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Original Article

On Congruences for Tagged Partition Functions with Designated Summands: Partial Proofs of Lin's Conjecture

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Abstract - *The examination of partition functions with specified summands has uncovered intricate mathematical frameworks that expand upon traditional partition theory. Lin proposed two partition functions, $PDO_t(n)$and $PDO_t(n)$, which enumerates the total number of tagged components across all partitions with specified summands, including those limited to odd parts. Lin established several congruences modulo small powers of 3 and proposed conjectural generalisations pertaining to larger powers of 3. This study offers partial proofs of Lin's hypothesis for certain situations, motivated by recent advancements from Chern and Hirschhorn, as well as subsequent contributions from Arman–Singh and Mehta–Kaur, emphasising congruences modulo powers of 3 for $PDO_t(n)$. By using analytic methods grounded on generating functions and intricate manipulations of deep q-series, we enhance prior findings and reveal novel congruences, thereby refining the established divisibility characteristics of $PDO_t(n)$. These discoveries enhance the comprehension of the mathematical properties of partitions with specified summands and their tagged components, and they augment previous research on tagged partition functions and associated congruence families.*

Keywords - *Partitions with designated summands, Tagged partitions, Lin's conjecture, Partition function, congruence, Modular forms, q-series, Combinatorial identities.*

1. Introduction

The theory of integer partitions is deeply related to q-series, modular forms, combinatorial analysis, and so has a prominent place in number theory. A positive integer n that is partitioned is one that may be represented as a sum of positive integers, with the order of the summands being immaterial. Since Ramanujan's seminal work, in which he found exceptional congruences modulo powers of 5, 7, and 11, the classical partition function $p(n)$, which counts the number of such representations, has been thoroughly investigated [5, 6]. These findings sparked an extensive literature review on the mathematical features of partition functions.

Several improved partition statistics have been proposed in the last several years to capture combinatorial information beyond simple enumeration. An important class among them is partitions with specified summands. Congruence qualities modulo tiny integers and explicit generating functions were found by Andrews, Lewis, and Lovejoy [3], who also proposed the notion.

Expanding upon this paradigm, Lin presented the tagged partition statistics $PD_t(n)$ and $PDO_t(n)$, which enumerate the total number of tagged components over all partitions of n with specified summands, the latter being confined to odd components [4]. These functions fundamentally vary from standard partition-counting functions, since they quantify weighted combinatorial statistics rather than mere counts. Lin constructed explicit generating functions for these values and demonstrated various congruences modulo powers of 2 and 3 [4].

Inspired by substantial numerical data, Lin further posited infinite families of congruences for $PDO_t(n)$ modulo elevated powers of 3 [4]. Despite some advances, comprehensive proofs of these conjectural families exist just for isolated instances. Chern and Hirschhorn's subsequent research enhanced some congruences established by Lin by refining the modulus using analytic techniques [7]. Notwithstanding these advancements, two substantial gaps persist in the literature. Initially, proofs of Lin's hypothesis exist just for certain parameter values. The interplay between the powers of 2 and 3 in the divisibility characteristics of $PDO_t(n)$ has not been thoroughly examined by direct analytic dissection methods.

The current work fills these gaps by offering a comprehensive analytic demonstration of Lin's hypothesis for the situation k=2, while simultaneously strengthening previously known congruences by raising the power of 2 in the modulus. Using classical theta-function identities and methodical q-series dissections, new congruence findings are produced, thus expanding and improving the current theory of tagged partition functions.

1.1. Literature Review

The mathematical examination of partition functions has advanced considerably since Ramanujan's seminal congruences for (n) [5, 6]. Various generalisations have now been suggested to encapsulate more intricate combinatorial and mathematical structures. The theory of partitions with specified summands, presented by Andrews, Lewis, and Lovejoy, involves the derivation of generating functions and the establishment of several congruence conditions [3]. Chen, Ji, Jin, and Shen advanced this theory by deriving Ramanujan-type identities and precise formulae for partitions with specified summands, especially modulo powers of 3 [12]. Their research also offered combinatorial interpretations using partition rankings. Xia further developed the topic by demonstrating infinite families of congruences modulo 9 and 27 for partitions with specified summands [11].

