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Original Article

Bounds on $rr(G+e)$ in terms of $rr(G)$ for Connected Graphs

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Abstract - *Let $G(\Gamma, E)$ be a simple, connected, and undirected graph. A radial radio labeling of G, $\psi: \Gamma \rightarrow \{1,2,3, \ldots\}$ is a function satisfying the condition for any two distinct vertices u and v that: $d(u,v)+|\psi(u)-\psi(v)| \geq 1+rad(G)$, where $d(u,v)$ denotes the distance between the vertices u and v, and $rad(G)$ is the radius of the graph G. The span of a radial radio labeling is the maximum integer that assigns to a vertex and radial radio number, $rr(G)$ is the minimum span taken overall radial radio labelings of G. This paper presents relationships between the radial radio number of a graph and its super-graph obtained by adding a new edge under some constraints.*

Keywords - *Radial radio labeling, Radial Radio number, Graph labeling, Frequency assignment.*

AMS Classification Code: *05C78*

1. Introduction

The study of graph labeling has emerged as a significant area within graph theory, offering insights into combinatorial properties and their applications in network design, communication systems, and optimization problems. The concept labeling was introduced by Rosa et al. Al. [8]. The detailed study of the Frequency Assignment Problem [7] by Chartrand et al. Al.[5] leads to the new concept of radio k coloring, and the definition is as follows: For any simple connected graph G, a function $f: V(G) \rightarrow \{1,2,3 \ldots\}$ is said to be a radio k coloring if it satisfies the condition that $d(u,v)+| f(u)-f(v) | \geq 1+$ k, where $1 \leq k \leq diam(G)$ and $diam(G)$ represents the diameter of the graph.

A radial radio labeling of a simple connected graph G, also known as a radio k-coloring where $k=\text{rad}(G)$, assigns positive integers to the vertices of Gsuch that for any two distinct vertices $u$and v, the condition $d(u,v)+| f(u)-f(v) | \geq \text{rad}(G)+1$ holds, where $d(u,v)$ is the distance between $u$and v, and rad$(G)$is the radius of G.The span of such a labeling, denoted span(f), is the largest integer in the range of f, while the radial radio number rr(G)represents the minimum span achievable over all valid radial radio labelings of G.

So far, many researchers have either determined the radial radio numbers of various families of graphs or used variants like radial radio mean labeling [10], radio mean labeling [9], etc. But [3], [11], [13] provide a detailed study on the interconnections among some graph theoretical parameters such as radial radio number, radio number, R-number [12], clique number, and chromatic number. There arises a question, "Is there any relationship between the radial radio numbers of a graph and its subgraphs?" The possible three categories are as follows:
1. graphs with $rr(G)=rr(H)$.
2. graphs with $rr(G)>rr(H)$.
3. graphs with $rr(G)<rr(H)$.

We may not be able to directly categorise all the connected graphs. As a slow and steady process, this paper provides the relationships between $rr(G))$ and $rr(G+e)$, where $G+e$ is the graph obtained from G by adding a new edge in a particular place.

For further details, one can refer to [1], [2], [4].

2. Relationship between $rr(G+e)$ and $rr(G)$

The following lemma gives the relationship between the radial radio number of a graph G and the radial radio number of its spanning subgraph H, such that $rad(G)=rad(H)$.

Lemma 2.1

Let G be a graph with radial radio number $rr(G)$. If H is a spanning subgraph of G with $rad(G) = rad(H)$, then $rr(H) \leq rr(G)$.

Proof

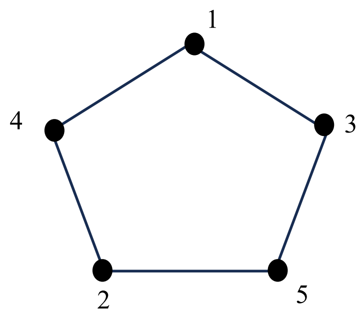
Let g be a $rr(G)$ – radial radio labeling of G. Then

$$d(u, v) + |g(u) - g(v)| \geq 1 + rad(G)$$

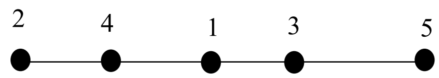
for any two distinct vertices u and v of G and $span\ g = rr(G)$.
Now, $d_G(u, v) \leq d_H(u, v)$, for any two distinct vertices u and v of G.
Then $|g(u) - g(v)| \geq 1 + rad(G) - d_G(u, v) \geq 1 + rad(H) - d_H(u, v)$.
This is true for any two distinct vertices u and v of G. Thus, g is also a radial radio labeling of H.
This forces that, $rr(H) \leq span\ g = rr(G)$.
Thus $rr(H) \leq rr(G)$. ■

