sum of Lucas numbers three indices apart on both sides of r gives 10 times the Fibonacci number at r.
$$L_{r+3} + L_{r-3} = 10F_r$$

And the definition of dual Lucas numbers, along with the identity for negative-indexed Fibonacci numbers, we have, by calculation,
$$BL_{-r} = (-1)^r BL_r + (-1)^{r+1} 5F_r\ i + (-1)^{r+1} 5F_r\ j + (-1)^{r+1} 10F_r k.$$
$$= (-1)^r BL_r + (-1)^{r+1} 5F_r\ (i + j + 2k).$$

Theorem 3.5. The Binet's formula for bicomplex Fibonacci and Lucas numbers
$$BF_n = \frac{(\overline{w_1}\ w_1{}^n - \overline{w_2}\, w_2^n)}{(w_1 - w_2)}$$

and $$BL_n = \overline{w_1}w_1^n + \overline{w_2}\, w_2{}^n$$

where $\overline{w_1} = 1 + iw_1 + jw_1^2 + kw_1^3$ *and* $\overline{w_2} = 1 + iw_2 + jw_2^2 + kw_2^3$

Proof: By using Binet's formula $(Koshy[10])$
$$F_n = \frac{w_1^n - w_2^n}{w_1 - w_2} \quad and \quad L_n = w_1{}^n + w_2^n$$

With
$$w_1 = \left(\frac{1 + \sqrt{5}}{2}\right) \quad and \quad w_2 = \frac{1 - \sqrt{5}}{2}$$

And $\overline{w_1} = 1 + iw_1 + j\,w_1^2 + kw_1^3$, $\overline{w_2} = 1 + iw_2 + jw_2^2 + kw_2^3$,
we have

$$BF_n = F_n + F_{n+1}\,i + F_{n+2}\,j + F_{n+3}\,k$$

Using Binet's formula $F_n = \frac{w_1^n - w_2^n}{w_1 - w_2}$, we can write

$$BF_n = \frac{w_1^n - w_2^n}{w_1 - w_2} + \frac{w_1^{n+1} - w_2^{n+1}}{w_1 - w_2}\, i + \frac{w_1^{n+2} - w_2^{n+2}}{w_1 - w_2}\, j + \frac{w_1^{n+3} - w_2^{n+3}}{w_1 - w_2}\, k$$

After simplification, we get
$$= \frac{w_1{}^n (1 + iw_1 + j\,w_1^2 + kw_1^3) - w_2^n\,(1 + iw_1 + j\,w_1^2 + kw_1^3)}{w_1 - w_2}$$
$$= \frac{(\overline{w_1}\, w_1^n - \overline{w_2}\, w_2^n)}{(w_1 - w_2)}$$

And $BL_n = L_n + L_{n+1}\,i + L_{n+2}\,j + L_{n+3}\,k$

And using $L_n = w_1^n + w_2^n$, we can write

$$BL_n = (w_1{}^n + w_2^n) + (w_1^{n+1} + w_2^{n+1})\, i + (w_1^{n+2} + w_2^{n+2})\, j + (w_1^{n+3} + w_2^{n+3})k$$

By simplifying, we get

$$= w_1^n(1 + i\, w_1 + j\, w_1^2 + kw_1^3) + w_2^n(1 + iw_2 + jw_2^2 + kw_2^3)$$
$$= \overline{w_1} w_1^n + \overline{w_2}\, w_2^n.$$

4. Conclusion

This article discussed the Fibonacci and Lucas numbers in a bi-complex framework and proved some identities and theorems. The future work behind bi-complex Fibonacci and Lucas numbers lies in expanding their theoretical foundation and exploring novel applications across mathematics, physics, and engineering. Since this is a relatively new and growing field, there are several open directions for research such as deepening more algebraic structures and identities in algebra, modelling quantum and wave systems in physics, developing cryptographic and signal applications in engineering, creating new geometric patterns in fractals, implementing bi-complex Fibonacci and Lucas algorithms in computation, analyzing new sequence spaces in topology and so on.

Acknowledgments

The first author would like to thank the anonymous referee(s) for their suggestions and comments on how to improve the paper.

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