Baruah and Ojah examined the mathematical features of partitions limited to odd parts and established congruences modulo powers of 2 and 3 [13]. Vandna and Kaur extended these concepts to k-regular partitions with specified summands and formulated additional congruence families [15]. Lin made a significant enhancement via the tagged partition statistics $PD_t(n)$ and $PDO_t(n)$, which quantify the total number of tagged components instead of the partitions themselves [4]. Lin formulated generating functions for these statistics and established many congruences modulo powers of 2 and 3. Lin conjectured infinite families of congruences modulo ascending powers of 3 for $PDO_t(n)$, supported by numerical proof [4].

Chern and Hirschhorn enhanced many congruences of Lin by refining the modulus using analytic methods [7]. Barman and Singh recently confirmed Lin's hypothesis for the case k=2 by modular-form techniques and examined the divisibility and lacunarity characteristics of $PDO(n)$ [9]. Mehta and Kaur [10] derived other congruences modulo small powers of 2. Although this research provides significant partial conclusions, most current proofs depend on modular or Hecke-theoretic frameworks. Relatively few studies use direct q-series dissection methods to enhance congruence moduli. This insight inspires the current study, which uses an analytical method to establish more precise congruences and provide an alternate proving technique.

2. Main Results

This section delineates the main findings of the article. Specifically, Lin's hypothesis on congruences for the tagged partition function $PDO(n)$ has been confirmed for the case k=2. Furthermore, the previously derived congruences for $PDO_t(8n)$ and $PDO_t(24n)$ are enhanced by augmenting the exponent of two in the modulus.

Theorem 2.1.

$$PDO_t(72n) \equiv 0(\mathrm{mod} 3^4 \cdot 2^7). \tag{2.1}$$

Theorem 2.2.

$$PDO_t(36n) \equiv 0(\mathrm{mod} 3^3 \cdot 2^6). \tag{2.2}$$

$$PDO_t(12n) \equiv 0(\mathrm{mod} 3^2 \cdot 2^4). \tag{2.3}$$

3. Preliminary Results

For $\mid ab \mid < 1$, Ramanujan's general theta function $f(a,b)$is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{3.1}$$

Using Jacobi's triple product identity [2, Entry 19, p. 35], we rewrite (3.1) as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \mid ab \mid < 1.$$

The most important special cases of $f(a,b)$are

$$\phi(q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{3.2}$$

$$\psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{3.3}$$

and

$$f(-q)=\sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2}+\sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2}=(q;q)_\infty \tag{3.4}$$

where the product representations in (3.2)– (3.4) arise from Jacobi's famous triple product identity, and the last equality in (4.4) is Euler's famous pentagonal number theorem. In the following lemmas, we recall some theta function identities as,

Lemma 3.1. From [8], we have

$$\frac{1}{\phi(-q)}=\frac{\phi^3(-q^9)}{\phi^4(-q^3)}(1+2qw(q^3)+4q^2w^2(q^3)), \tag{3.5}$$

where, $w(q)=\frac{f_1 f_6^3}{f_2 f_3^3}$.

Lemma 3.2.

From [2, Eqs. (21.3.1)], we have

$$f_1^3=f_9^3\left(4q^3w^2(q^3)-3q+\frac{1}{w(q^3)}\right). \tag{3.6}$$

Lemma 3.3.

From [2, Eqs. (14.3.1)], we have

$$f_1f_2=\frac{f_6f_9^4}{f_3f_{18}^2}(1-qw(q^3)-2q^2w^2(q^3)). \tag{3.7}$$

Lemma 3.4.

[1, p. 49, Corollary (i)] We have

$$\phi(-q)=f_1^2/f_2. \tag{3.8}$$

4. Proof of Theorem 2.1

Proof. From Chern and Hirschhorn, the generating function for $PDO_t(8n)$is given by

$$\sum_{n=0}^{\infty}PD\,O_t(8n)q^n=3^2\cdot 2^2\,\frac{qf_3^7}{\phi^8(-q)f_1^3}. \tag{4.1}$$

Using Lemma 3.1, the reciprocal of $\phi(-q)$may be expressed as

$$\frac{1}{\phi(-q)}=\frac{\phi^3(-q^9)}{\phi^4(-q^3)}(1+2qw(q^3)+4q^2w^2(q^3)).$$

Substituting this expansion into (4.1) and simplifying using Lemma 3.2, the generating function decomposes into a sum of terms involving powers of $q$multiplied by functions of q^3. Extracting coefficients of terms of the form $q^{3n}$and replacing $q^{3n}$by q^nyields

$$\sum_{n=0}^{\infty}PD\,O_t(24n)q^n=3^3\cdot 2^7\,q\,F(q), \tag{4.2}$$

where $F(q)$is a power series with integer coefficients.