Remark 2.2

Note that, $rr(P_5) = rr(C_5) = 5$. But P_5 is a spanning subgraph of C_5. Thus, the bound in Lemma 2.1 is sharp.



C_5



P_5

The following two theorems in this section provide the relationships between $rr(G)$ and $rr(G + e)$ under various conditions.

Theorem 2.3

Let G be a graph of order n and let u be a vertex of G such that $deg_G(u) = n - 2$. Suppose v is the vertex that is not adjacent to u. Then $rr(G + uv) \leq rr(G)$.

Proof

Clearly, $\Delta(G + e) = n - 1$ and so $rad(G + e) = 1$. Let f be a $rr(G) -$ radial radio labeling of G.
Then the label difference between any two adjacent vertices is 2, and the label difference between the vertices at distance 2 is 1.
That is, for any two distinct vertices x and y of G, we have
$|f(x) - f(y)| \geq 2$, if $d_G(x, y) = 1$
$|f(x) - f(y)| \geq 1$, if $d_G(x, y) = 2$
Since $deg_G(u) = n - 2$, there exists only one vertex v (say), which is not adjacent to u.
Since f is a radial radio labeling of G, $|f(u) - f(v)| \geq 1$.
Without changing any label given by f, the function f itself satisfies the radial radio condition of $G + e$.
Thus $rr(G + e) \leq span\ f = rr(G)$ and hence $rr(G + e) \leq rr(G)$.

Theorem 2.4

Let G be a graph on n vertices with radius 2 and let $\Delta < n - 2$. Then $rr(G + uv) \geq rr(G)$, where u and v are not adjacent in G.

Proof

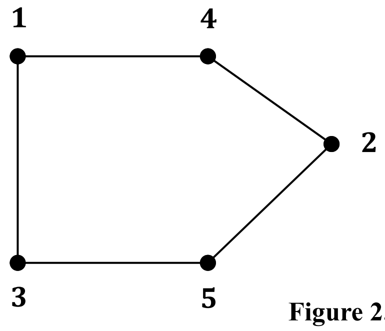
Since $\Delta < n - 2$, $rad(G + e) = 2$, for any e not in G.
Also, for any two distinct vertices x and y of $G + e$, $d_{G+e}(x, y) \leq d_G(x, y)$.
Assume that f is a radial radio labeling for $G + e$ such that $span\ f = rr(G + e)$.
Then f satisfies the following condition for any two distinct vertices x and y of $G + e$:

$$d_{G+e}(x, y) + |f(x) - f(y)| \geq 1 + 2.$$

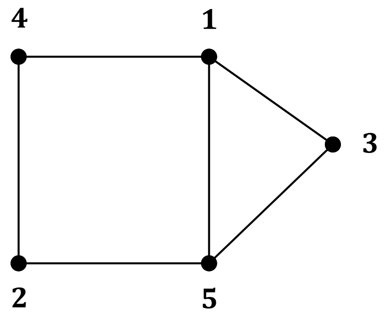
This implies that, $|f(x) - f(y)| \geq 1 + 2 - d_{G+e}(x, y) \geq 1 + 2 - d_G(x, y)$.

This forces that f to be a radial radio labeling of G and hence $rr(G) \leq span\ f = rr(G+e)$. This completes the proof.

Remark 2.5
The bound obtained in Theorem 2.4 is sharp. For example, consider the graphs drawn in Figure 2.2.



G



H

Figure 2.2

Here, $\boldsymbol{rr(G) = rr(H) = 5}$.

3. Conclusion

In this paper, we determined relationships between the radial radio number of a graph and its super-graph obtained by adding a new edge under some constraints. We noted that the radial radio number of a graph and its subgraph are not always equal. The relationship between any graph and its subgraph is still open to finding.

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