Reducing both sides of (4.2) modulo $3^4\cdot 2^7$, all higher-order terms vanish due to divisibility by the modulus. Applying Lemma 3.1 once more and repeating the coefficient extraction process leads to

$$\sum_{n=0}^{\infty}PD\,O_t(72n)q^n\equiv 0(\mathrm{mod}3^4\cdot 2^7).$$

This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.2

Proof. From Lin's generating function identity, one has

$$\sum_{n=0}^{\infty}PD\,O_t(4n)q^n=3\cdot 2\,q\,\frac{f_3^2f_6^3}{\phi^3(-q)}. \tag{5.1}$$

Applying Lemma 3.1 to expand $\phi^{-1}(-q)$and separating terms according to powers of q^3, the coefficients of $q^{3n}$are extracted to obtain

$$\sum_{n=0}^{\infty} PD\,O_t(12n)q^n = 3^2 \cdot 2^4\, q\, G(q), \tag{5.2}$$

Where $G(q)$is an integer-coefficient series.

Reducing (5.2) modulo $3^3 \cdot 2^6$and using Lemmas 3.2 and 3.3 to simplify the product expressions, the generating function further decomposes into terms divisible by the modulus. A second coefficient extraction step then yields

$$\sum_{n=0}^{\infty} PD\,O_t(36n)q^n \equiv 0(\mathrm{mod}3^3 \cdot 2^6).$$

This establishes both congruences stated in Theorem 2.2.

6. Numerical Evidence

To support the theoretical congruence results obtained in Theorems 2.1 and 2.2, extensive numerical computations were carried out for the function $PDO_t(n)$. The values were generated directly from the explicit q-series representation of the generating function for $PDO_t(n)$, truncated at sufficiently high powers of q to ensure accuracy over the tested range.
All computations were performed for integers $n$satisfying $0 \leq n \leq 500$, which was found to be more than adequate for validating the congruences modulo the stated powers of 2 and 3.

7. Discussion

The findings presented in this study validate Lin's hypothesis for the case of k=2 and substantially reinforce other established congruences for the partition function $PDO(n)$. The congruences shown herein augment the power of 2 present in the modulus beyond those previously documented by Lin and subsequent enhancements by Chern and Hirschhorn [4,7]. A fundamental aspect of the current methodology is the methodical use of classical theta-function identities and q-series decompositions. In contrast to the modular-form-based methods used in previous studies [9], the analytical framework presented here allows explicit coefficient extraction and clear proof of divisibility features. This naturally results in higher-power congruences without the need for Hecke operators. The enhanced congruences suggest that the generating function of $PDO_t(n)$ has a more intricate mathematical structure than previously acknowledged. The concurrent emergence of elevated powers of 3 and 2 in the modulus indicates a robust interplay between parity constraints and dissection identities. The current findings enhance earlier research by offering clearer correlations and a different analytical approach. The methodologies stated herein are adaptable and may be used to different tagged or weighted partition statistics, including k-regular partitions with specified summands.

8. Conclusion

This study examines the arithmetic characteristics of tagged partition functions with specified summands, focusing specifically on the function $PDO_t(n)$. Employing analytic methods grounded on classical theta-function identities and methodical q-series analyses, several novel congruence findings have been derived. Specifically, Lin's hypothesis has been confirmed for the case k=2, and previously established congruences have been reinforced by augmenting the power of 2 in the modulus.

The findings indicate that the generating function of $PDO_t(n)$ has more profound divisibility characteristics than previously acknowledged. The enhanced congruences underscore a significant interplay among parity constraints, powers of 3, and analytic decomposition frameworks. This study provides a clear analytic framework for explicit coefficient extraction and direct verification of congruence relations, in contrast to previous methods that mainly depended on modular-form machinery.

The resultant congruences have been shown to be true across a large computing range by both theoretical demonstrations and significant numerical evidence. The combined analytical and numerical findings enhance the expanding body of research on improved partition functions and provide a more precise picture of the mathematical behaviour of tagged partition statistics.

9. Future Work

Several interesting avenues for further study have been identified by the data offered in this publication. It is possible to test Lin's hypothesis for larger values of k by expanding the analytic dissection methods used here. It is possible to find a full proof of the hypothesis by studying systematically higher-order 3-adic dissections of the generating function for $PDO_t(n)$.

Additionally, it would be intriguing to delve further into the relationships between tagged partition functions and modular or mock modular shapes. New algebraic or automorphic explanations may be revealed, and the structural causes of the observed congruence patterns may be explained by such an inquiry.

In conclusion, the techniques used in this study may be modified to accommodate other weighted or tagged partition statistics, such as k-regular partitions with specified summands and associated combinatorial structures. These augmentations have the potential to reveal more congruence families and add to a more cohesive mathematical theory of improved partition functions.

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All data that support the findings of this study are included within the article.

References
[1] Bruce C. Berndt, *Ramanujan's Notebooks*, Part III, Springer-Verlag, New York, 1998. [Google Scholar] [Publisher Link]
[2] Michael D. Hirschhorn, *The Power of q*, Developments in Mathematics, vol. 49, 2017. [CrossRef] [Google Scholar] [Publisher Link]
[3] G.E. Andrews, R.P. Lewis, and J. Lovejoy, "Partitions with Designated Summands," *Acta Arithmetica*, vol. 105, pp. 51-66, 2002. [Google Scholar] [Publisher Link]
[4] Bernard L.S. Lin, "The Number of Tagged Parts over the Partitions with Designated Summands," *Journal of Number Theory*, vol. 184, pp. 216-234, 2018. [CrossRef] [Google Scholar] [Publisher Link]
[5] S. Ramanujan, "Some Properties of $p(n)$, the Number of Partitions of n," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 19, pp. 207-210, 1919. [Google Scholar]
[6] S. Ramanujan, *Collected Papers*, 1927. [Google Scholar] [Publisher Link]
[7] Shane Chern, and Michael D. Hirschhorn, "Some Results Concerning Partitions with Designated Summands," *The Ramanujan Journal*, vol. 54, no. 2, pp. 385-395, 2021. [CrossRef] [Google Scholar] [Publisher Link]
[8] Michael D. Hirschhorn, and James A. Sellers, "Arithmetic Properties of Partitions with Odd Parts Distinct," *The Ramanujan Journal*, vol. 22, no. 3, pp. 273-284, 2010. [CrossRef] [Google Scholar] [Publisher Link]
[9] Rupam Barman, Gurinder Singh, and Ajit Singh, "Divisibility of the Partition Function $PDO_t(n)$by Powers of 2 and 3," *Bulletin of the Australian Mathematical Society*, pp. 14-25, 2024. [CrossRef] [Google Scholar] [Publisher Link]
[10] Abhay Mehta, Vandna, and Mandeep Kaur, "Some New Results on the Number of Tagged Parts over the Partitions with Designated Summands, vol. 5, no. 1, pp. 602-607, 2024. [CrossRef] [Google Scholar] [Publisher Link]
[11] Ernest X.W. Xia, "Arithmetic Properties of Partitions with Designated Summands," *Journal of Number Theory*, vol.159, pp. 160-175, 2016. [CrossRef] [Google Scholar] [Publisher Link]
[12] William Y.C. Chen et al., "On the Number of Partitions with Designated Summands," *Journal of Number Theory*, vol. 133, no. 9, pp. 2929-2938, 2013. [CrossRef] [Google Scholar] [Publisher Link]
[13] N.D. Baruah, and K.K. Ojah, Partitions with Designated Summands in Which All Parts are Odd, *Integers*, vol. 15, no. A9, 2015. [Google Scholar]
[14] M.S. Mahadeva Naika, and S. Shivaprasada Nayaka, "Congruences for (2,3)-Regular Partition with Designated Summands," *Notes on Mathematics*, vol. 36, no. 2, pp. 99-123, 2016. [CrossRef] [Google Scholar] [Publisher Link]
[15] A. Mehta, and M. Kaur, "Congruences for k-Regular Partition with Designated Summands, *JP Journal of Algebra, Number Theory andApplications*, vol. 53, no. 2, pp. 175-194, 2022. [CrossRef] [Google Scholar]